
CRIME SCENE DO NOT CR

Chapter

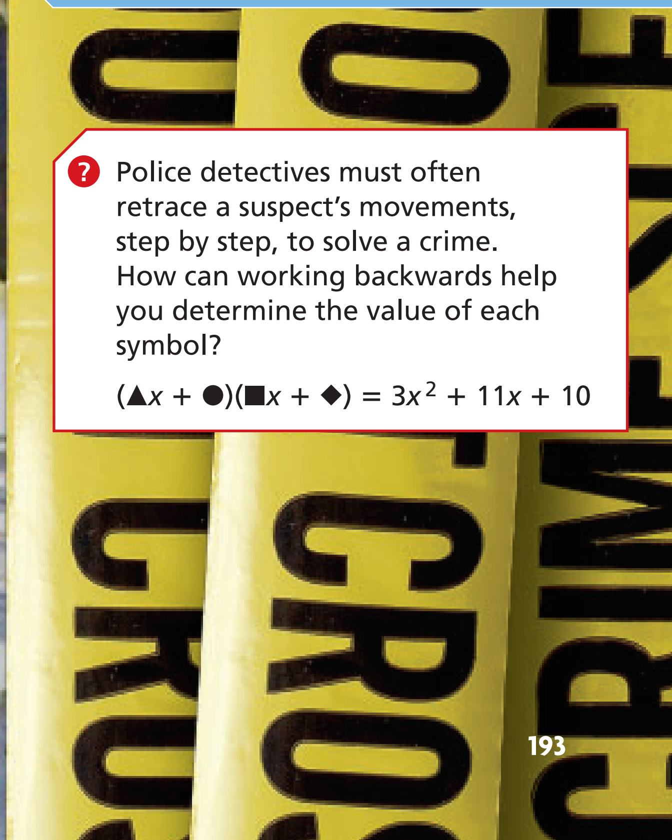
4

Factoring Algebraic Expressions

GOALS

You will be able to

- Determine the greatest common factor in an algebraic expression and use it to write the expression as a product
- Recognize different types of quadratic expressions and use appropriate strategies to factor them



? Police detectives must often retrace a suspect's movements, step by step, to solve a crime. How can working backwards help you determine the value of each symbol?

$(\blacktriangle x + \bullet)(\blacksquare x + \blacklozenge) = 3x^2 + 11x + 10$

4 Getting Started

WORDS YOU NEED to Know

1. Match each word with the mathematical expression that best illustrates its definition.

a) binomial	**d)** monomial	**g)** expanding
b) coefficient	**e)** variable	**h)** like terms
c) factoring	**f)** trinomial	

i) $2x - 5$	**v)** $24 = 2 \times 2 \times 2 \times 3$
ii) $4(2x - 3) = 8x - 12$	**vi)** $4xy$
iii) $7x^2 + 3x - 1$	**vii)** x
iv) $8x^2$	**viii)** $6y$ and $-8y$

SKILLS AND CONCEPTS You Need

Simplifying an Algebraic Expression

Study *Aid*

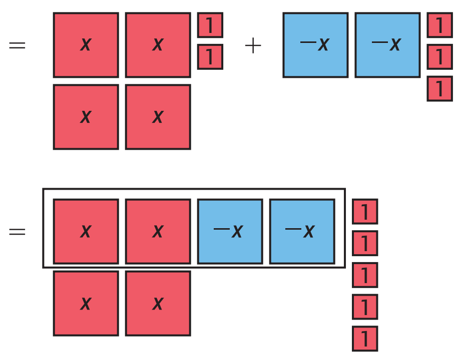
- For more help and practice, see Appendix A-8.

To simplify an algebraic expression, create a simpler equivalent expression by collecting like terms.

EXAMPLE

Simplify $(4x^2 + 2) + (-2x^2 + 3)$.

Solution

Using Symbols	Using an Algebra Tile Model
$(4x^2 + 2) + (-2x^2 + 3)$ $= 4x^2 + 2 - 2x^2 + 3$ $= 4x^2 - 2x^2 + 2 + 3$ $= 2x^2 + 5$	$(4x^2 + 2) + (-2x^2 + 3)$  $= 2x^2 + 5$

2. Simplify each expression.

a) $4x - 6y + 8y - 5x$

b) $5ab - 6a^2 + 6ab - 3a^2 - 11ab + 9b^2$

c) $(2x - 5y) + (7x + 4) - (5x - y)$

d) $(7a - 2ab) - (4b + 5a) + (ab - 3a)$

Expanding an Algebraic Expression

To expand an algebraic expression, multiply all parts of the expression inside the brackets by the appropriate factor. You can use the distributive property algebraically or geometrically with a model.

Multiplying by a Monomial	Multiplying by a Binomial
$a(b + c) = ab + ac$	$(a + b)(c + d) = a(c + d) + b(c + d)$ $= ac + ad + bc + bd$

Study | Aid

- For more help and practice, see Appendix A-8.

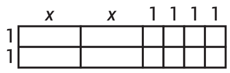
EXAMPLE

Expand and simplify.

