Name: ______________________________

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MATH@TD

ANALYTIC GEOMETRY

Unit Outline:

a. Overview: Terms and Formulas
b. Midpoint and Perpendicular Bisector of a Line
c. Length of a line segment
d. The Equation of a Circle
e. Classifying Geometric Figures

Analytic Geometry: Terms and Formulas

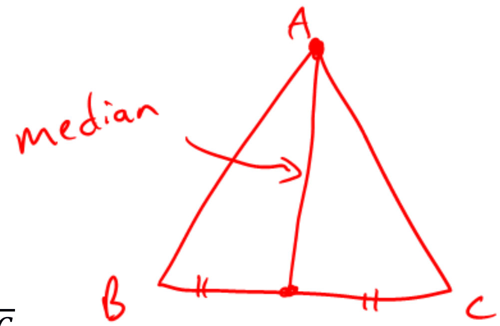
“Analytic Geometry” is using algebra to analyze geometric properties of shapes. The connection between the algebra and the geometry is through formulas which use the coordinates of points.

Some Terms

Line Segment – A part of a line between two points. For example

shows line segment $\overline{AB}$

Midpoint – The point in the middle of a line segment



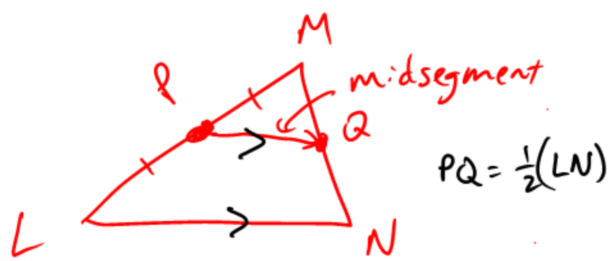
$M_{\overline{AB}} = D(x, y)$

Median – A line segment in a triangle from one vertex to the midpoint of the opposite side

$\overline{AD}$ is a median of triangle *ABC*. *D* is the midpoint of $\overline{BC}$

Midsegment – A midsegment is a line segment inside a triangle which joins the midpoints of two sides of the triangle.

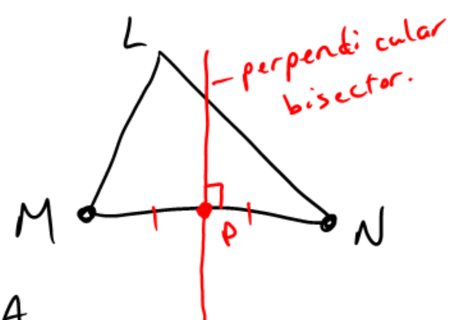
If P is the midpoint of $\overline{LM}$, and
Q is the midpoint of $\overline{MN}$, then
$\overline{PQ}$ is a midsegment of triangle *LMN*



Note: The slope of $\overline{PQ}$ is equal to the slope of $\overline{LN}$

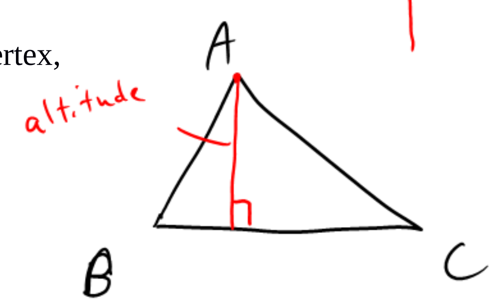
90° cut in two

Perpendicular Bisector – A line which cuts a line segment in half, and which is also perpendicular to that line segment.



Note that point P is the midpoint of $\overline{MN}$, and that the slope of line *l* is the negative reciprocal of the slope of $\overline{MN}$

Altitude – A line segment inside a triangle from one vertex, and perpendicular to the opposite side



$\overline{AD}$is an altitude of triangle *ABC*

The slope of $\overline{AD}$ is the negative reciprocal of the slope of $\overline{BC}$

Formulas

Slope of a line (or line segment) – Given two points on a line $A(x_1, y_1)$and $B(x_2, y_2)$, then

$$m_{\overline{AB}} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Equation of a line – To determine the equation of a line you need two pieces of information: you need a slope and a point (or two points, and then you would calculate the slope using the slope formula). The equation is:

$y = mx + b$ (slope-intercept form), or

$\boxed{y - y_1 = m(x - x_1)}$ (slope-point form)

Midpoint – Given a line segment $\overline{AB}$ with endpoints $A(x_1, y_1)$and $B(x_2, y_2)$, then

$$M_{\overline{AB}} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Length of a line segment (or distance between two points) - Given a line segment $\overline{AB}$ with endpoints $A(x_1, y_1)$and $B(x_2, y_2)$, then the length of $\overline{AB}$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

change in x

Δ = delta

Equation of a Circle (with centre $(0,0)$) – A circle centered at the origin, and with radius r has the equation

$$x^2 + y^2 = r^2.$$

2.0 Writing Equations

Determine the equation of the line that: put in slope-intercept form. Standard form: $Ax+By+C=0$

a) passes through $(\overset{x_1, y_1}{-1, 7})$ and $(\overset{x_2\ y_2}{2, 14})$

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{14-7}{2-(-1)} = \frac{7}{3}$