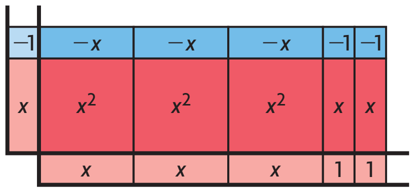
a) $2(2x + 4)$ **b)** $(x - 1)(3x + 2)$

Solution

a)

Distributive Property	Area Model
$2(2x + 4)$ $= 4x + 8$	 The area is $4x + 8$.

b)

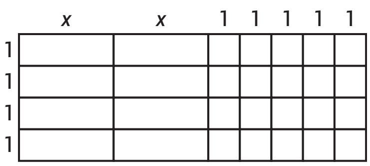
Distributive Property	Algebra Tile Model
$(x - 1)(3x + 2)$ $= x(3x + 2) - 1(3x + 2)$ $= 3x^2 + 2x - 3x - 2$ $= 3x^2 - x - 2$	 The area is $3x^2 - x - 2$.

3. Expand and simplify.

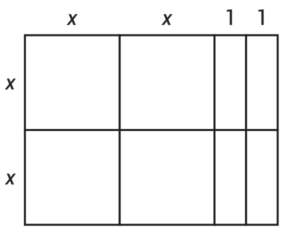
a) $7(2x - 5)$
b) $-5x(3x^2 - 4x + 5)$
c) $2(4x^2 + 3x + 1) - 2x(8x - 3)$
d) $(d - 6)(d + 2)$
e) $(3a - 7b)(4a - 3b)$
f) $6x(2x + 1)^2$

4. Determine the multiplication expression and product for each model.

a)



b)



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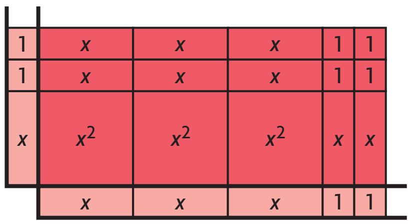
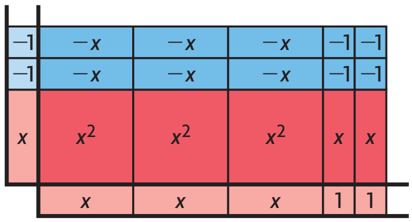


d)



Study Aid

- For help, see the Review of Essential Skills and Knowledge Appendix.

Question	Appendix
5	A-3

PRACTICE

5. Simplify.

a) $(x^5)(x^7)$ **c)** $(4y)(3y^2) \div 2y^3$

b) $(-6a^2)(3a^4)(2a)$ **d)** $20z^5 \div (-4z^3)(-z^2)$

6. Create an expression for each description.

a) a monomial with a coefficient of 5

b) a binomial with coefficients that are even numbers

c) a trinomial with coefficients that are three consecutive numbers

d) a quadratic trinomial

7. Identify the greatest common factor for each pair.

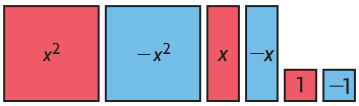
a) 28, 35 **c)** 99, 90 **e)** $25, 5x^2$

b) 36, 63 **d)** $4x$, 8 **f)** $12y, 6x$

8. a) Determine the x-intercepts, the equation of the axis of symmetry, and the vertex of $y = (x - 4)(x + 8)$.

b) Use the information you determined for part a) to sketch this parabola.

c) Express the equation in standard form.



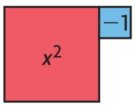
9. Sketch algebra tiles, like those shown at the left, to represent each expression.

a) $3x + 2$ **c)** $2x^2 - x + 3$ **e)** $2x^2 - x - 1$

b) $-2x - 4$ **d)** $-2x^2 - 1$ **f)** $1 + 2x - 3x^2$

10. Match each expression with the correct diagram.

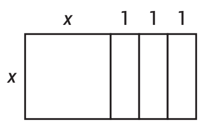
a) $x + 3$ **b)** $-3x + 2$ **c)** $x^2 - 1$

i)


ii)


iii)


11. The area model shown below represents $x^2 + 3x$.



Sketch an area model to show each expression.

a) $2x^2 - 3x$ **c)** $-x^2 - 3x$

b) $3x + 6$ **d)** $2x^2 + x$

12. Decide whether you agree or disagree with each statement. Explain why.

a) All even numbers contain the number 2 in their prime factorization.

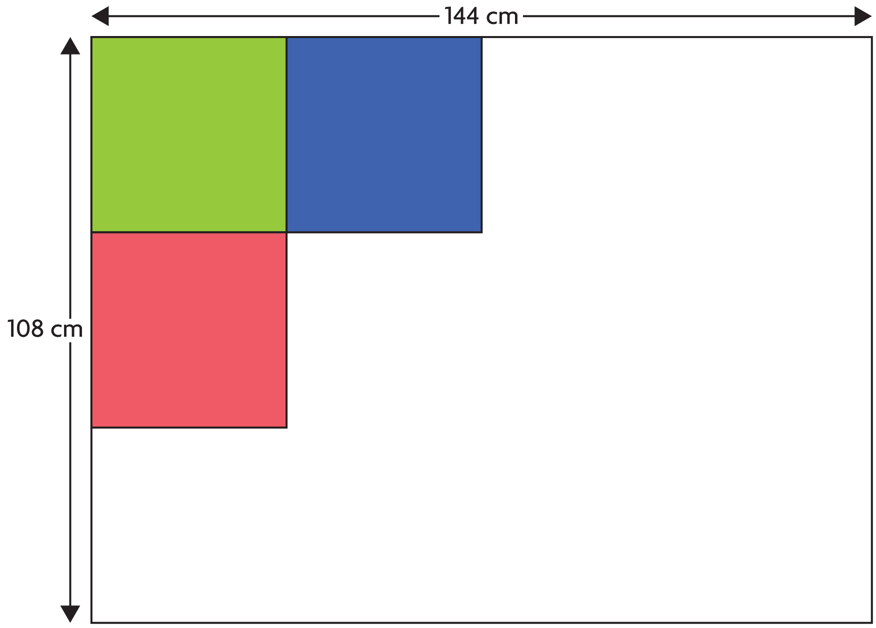
b) $6x^2y^3$ can be written as a product of two or more algebraic expressions.

c) The only way to factor 100 is 10×10.