$y - y_1 = m(x-x_1)$

plug in $m=\frac{7}{3}$ pt $(\underset{x_1, y_1}{-1, 7})$

$y-7 = \frac{7}{3}(x+1)$

$y-7^{+7} = \frac{7}{3}x + \frac{7}{3}^{+7}$

$\boxed{y = \frac{7}{3}x + \frac{28}{3}}$

$\frac{7}{3}+7 = \frac{7}{3}+\frac{21}{3} = \frac{28}{3}$

Standard Form

$\left[0 = \frac{7}{3}x - y + \frac{28}{3}\right]\times 3$

$\boxed{0 = 7x - 3y + 28}$

b) is Perpendicular to $y = -2x - 3$ and passes through $(\underset{x_1, y_1}{2, -5})$

$\rightarrow m = -2 \perp \boxed{\frac{1}{2}}$

$\rightarrow$ perpendicular

$y - y_1 = m(x-x_1)$

$y+5 = \frac{1}{2}(x-2)$

$y+5 = \frac{1}{2}x - 1$

$\boxed{y = \frac{1}{2}x - 6}$

$\left[0 = \frac{1}{2}x - y - 6\right]\times 2$

$\boxed{0 = x - 2y - 12}$

Practice:

c) passes through (3, 4) and (-2, -7)

d) parallel to m = 3 and passes through (1, 3)

e) is perpendicular to $y = -2x - 3$ and passes (3, 4)

f) is parallel to $2x - 3y = 8$ and passes through (2, -5)

2.1 Midpoint and Perpendicular Bisector of a Line

Find the Midpoint of a line – The point in the middle of a line segment

Question: If you scored a 70% on a test and then an 82% on the next test, what is the average of those tests?

$$\frac{70+82}{2} = \frac{152}{2} = 76$$

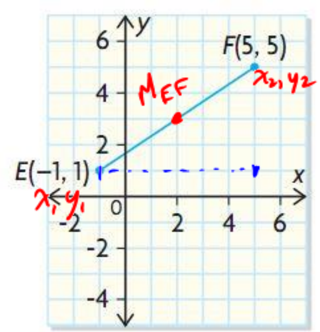
$M_{\overline{AB}} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$Similarly, the coordinates of the midpoint (M) of a line is the midpoint (average) of the x-values and the midpoint of the y-values

$$\cancel{M_{\overline{AB}} = D(x, y)}$$

Examples

1) From your text: Pg. 78 #2a

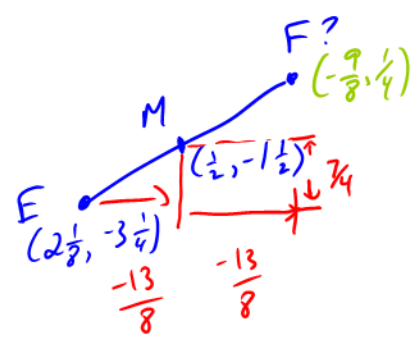
Determine the coordinates of the midpoint of the line segment.



$$M_{EF} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-1+5}{2}, \frac{1+5}{2}\right) = (2, 3)$$

2) From your text: Pg. 74 Example 2

Line segment EF has an endpoint at $E\left(2\frac{1}{8}, -3\frac{1}{4}\right)$. Its midpoint is located at $M\left(\frac{1}{2}, -1\frac{1}{2}\right)$. Determine the coordinates of endpoint F.



$$x_2 - x_1 = \frac{1}{2} - 2\frac{1}{8} = \frac{1}{2} - \frac{17}{8} = \frac{4}{8} - \frac{17}{8} = \frac{-13}{8}$$

$$\frac{1}{2} + -\frac{13}{8} = \frac{4}{8} - \frac{13}{8} = \boxed{-\frac{9}{8}}$$

$$y_2 - y_1 = -1\frac{1}{2} - \left(-3\frac{1}{4}\right) = -\frac{3}{2} + \frac{13}{4} = -\frac{6}{4} + \frac{13}{4} = \frac{7}{4}$$

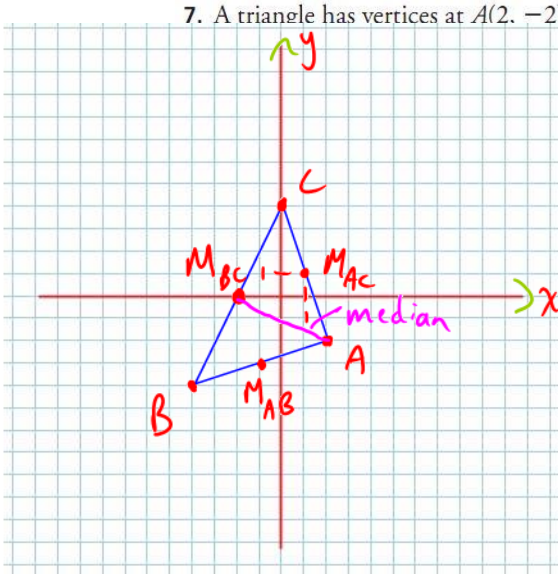
$$-1\frac{1}{2} + \frac{7}{4} = -\frac{3}{2} + \frac{7}{4} = -\frac{6}{4} + \frac{7}{4} = \boxed{\frac{1}{4}}$$

Do pg 78: 2-8

Find the equation of a Median
A line segment in a triangle from one vertex to the midpoint of the opposite side.