APPLYING What You Know

Designing a Geometric Painting

Sara and Josh are creating a large square painting for their school's Art Fair. They are planning to start with a large rectangle, measuring 108 cm by 144 cm. They will divide the rectangle into **congruent** squares, which they will paint different colours. They want these squares to be as large as possible. To create the final painting, they will use copies of the 108 cm by 144 cm rectangle to create a large square with the least possible whole number dimensions.



? What are the dimensions of the small squares and the final large square painting?

A. How do you know that the side length of the small squares, inside the 108 cm by 144 cm rectangle, cannot be 8 cm?

B. Why must the side length of the small squares be a factor of 108 and 144?

C. What is the side length of the largest square that can be used to divide the 108 cm by 144 cm rectangle?

D. Why does the side length of the final large square painting (created using copies of the 108 cm by 144 cm rectangle) have to be a multiple of both 108 and 144?

E. What is the side length of the smallest final square painting that can be created using copies of the 108 cm by 144 cm rectangle?

4.1 Common Factors in Polynomials

YOU WILL NEED

- algebra tiles

GOAL

Factor algebraic expressions by dividing out the greatest common factor.

LEARN ABOUT the Math

Yasmine squared the numbers 5 and 6, and then added 1 to get a sum of 62.

$$5^2 + 6^2 + 1 = 25 + 36 + 1$$
$$= 62$$

She repeated this process with the numbers 8 and 9, and got a sum of 146.

$$8^2 + 9^2 + 1 = 64 + 81 + 1$$
$$= 146$$

Both 62 and 146 are divisible by 2.

$$\frac{62}{2} = 31 \text{ and } \frac{146}{2} = 73$$

? Is a number that is 1 greater than the sum of the squares of two consecutive integers always divisible by 2?

EXAMPLE 1 **Selecting a strategy** to determine the greatest common factor

Lisa's Solution: Selecting an algebraic strategy

$$n^2 + (n + 1)^2 + 1$$

I let n represent the first integer. Then two consecutive integers are n and $n + 1$. I wrote an expression for the sum of their squares and added 1.

$$= n^2 + (n + 1)(n + 1) + 1$$
$$= n^2 + (n^2 + 2n + 1) + 1$$
$$= 2n^2 + 2n + 2$$
$$= 2(n^2 + n + 1)$$

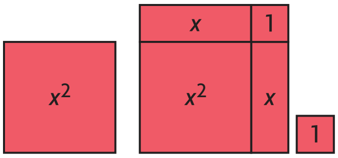
I simplified the expression by expanding and collecting like terms.
I factored the algebraic expression by determining the GCF of its terms. The GCF is 2.
I wrote an equivalent expression as a product.

2 is always a factor of this expression. Therefore, 1 greater than the sum of the squares of two consecutive integers is always divisible by 2.

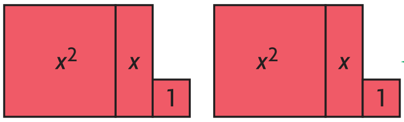
Communication *Tip*

GCF is an abbreviation for Greatest Common Factor.

Abdul's Solution: Representing with an area model



I used algebra tiles to show x^2 and $(x + 1)^2$. Then I added one unit tile. I wanted to see if I could make two equal groups. If I could, this would show that the number is divisible by 2.



I moved tiles around to get two equal groups.

The result will always be divisible by 2.

Since there are two equal groups, the total value is always a multiple of 2.

Reflecting

A. Why did both Lisa and Abdul use a variable to prove that the result is always even?

B. Why did Lisa need to **factor** in order to solve this problem?

C. Which strategy do you prefer? Explain why.

factor
to express a number as the product of two or more numbers, or express an algebraic expression as the product of two or more terms

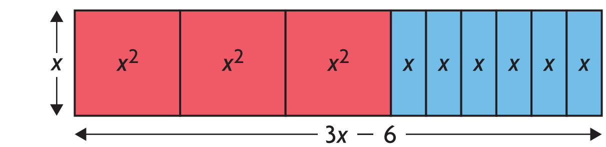
APPLY the Math

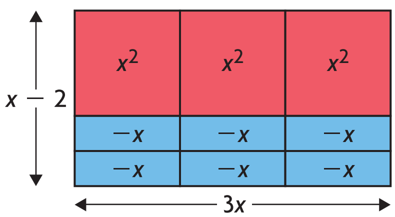
EXAMPLE 2 **Selecting a strategy** to factor a polynomial

Factor $3x^2 - 6x$ over the set of integers.

Communication Tip
Factoring over the set of integers means that all the numbers used in each factor of an expression must be integers. When you are asked to factor, this is implied.