3) From your text: Pg. 79 #7

7. A triangle has vertices at $A(2, -2)$, $B(-4, -4)$, and $C(0, 4)$.
…nine the coordinates of the midpoints
…A, and determine its equation.



a) $M_{AB} = \left(\frac{2+-4}{2}, \frac{-2+-4}{2}\right) = (-1,-3)$

$M_{AC} = \left(\frac{2+0}{2}, \frac{-2+4}{2}\right) = (1,1)$

$M_{BC} = \left(\frac{-4+0}{2}, \frac{-4+4}{2}\right) = (-2,0)$

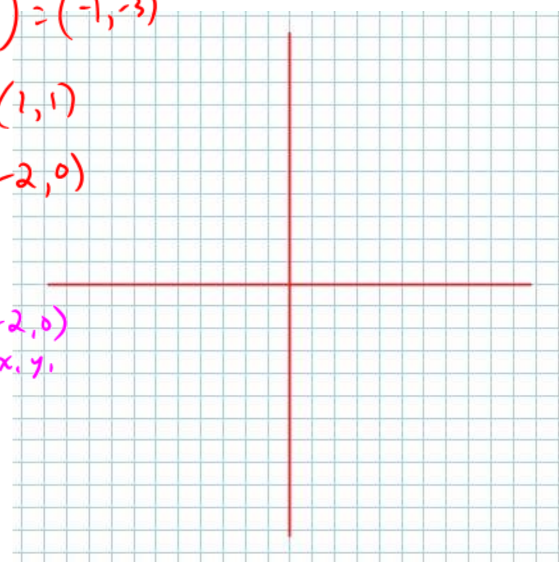
b) eqn of $\overline{AM_{BC}}$

$m = \frac{-2}{4} = -\frac{1}{2}$ pt $(-2,0)$ x_1, y_1

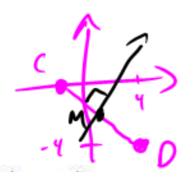
$y - y_1 = m(x - x_1)$

$y - 0 = -\frac{1}{2}(x+2)$

$\boxed{y = -\frac{1}{2}x - 1}$



Find the Perpendicular Bisector – A line which cuts a line segment in half, and which is also perpendicular to that line segment. Perpendicular therefore have slopes where m_2 is the negative reciprocal of m_1 $m_1 = \frac{3}{2}$ $m_2 = -\frac{2}{3}$



From your text: Pg. 80 #13a

13. Determine an equation for the perpendicular bisector of a line segment with each pair of endpoints.

a) $C(-2, 0)$ and $D(4, -4)$
x_1, y_1 x_2 y_2

$M_{CD} = \left(\frac{-2+4}{2}, \frac{0+-4}{2}\right)$

$= \boxed{(1,-2)}$
x_1, y_1

$m_{CD} = \frac{-4-0}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3} \perp \boxed{\frac{3}{2}} m_\perp$

$y - y_1 = m(x - x_1)$

$y + 2 = \frac{3}{2}(x - 1)$

$y + 2 = \frac{3}{2}x - \frac{3}{2}$

$y = \frac{3}{2}x - \frac{3}{2} - 2$

$y = \frac{3}{2}x - \frac{7}{2}$

Class/Homework: Pg78 – 79 # 2-8, 11, 13

2.2 – Length of a line segment (distance between two points)

Given a line segment $\overline{AB}$ with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$, then the length of AB is given by:

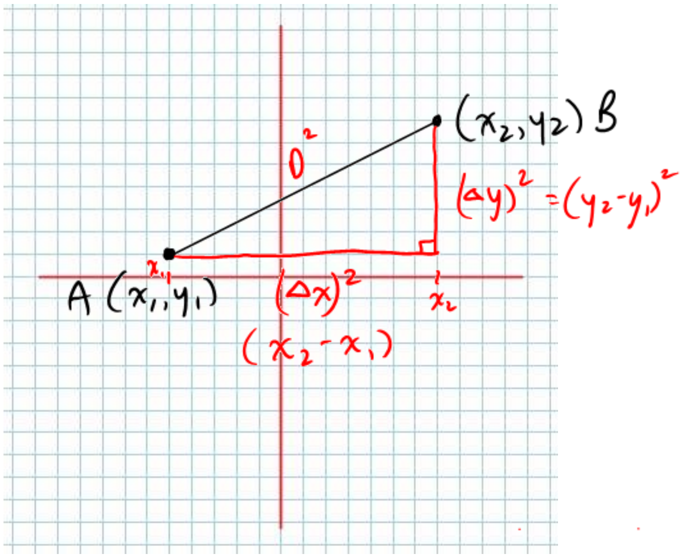
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

think Pythagorean Theorem

$\boxed{a^2 + b^2 = c^2} \rightarrow c = \sqrt{a^2 + b^2}$

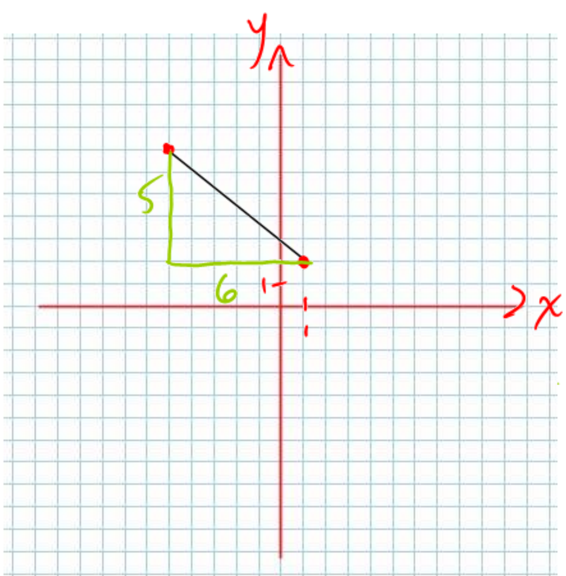
$D_{\overline{AB}} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$D_{\overline{AB}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Example: Find the length of the line from (1, 2) to (-5 , 7) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$x_1 \; y_1 \qquad x_2 \; y_2$