Noel's Solution: Representing with an area model





I used algebra tiles to represent each term. I used red tiles for positive terms and blue tiles for negative terms. I arranged these tiles to form two different rectangles.

$3x^2 - 6x = x(3x - 6)$
and
$3x^2 - 6x = 3x(x - 2)$

The dimensions of each rectangle are factors of the expression.

$$3x^2 - 6x = x(3x - 6)$$
$$= x(3)(x - 2)$$
$$= 3x(x - 2)$$

I noticed that $x(3x - 6)$ could be factored again, since $3x - 6$ contains a common factor of 3. The greatest common factor is $3x$, which is the first factor in $3x(x - 2)$.

The polynomial is factored fully since one of the factors is the greatest common factor of the polynomial.

Communication *Tip*

An algebraic expression is factored fully when only 1 or −1 remains as a factor of every term in the factorization. When you are asked to factor, you are expected to factor fully.

Connie's Solution: Reasoning symbolically

$$3x^2 - 6x$$
$$= 3(x^2) + 3(-2x)$$

To factor, I need to determine the greatest common factor. The GCF of the coefficients 3 and 6 is 3.

$$= 3x(x) + 3x(-2)$$

The GCF of the variable parts x^2 and x is x.

The greatest common factor of both terms in $3x^2 - 6x$ is $3x$.
$3x^2 - 6x = 3x(x - 2)$

I used the distributive property to write an equivalent expression as the product of the factors.

EXAMPLE 3 **Connecting** the distributive property with factoring

Factor.

a) $10a^3 - 25a^2$ **b)** $9x^4y^4 + 12x^3y^2 - 6x^2y^3$

Antonio's Solution

a) The greatest common factor of the terms in $10a^3 - 25a^2$ is $5a^2$.

The GCF of the coefficients 10 and 25 is 5. The GCF of the variable parts a^3 and a^2 is a^2.

$10a^3 - 25a^2 = 5a^2(2a - 5)$

I used the distributive property to write an equivalent expression as the product of the factors.

b) The greatest common factor of the terms in $9x^4y^4 + 12x^3y^2 - 6x^2y^3$ is $3x^2y^2$.

The GCF of the coefficients 9, 12, and 6 is 3. The GCF of the variable parts is x^2y^2.

$$9x^4y^4 + 12x^3y^2 - 6x^2y^3$$
$$= 3x^2y^2(3x^2y^2 + 4x - 2y)$$

I used the distributive property to write an equivalent expression as the product of the factors.

EXAMPLE 4 | **Reasoning** to factor a polynomial

Factor each expression.

a) $5x(x - 2) - 3(x - 2)$ **b)** $ax - ay - 5x + 5y$

Joanne's Solution

a) $5x(x - 2) - 3(x - 2)$

Both $5x$ and -3 are multiplied by $x - 2$, so $x - 2$ is a common factor.

$= (x - 2)(5x - 3)$

I wrote an equivalent expression using the distributive property.

b) $ax - ay - 5x + 5y$

I noticed that the first two terms contain a common factor of a and the last two terms contain a common factor of -5.

To factor the expression, I used a grouping strategy. I grouped the terms that have the same common factor.

$= a(x - y) - 5(x - y)$

I wrote an equivalent expression using the distributive property. Both a and -5 are multiplied by $x - y$, so $x - y$ is a common factor. Since this binomial is the same in both terms, I can factor further.

$= (x - y)(a - 5)$

I wrote an equivalent expression using the distributive property.

In Summary

Key Ideas

- Factoring is the opposite of expanding. Expanding involves multiplying, and factoring involves looking for values to multiply.
- One way to factor an algebraic expression is to look for the greatest common factor of the terms in the expression. For example, $5x^2 + 10x - 15$ can be factored as $5(x^2 + 2x - 3)$ since 5 is the greatest common factor of all the terms.

expanding →

$4x(2x - 3) = 8x^2 - 12x$

← factoring

Need to Know

- It is possible to factor an algebraic expression by dividing by a common factor that is not the greatest common factor. This will result in another expression that still has a common factor. For example,

$$\begin{aligned} 8x + 16 &= 4(2x + 4) \\ &= 4(2)(x + 2) \\ &= 8(x + 2) \end{aligned}$$

- An algebraic expression is factored fully when only 1 or -1 remains as a factor of every term in the factorization. In the example above, $8x + 16$ is factored fully when it is written as $8(x + 2)$ or $-8(-x - 2)$.
- A common factor can have any number of terms. For example, a common factor of the terms in $9x^2 + 6x$ is $3x$, which is a monomial. A common factor of the terms in $(3x - 2)^2 - 4(3x - 2)$ is $(3x - 2)$, which is a binomial.

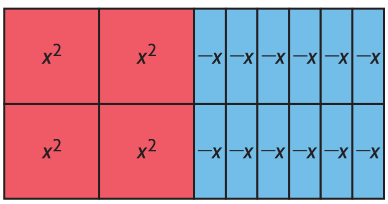
CHECK *Your Understanding*

1. In each algebra tile model below, two terms in an algebraic expression have been rearranged to show a common factor.

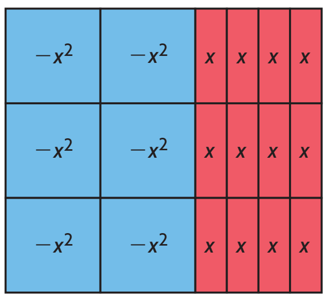
i) Identify the algebraic expression, and name the common factor that is shown.

ii) Determine the greatest common factor of each expression.

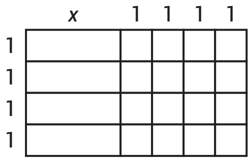
a)

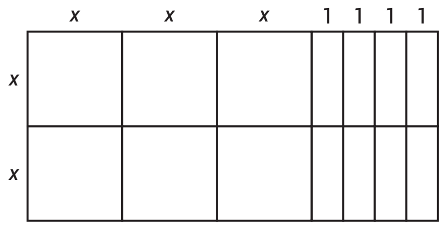


b)



2. Each area diagram represents a polynomial.
   i) Identify the polynomial, and determine a common factor of its terms.
   ii) Determine any other common factors of the terms in the polynomial.

a) 

b) 

3. State the GCF of each pair of terms.
   a) $6x$ and $10x$
   b) $15a^3$ and $20a^2$
   c) ab and a^2b^2
   d) $-2x^4y^4$ and $8x^3y^5$

PRACTISING

4. Identify the greatest common factor of the terms in each expression.
   a) $3x^2 - 9x + 12$
   b) $5x^2 + 3x$
   c) $x^2y - xy^2$
   d) $4x(x - 1) + 3(x - 1)$

5. Determine the missing factor.
   a) $8xy = (■)(2xy)$
   b) $-6x^2 = (3x^2)(■)$
   c) $15x^4z = (5x^2)(■)$
   d) $-49a^2b^5 = (■)(7b^3)$
   e) $-12x^3y^3 = (3y^3)(■)$
   f) $30m^2n^3 = (-5m^2n)(■)$

6. Determine the missing factor.
   a) $4x - 4y = (■)(x - y)$
   b) $8x - 2y = (2)(■)$
   c) $5a + 10b = (■)(a + 2b)$
   d) $36x^2 - 32y^3 = (4)(■)$
   e) $-24x^2 - 6y = (■)(4x^2 + y)$
   f) $45a^4 - 54a^3 = (9a^3)(■)$

7. Determine the greatest common factor of each expression.
   a) $7x^2 + 14x - 21$
   b) $3b^2 + 15b$
   c) $12c^2 - 8c + 16$
   d) $-25m^2 - 10m$
   e) $3d^4 - 9d^2 + 15d^3$
   f) $y^3 + y^5 - y^2$

8. Factor each expression. Then choose one expression, and describe the strategy you used to factor it.
   a) $9x^2 - 6x + 18$
   b) $25a^2 - 20a$
   c) $27y^3 - 9y^4$
   d) $2b(b + 4) + 5(b + 4)$
   e) $4c(c - 3) - 5(c - 3)$
   f) $x(3x - 5) + (3x - 5)(x + 1)$

9. Factor each polynomial. Then identify the two polynomials that have the same trinomial as one of their factors.
   a) $dc^2 - 2acd + 3a^2d$
   b) $-10a^2c + 20ac - 5ac^3$
   c) $10ac^2 - 15a^2c + 25$
   d) $2a^2c^4 - 4a^3c^3 + 6a^4c^2$
   e) $3a^5c^3 - 2ac^2 + 7ac$
   f) $10c^3d - 8cd^2 + 2cd$

10. Factor each expression.

a) $ax - ay + bx - by$

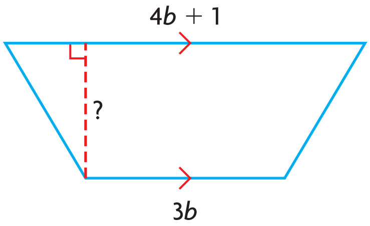
b) $10x^2 + 5x - 6xy - 3y$

c) $3mx + 3my + 2x + 2y$

d) $5my + tm + 5ny + tn$

e) $5wx - 10w - 3tx + 6t$

f) $4mnt - 16mn - t + 4$



11. The area of the trapezoid at the left is $A = 70b + 10$. Determine the height.

12. Examine each quadratic relation below.

i) Express the relation in factored form.

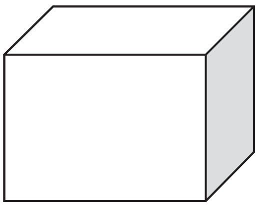
ii) Determine the zeros and the equation of the axis of symmetry.

iii) Determine the coordinates of the vertex.

iv) Sketch the graph of the relation.

a) $y = 2x^2 - 10x$

b) $y = -x^2 - 8x$



13. Determine an expression, in factored form, that can be used to determine the surface area of any rectangular prism.

14. **A** Two parabolas are defined by $y = 10x - x^2$ and $y = 3x^2 - 30x$. What is the distance between their maximum and minimum values?

15. **K** a) Write three quadratic binomials whose greatest common factor is $5x$. Then factor each binomial.

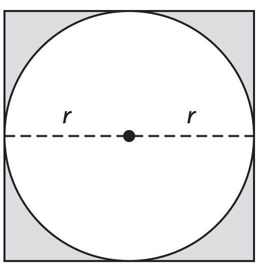
b) Write three quadratic trinomials whose greatest common factor is $3x$. Then factor each trinomial.

16. **C** Marek says that the greatest common factor of $-5x^3 + 10x^2 - 20x$ is $5x$. Jen says that the greatest common factor is $-5x$. Explain why both Marek and Jen are correct.