$d = \sqrt{(-5-1)^2 + (7-2)^2}$

$= \sqrt{(-6)^2 + (5)^2}$

$= \sqrt{36 + 25}$

$= \sqrt{61} \approx 7.8$

or

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$= \sqrt{6^2 + 5^2}$

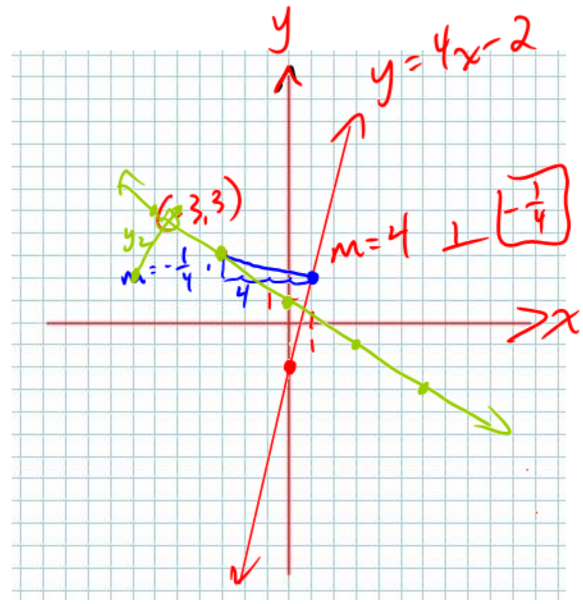
$= \sqrt{36 + 25}$

$= \sqrt{61} \approx 7.8$

Example (From your Text: Pg. 87 #12a)
Calculate the distance between the line and the point. y = 4x – 2 (-3, 3)

The shortest distance is a line perpendicular to y=4x - 2

Step 1 find m_1 and m_2
Step 2 find the equation of the perpendicular line
$y = mx + b$ (slope-intercept form), or
$y - y_1 = m(x - x_1)$ (slope-point form)



Step 3 Find the POI of the two lines. Solve the system by graphing, substitution or elimination

Step 4 Find the length of the line from the POI to (-3, 3)
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{1^2 + 4^2}$$
$$= \sqrt{17} \approx 4.1$$

Tough: Distance between $y = -\frac{2}{3}x + 1$ and $(-7, 2)$

$m = -\frac{2}{3} \perp \frac{3}{2}$

y_2 : $m = \frac{3}{2}$ pt $(-7, 2)$ (x_1, y_1)

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{2}(x + 7)$

$y - 2 = \frac{3}{2}x + \frac{21}{2}$

$y = \frac{3}{2}x + \frac{21}{2} + 2$

$\boxed{y = \frac{3}{2}x + \frac{25}{2}}$

and $\boxed{y = -\frac{2}{3}x + 1}$

Substitution

$\left[\frac{3}{2}x + \frac{25}{2} = -\frac{2}{3}x + 1\right] \times \frac{6}{1}$

$9x + 75 = -4x + 6$

$\frac{13x}{13} = \frac{-69}{13}$

$x = -\frac{69}{13}$

$y?$: $y = -\frac{2}{3}\left(-\frac{69}{13}\right) + 1$

$y = \frac{138}{39} + 1 = \frac{138}{39} + \frac{39}{39} = \frac{177}{39} = \frac{59}{13}$

POI is $\left(-\frac{69}{13}, \frac{59}{13}\right)$

Distance b/w $(-7, 2)$ and $\left(-\frac{69}{13}, \frac{59}{13}\right)$

$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(-\frac{69}{13} + 7\right)^2 + \left(\frac{59}{13} - 2\right)^2} \approx 3.05$

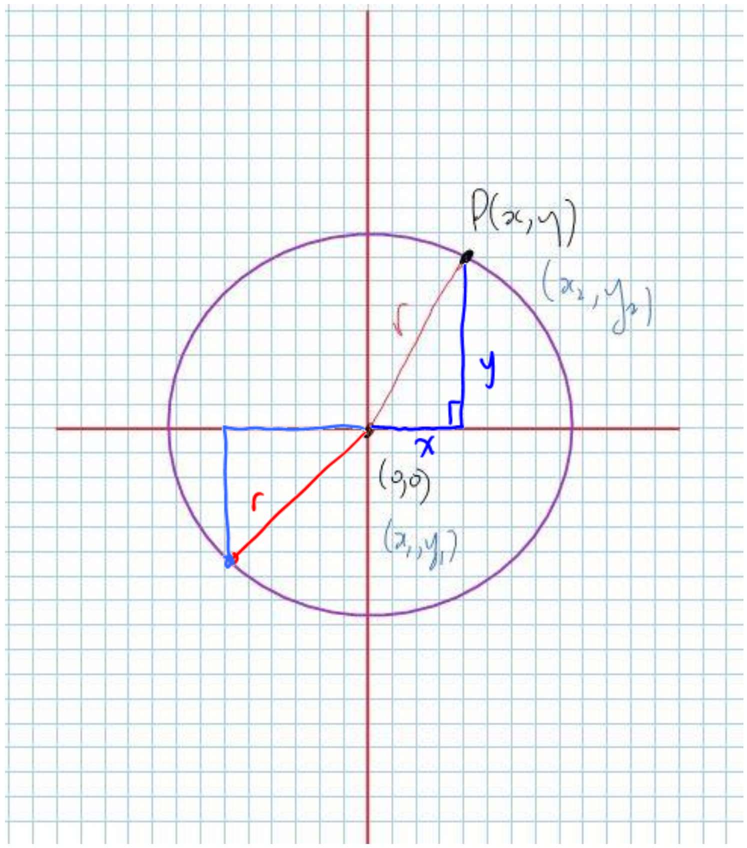
Class/Homework: Pg 86 – 87 #2bc, 3, 5ace, 7 (Calculate two lengths!), 9, 12, 14 Draw Pictures for the problems that need them

2.3 – The Equation of a Circle centered at $(0,0)$

Analytic Definition of a Circle (i.e. the equation)

A **Circle** is a set of points which are all the same **distance** from a central point.

$$x^2 + y^2 = r^2$$



Pythag. Thm

$$\boxed{x^2 + y^2 = r^2}$$

Example 2.3.1

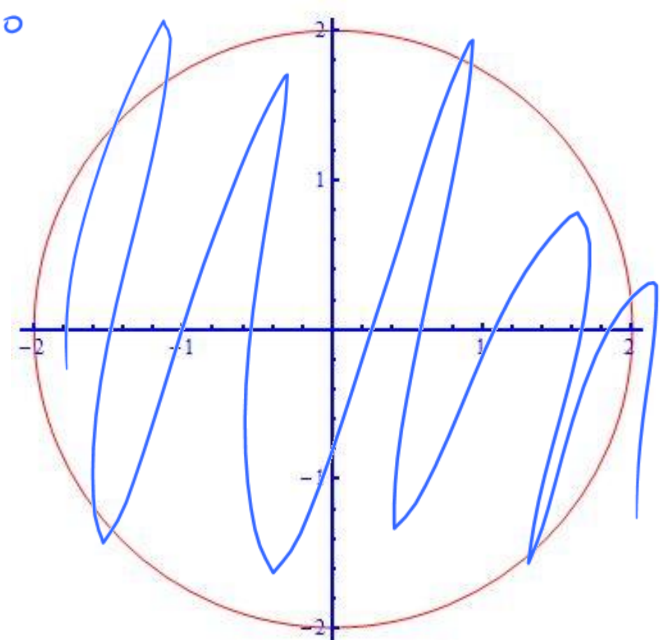
Determine the radius of the circle. Set y = 0
=0.

$x^2 + y^2 = 16$

$x^2 + y^2 = r^2$

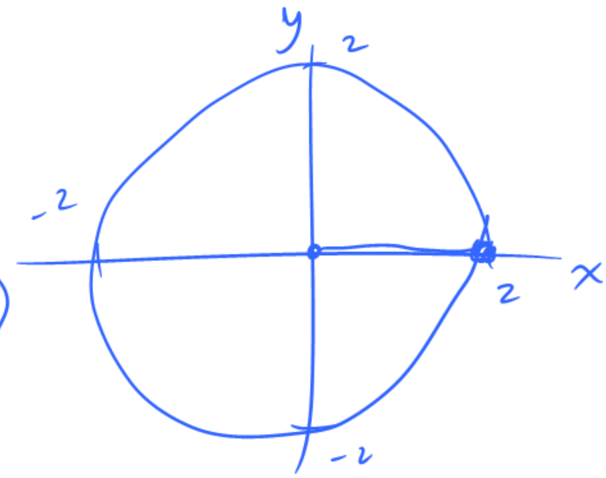
$r^2 = 16$

$r = 4$



Example 2.3.2

Consider the sketch of a circle. Determine:



a) x intercepts $\rightarrow x = -2$ and 2

b) y intercepts $\rightarrow y = -2$ and 2

c) the radius of the circle 2

d) the equation of the circle

$$x^2 + y^2 = r^2 \quad (\text{centre } (0,0))$$

$$x^2 + y^2 = 2^2$$

$$\therefore x^2 + y^2 = 4$$

Example 2.3.3

Determine the equation of a circle with radius $r = 5\frac{2}{3}$. $= \frac{17}{3}$