17. **T** Show that 1 greater than the sum of the squares of any three consecutive integers is always divisible by 3.

18. Once you have factored an algebraic expression, how can you check to ensure that you have factored correctly? Explain why your strategy will always work.

Extending



19. Determine the expression, in factored form, that represents the shaded area between the circle and the square in the diagram at the left.

20. Factor the numerator in each expression, and then simplify the expression. Assume that no variable equals zero.

a) $\dfrac{2x^2y + 3xy^2}{xy}$

b) $\dfrac{6x^3y + 12x^3y^2}{6x^3y}$

c) $\dfrac{-12x^3y^2 - 18x^2y^3}{6x^2y^2}$

d) $\dfrac{3x^4 + 6x^3 + 9x^2}{3x^2}$

4.2 Exploring the Factorization of Trinomials

GOAL

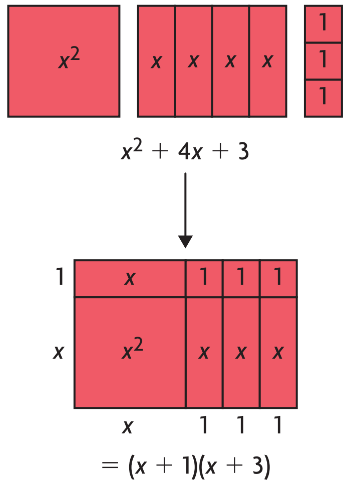
Discover the relationship between the coefficients and constants in a trinomial and the coefficients and constants in its factors.

YOU WILL NEED

- algebra tiles

EXPLORE the Math

A trinomial can be represented using algebra tiles. If you can arrange the tiles as a rectangle, then the expression represents the area of the rectangle. The dimensions of the rectangle represent its factors. For example, to factor $x^2 + 4x + 3$, arrange one x^2 tile, four x tiles, and three unit tiles into a rectangle. The diagram shows that $x^2 + 4x + 3 = (x + 1)(x + 3)$.



? What is the relationship between the terms in a trinomial and the terms in its factors?

A. Factor each trinomial using algebra tiles. Sketch your model, and record the two factors.

i) $x^2 + 2x + 1$ **iii)** $x^2 + 6x + 8$ **v)** $x^2 - 4x + 3$
ii) $x^2 + 5x + 6$ **iv)** $x^2 - 2x + 1$ **vi)** $x^2 - 3x + 2$

B. Factor each trinomial using algebra tiles. Sketch your model, and record the two factors.

i) $x^2 - 2x - 3$ **iii)** $x^2 - 2x - 8$ **v)** $x^2 - 3x - 10$
ii) $x^2 + 3x - 4$ **iv)** $x^2 + x - 6$ **vi)** $x^2 + 4x - 5$

C. If the coefficient of x^2 in a trinomial is a value other than 1, it may be possible to factor it using rectangular arrangements of algebra tiles. Factor each trinomial using algebra tiles. Sketch your model, and record the two factors.

i) $2x^2 + 3x + 1$ **iii)** $3x^2 - 4x + 1$ **v)** $2x^2 - 7x - 4$
ii) $2x^2 + 5x + 2$ **iv)** $2x^2 - 7x + 6$ **vi)** $3x^2 + 5x - 2$

D. Examine the trinomials in parts A, B, and C.

i) Compare the coefficient of x^2 in each trinomial with the coefficients of x in the factors. What do you notice?

ii) Compare the constant term in each trinomial with the constant terms in the factors. What do you notice?

iii) Compare the coefficient of x in each trinomial with the coefficients and constants in the factors. What do you notice?

Communication *Tip*

When negative tiles are used, add tiles to make a rectangle if necessary. To do this without changing the value of the trinomial, add positive and negative tiles of equal value.

Reflecting

E. What type of polynomial are the factors of trinomials of the form $ax^2 + bx + c$?

F. How can you use the values of a, b, and c in $ax^2 + bx + c$ to help you factor a trinomial, without using an algebra tile model?

G. Can a rectangular arrangement of algebra tiles be created for $x^2 + 3x + 1$ or $2x^2 + x + 1$? What does this imply?

In Summary

Key Idea

- To factor a trinomial of the form $ax^2 + bx + c$ using algebra tiles, you need to form a rectangle. The factors are the dimensions of the rectangle.

Need to Know

- A factorable trinomial of the form $ax^2 + bx + c$ may have two binomials as its factors.
- If a rectangle cannot be created for a given trinomial, the trinomial cannot be factored.
- When factoring a trinomial using algebra tiles, you may need to add tiles to create a rectangular model. This requires you to add positive and negative tiles of equal value.

FURTHER Your Understanding

1. Use algebra tiles to factor each polynomial. Sketch your model, and record the two factors.

a) $x^2 + 7x + 12$
b) $x^2 - x - 12$
c) $x^2 - 4x + 4$
d) $x^2 - x - 6$
e) $x^2 - 5x + 6$
f) $x^2 - 4$

2. Check your results for each expression in question 1 by expanding.

3. Use algebra tiles to factor each polynomial. Sketch your model, and record the two factors.

a) $2x^2 + 7x + 3$
b) $4x^2 + 4x - 3$
c) $4x^2 - 4x + 1$
d) $3x^2 + 7x - 6$
e) $2x^2 - 9x + 4$
f) $6x^2 + 7x + 2$

4. Check your results for each expression in question 3 by expanding.

4.3 Factoring Quadratics: $x^2 + bx + c$

GOAL

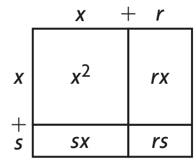
Factor quadratic expressions of the form $ax^2 + bx + c$, where $a = 1$.