$$x^2 + y^2 = \left(\frac{17}{3}\right)^2$$

$$\boxed{x^2 + y^2 = \frac{289}{9}}$$

Example 2.3.4

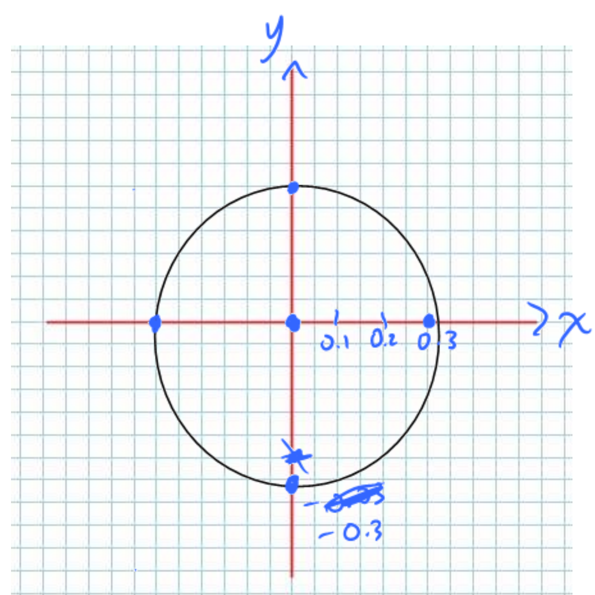
Sketch the circle with equation

$$x^2 + y^2 = 0.09$$

$x^2 + y^2 = \boxed{r^2}$ with centre $(0,0)$

$$r^2 = 0.09$$

$$r = 0.3$$



Example 2.3.5

Determine the equation of a circle with center at $(0,0)$ and a **diameter** of 14 units.

$r = 7$

$x^2 + y^2 = 7^2$

$x^2 + y^2 = 49$

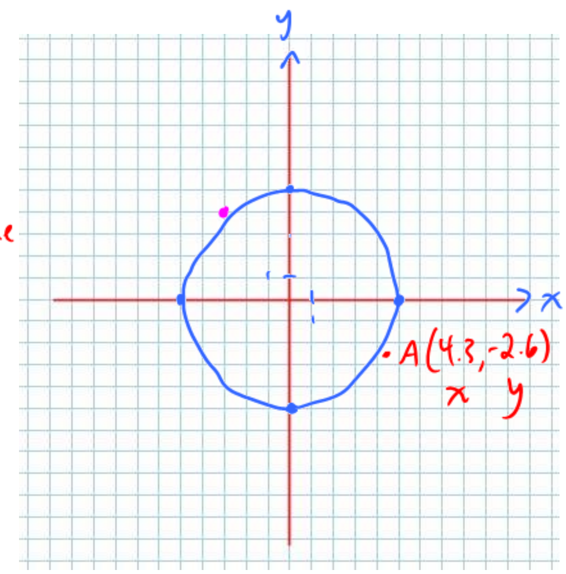
Example 2.3.6

Determine whether the point $A(4.3,-2.6)$ is inside, outside or on the circle with equation $x^2 + y^2 = 25$

$x^2 + y^2 = \boxed{r^2}$ $r = 5$

$(4.3)^2 + (-2.6)^2 = 25.25 > 25$

outside of the circle.



What about the point $(-3,4)$?

$(-3)^2 + (4)^2 = 25 = 25$

on the circle.

Example 2.3.7

Determine the equation of a circle with center $(0,0)$ which passes through the point $(7,-3)$.

$x^2 + y^2 = r^2$

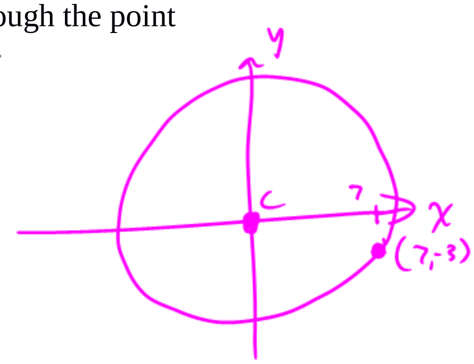
$(7)^2 + (-3)^2 = r^2$

$49 + 9 = r^2$

$58 = r^2$

$\sqrt{58} = r$

eqn: $x^2 + y^2 = 58$



1-7,11-15

Class/Homework

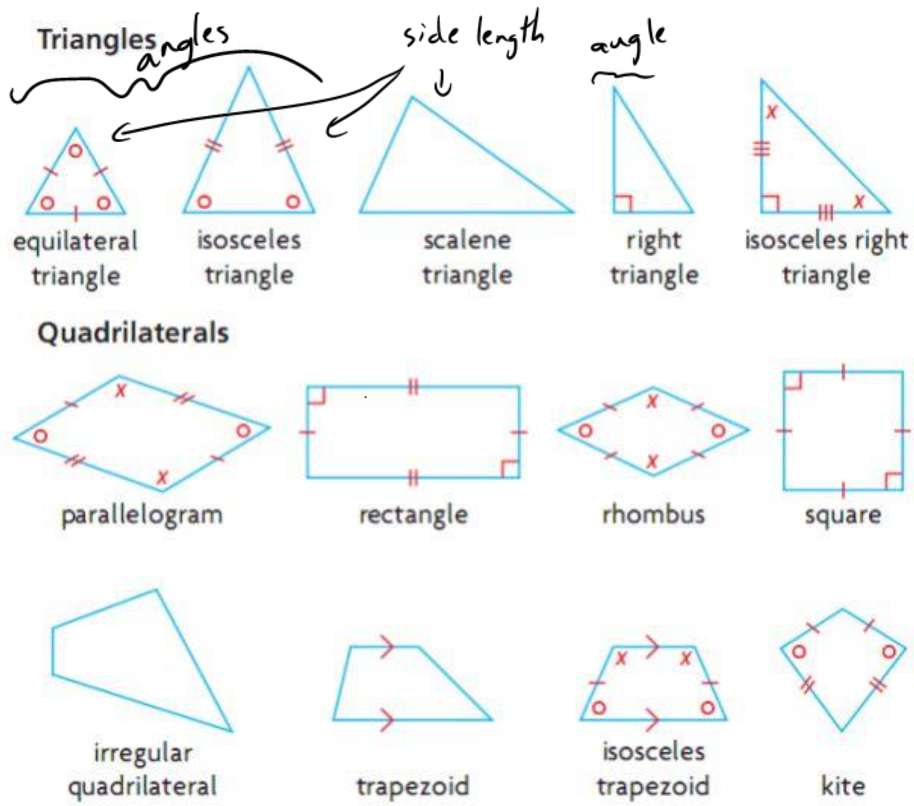
Pg. 91 – 93 #1-7

10 (Remember: The Circumference of a circle is given by), **11, 12, 13, 15, 16** (Optional)

2.4 – Classifying Geometric Figures

There are so many geometric figures that it's ridiculous. But we now know enough Analytic Geometry that we can easily do the "classification". We are really only going to worry about two "classes": Triangle and Quadrilaterals

You need to know the following types of Triangles and Quadrilaterals:



Note: When trying to "classify" some geometric figure **using analytic geometry**, keep in mind the following three "rules of thumb":

- To solve a problem that involves a geometric figure, it is a good idea to start by drawing a diagram of the situation on a coordinate grid.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are negative reciprocals.

Example 2.4.1

From your text – Pg. 101 #2

2. Show that *TU*, *T*(−1, 7) and *U*(3, 5), is perpendicular to *VW*, *V*(−4, 1) and *W*(−1, 7).

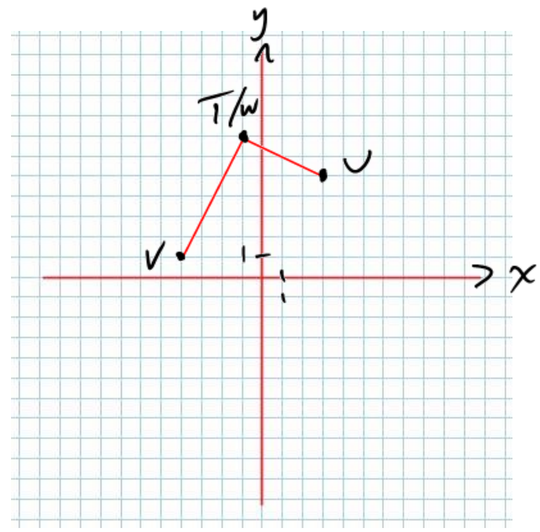
↳ slope

$$m_{TU} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$$

$$m_{VW} = \frac{6}{3} = \boxed{2}$$

$\perp$

$\therefore TU \perp VW$



Example 2.4.2

From your text – Pg. 101 #3

3. The sides of quadrilateral *ABCD* have the following slopes.

Side	*AB*	*BC*	*CD*	*AD*
Slope	−5	$-\frac{1}{7}$	−5	$-\frac{1}{7}$

What types of quadrilateral could *ABCD* be? What other information is needed to determine the exact type of quadrilateral?

Two opposite parallel sides! Adjacent sides are not perpendicular $-5 \not\perp -\frac{1}{7}$

It is a rhombus or parallelogram. We need to know lengths.

Example 2.4.3

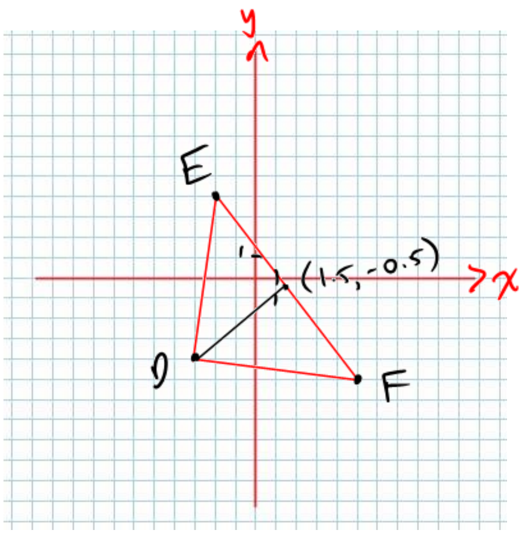
From your text – Pg. 101 #4

4. $\triangle DEF$ has vertices at $D(-3, -4)$, $E(-2, 4)$, and $F(5, -5)$.

a) Show that $\triangle DEF$ is isosceles. → 2 equal angles $\overset{\text{iff}}{\Longleftrightarrow}$ 2 equal sides

b) Determine the length of the median from vertex D. Ls? length? Distance

c) Show that this median is perpendicular to EF.



a) $D_{DE} = \sqrt{8^2 + 1^2} = \sqrt{65}$

$D_{DF} = \sqrt{1^2 + 8^2} = \sqrt{65}$

$D_{EF} = \sqrt{7^2 + 9^2} = \sqrt{49+81} = \sqrt{130}$

$\therefore \triangle DEF$ is isosceles since $DE = DF$

b) median: from D to M_{EF}

$M_{EF} = \left(\frac{-2+5}{2}, \frac{4+(-5)}{2}\right) = (\underset{x_1}{1.5}, \underset{y_1}{-0.5})$ $D(\underset{x_2}{-3}, \underset{y_2}{-4})$