YOU WILL NEED

- algebra tiles

INVESTIGATE the Math

Brigitte remembered that an area model can be used to multiply two binomials. To multiply $(x + r)(x + s)$, she created the model at the right and determined that the product is quadratic.



? How can an area model be used to determine the factors of a quadratic expression?

A. Use algebra tiles to build rectangles with the dimensions shown in the table below. Copy and complete the table, recording the area in the form $x^2 + bx + c$.

Length	Width	Area: $x^2 + bx + c$	Value of b	Value of c
$x + 3$	$x + 4$			
$x + 3$	$x + 5$			
$x + 3$	$x + 6$			
$x + 4$	$x + 4$			
$x + 4$	$x + 5$			

B. Look for a pattern in the table for part A. Use this pattern to predict the length and width of a rectangle with each of the following areas.

i) $x^2 + 8x + 12$
ii) $x^2 + 10x + 21$
iii) $x^2 + 11x + 30$
iv) $x^2 + 11x + 18$

C. The length of a rectangle is $x + 4$, and the width is $x - 3$. What is the area of the rectangle?

D. Repeat part A for rectangles with the following dimensions.

Length	Width	Area: $x^2 + bx + c$	Value of b	Value of c
$x - 3$	$x + 5$			
$x + 3$	$x - 6$			
$x - 2$	$x - 2$			
$x - 1$	$x - 5$			

E. Look for a pattern in the table for part D. Use this pattern to predict the length and width of a rectangle with each of the following areas.

i) $x^2 - 2x - 15$ **iii)** $x^2 - x - 30$
ii) $x^2 + 2x - 24$ **iv)** $x^2 - 8x + 7$

F. The expression $x^2 + bx + c$ represents the area of a rectangle. How can you factor this expression to predict the length and width of the rectangle?

Reflecting

G. Examine the length, width, and area of each rectangle in parts A and D. Explain how the signs in the area expression can be used to determine the signs in each dimension.

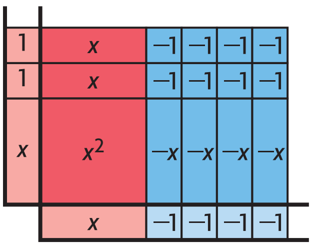
H. Can all quadratic expressions of the form $x^2 + bx + c$ be factored as the product of two binomials? Explain.

APPLY the Math

EXAMPLE 1 Selecting an algebra tile strategy to factor a quadratic expression

Factor $x^2 - 2x - 8$.

Timo's Solution



$x^2 - 2x - 8 = (x - 4)(x + 2)$

I arranged one x^2 tile, four $-x$ tiles, two x tiles, and eight negative unit tiles in a rectangle to create an area model.

I placed the x tiles and unit tiles so the length and width were easier to see. The dimensions of the rectangle are $x - 4$ and $x + 2$.

The sum $2 + (-4) = -2$ determines the number of x tiles, and the product $2 \times (-4) = -8$ determines the number of unit tiles in the original expression.

EXAMPLE 2 Connecting the factors of a trinomial to its coefficients and constants

Factor $x^2 + 12x + 35$.

Chaniqua's Solution

$x^2 + 12x + 35$
$= (x\ ?\ )(x\ ?\ )$

The two factors of the quadratic expression must be binomials that start with x. I need two numbers whose sum is 12 (the coefficient of x) and whose product is 35 (the constant).

$= (x + ?)(x + ?)$

I started with the product. Since 35 is positive, both numbers must be either positive or negative.

Since the sum is positive, both numbers must be positive.

$= (x + 7)(x + 5)$

The numbers are 7 and 5.

Check:

$(x + 7)(x + 5) = x^2 + 5x + 7x + 35$
$= x^2 + 12x + 35$

I checked by multiplying.

EXAMPLE 3 **Reasoning** to factor quadratic expressions

Factor each expression, if possible.

a) $x^2 - x - 72$ **b)** $a^2 - 13a + 36$ **c)** $x^2 + x + 6$

Ryan's Solution

a) $x^2 - x - 72$

$= (x - 9)(x + 8)$

I needed two numbers whose sum is −1 and whose product is −72.

The product is negative, so one of the numbers must be negative.

Since the sum is negative, the negative number must be farther from zero than the positive number.

The numbers are −9 and 8.

b) $a^2 - 13a + 36$

$= (a - 9)(a - 4)$

I needed two numbers whose sum is −13 and whose product is 36.

The product is positive, so both numbers must be either positive or negative.

Since the sum is negative, both numbers must be negative.

The numbers are −9 and −4.

c) $x^2 + x + 6$

This cannot be factored.

I needed two numbers whose sum is 1 and whose product is 6.

There are no such numbers because the only factors of 6 are 1 and 6, and 2 and 3. The sum of 1 and 6 is 7, and the sum of 2 and 3 is 5. Neither sum is 1.

EXAMPLE 4 Reasoning to factor a quadratic that has a common factor

Factor $3y^3 - 21y^2 - 24y$.