Distance of D to M_{EF}

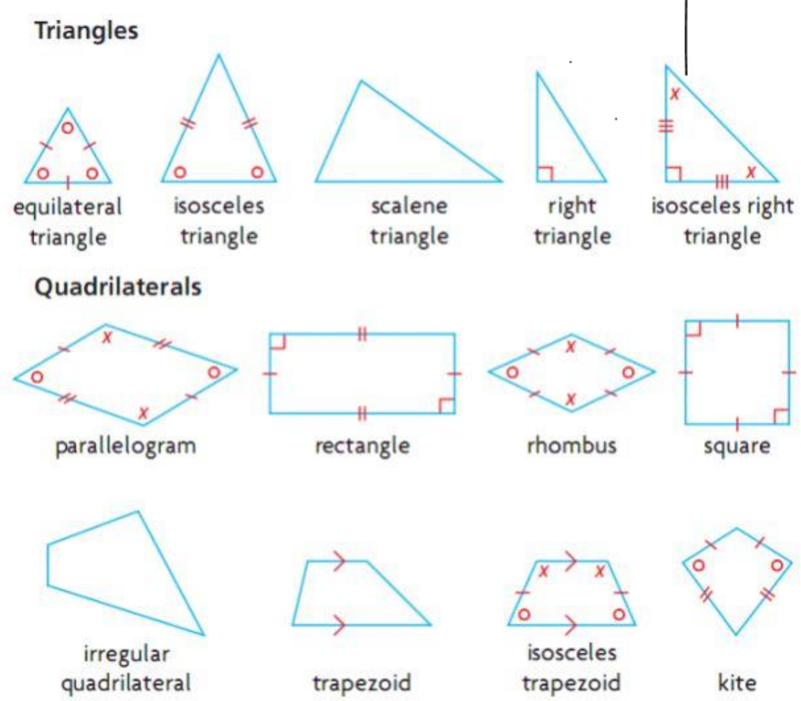
$D = \sqrt{(-3-1.5)^2 + (-4+0.5)^2}$

$= \sqrt{(-4.5)^2 + (-3.5)^2}$

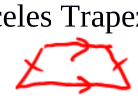
$= \sqrt{32.5} \approx 5.7$

c) $m_{median} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4+0.5}{-3-1.5} = \frac{-3.5}{-4.5} = \frac{-35}{-45} = \boxed{\frac{7}{9}}$

$m_{EF} = \boxed{-\frac{9}{7}}$ ← They are perpendicular! This feels so right.



Classifying Geometric Figures

Shape	What are you looking for when trying to classify each geometric shape? What formulas/calculations would you use to prove it?
Equilateral Triangles	① 3 equal sides → Distance formula or ② 3 equal angles → we don't know.
Isosceles Triangle	① 2 equal sides → Distance formula or ② 2 equal angles → we don't know.
Scalene Triangles	① 3 non-equal sides → Distance formula.
Right angle Triangles	① Right angles → slopes : ⊥ line segments.
Parallelogram	① 2 opposite parallel sides → slope or ② 2 opposite equal sides → distance
Rectangle	① 4 Right angles → slope
Rhombus	① 4 equal sides → distance
Square	① 4 Right angles and 4 equal lengths → slope and distance.
Irregular quadrilateral 	① 4 non-equal sides : distance.
Trapezoid	① 1 set of opposite parallel sides → slope.
Isosceles Trapezoid 	① 1 set of opposite parallel sides → slope and 2 equal lengths → distance.
Kite 	① 2 sets of adjacent equal sides → distance.

Class/Homework **Pg. 102 - 103 #5 – 14, 16 – 18**

Do: 5-9

Triangles

Sides

Angles

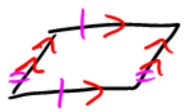
right

Obtuse

Acute

Quadrilaterals - 4 sides

Parallelogram



all sides equal

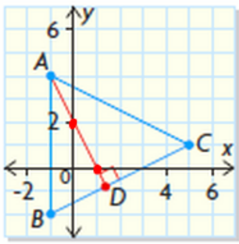
Rhombus

perpendicular adjacent sides

Rectangle

Square

Pg 120: #1.



a) $m_{BC} = \frac{3}{6} = \frac{1}{2}$

b) $m_{AD} = \frac{-2}{1} = -2$

c) $y = mx + b$

$y = -2x + 2$

2. $m_{BC} = \frac{1}{2}$ $(\overset{x_1}{-1}, \overset{y_1}{-2})$

$y - y_1 = m(x - x_1)$

$y + 2 = \frac{1}{2}(x + 1)$

$y + 2^{-2} = \frac{1}{2}x + \frac{1}{2}^{-2}$

$y = \frac{1}{2}x - \frac{3}{2}$

3. D? $y = \frac{1}{2}x - \frac{3}{2}$ and $y = -2x + 2$

find the POI by substitution

$\left[\frac{1}{2}x - \frac{3}{2} = -2x + 2\right] \times 2$

$x - 3 = -4x + 4$

$5x = 7$

$x = \frac{7}{5}$

$y = \frac{1}{2}\left(\frac{7}{5}\right) - \frac{3}{2}$

$= \frac{7}{10} - \frac{3}{2}$

$= \frac{7}{10} - \frac{15}{10}$

$= \frac{-8}{10} = \frac{-4}{5}$

$D\left(\frac{7}{5}, -\frac{4}{5}\right)$

To do: pg 120: 9, 10