Sook Lee's Solution

$3y^3 - 21y^2 - 24y$

First, I divided out the greatest common factor. The GCF is $3y$, since all the terms are divisible by $3y$.

$= 3y(y^2 - 7y - 8)$

$= 3y(y - 8)(y + 1)$

To factor the trinomial, I needed two numbers whose sum is -7 and whose product is -8. These numbers are -8 and 1.

In Summary

Key Idea

- If a quadratic expression of the form $x^2 + bx + c$ can be factored, it can be factored into two binomials, $(x + r)$ and $(x + s)$, where $r + s = b$ and $r \times s = c$, and r and s are integers.

Need to Know

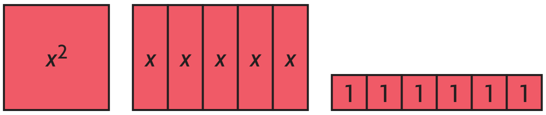
- To factor $x^2 + bx + c$ as $(x + r)(x + s)$, you can use the signs in the trinomial to determine the signs in the factors.

Trinomial	Factors
b and c are positive.	$(x + r)(x + s)$
b is negative, and c is positive.	$(x - r)(x - s)$
b and c are negative.	$(x - r)(x + s)$, where $r > s$
b is positive, and c is negative.	$(x + r)(x - s)$, where $r > s$

- It is easier to factor an algebraic expression if you first divide out the greatest common factor.

CHECK Your Understanding

1. a) Write the trinomial that is represented by these algebra tiles.

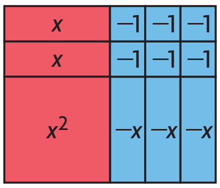


b) Sketch what the tiles would look like if they were arranged in a rectangle.

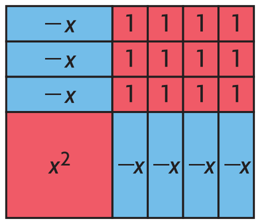
c) Use your sketch to determine the factors of the trinomial.

2. The tiles in each model represent an algebraic expression. Identify the expression and its factors.

a)



b)



3. One factor is given, and one factor is missing. What is the missing factor?

a) $x^2 - 10x + 21 = (x - 7)\,(\blacksquare)$

b) $x^2 + 4x - 32 = (x - 4)(\blacksquare)$

c) $x^2 - 2x - 63 = (\blacksquare)(x + 7)$

d) $x^2 + 14x + 45 = (\blacksquare)(x + 9)$

PRACTISING

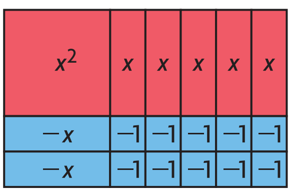
4. Factor each expression.

a) $x^2 + 2x + 1$ **c)** $x^2 - 2x - 3$ **e)** $x^2 - 4x + 4$

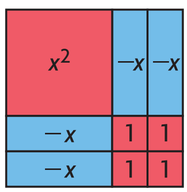
b) $x^2 - 2x + 1$ **d)** $x^2 + 6x + 9$ **f)** $x^2 - 4x - 12$

5. The tiles in each model represent a quadratic expression. Identify the expression and its factors.

a)



b)



6. One factor is given, and one factor is missing. What is the missing factor?

a) $x^2 + 11x + 24 = (x + 3)(\blacksquare)$
b) $c^2 - 15c + 56 = (c - 7)(\blacksquare)$
c) $a^2 - 11a - 60 = (a - 15)(\blacksquare)$
d) $y^2 - 20y - 44 = (\blacksquare)(y + 2)$
e) $b^2 + 2b - 48 = (\blacksquare)(b + 8)$
f) $z^2 - 19z + 90 = (\blacksquare)(z - 10)$

7. Factor each expression.

a) $x^2 + 4x + 3$
b) $a^2 - 9a + 20$
c) $m^2 - 8m + 16$
d) $n^2 + n - 6$
e) $x^2 + 6x - 16$
f) $x^2 + 15x - 16$

8. Factor.

a) $x^2 - 10x + 16$
b) $y^2 + 6y - 40$
c) $a^2 - a - 56$
d) $w^2 - 5w - 14$
e) $m^2 - 12m + 32$
f) $n^2 + n - 42$

9. Factor.

a) $3x^2 + 24x + 45$
b) $2y^2 - 2y - 60$
c) $3v^2 + 9v + 6$
d) $6n^2 + 24n - 30$
e) $x^3 + 5x^2 + 4x$
f) $7x^4 + 28x^3 - 147x^2$

10. **K** Write three different quadratic trinomials that have $(x - 2)$ as a factor.

11. **C** Nathan factored $x^2 - 15x + 44$ as $(x - 4)(x - 11)$. Martina factored the expression another way and found different factors. Identify the factors that Martina found, and explain why both students are correct.

12. Factor.

a) $a^2 + 8a + 15$
b) $3x^2 - 21x - 54$
c) $z^2 - 16z + 55$
d) $x^2 + 5x - 50$
e) $x^3 - 3x^2 - 10x$
f) $2xy^2 - 26xy + 84x$

13. Examine each quadratic relation below.

i) Express the relation in factored form.
ii) Determine the zeros.
iii) Determine the coordinates of the vertex.
iv) Sketch the graph of the relation.

a) $y = x^2 + 2x - 8$
b) $y = x^2 - 2x - 24$
c) $y = x^2 - 8x + 15$
d) $y = -x^2 - 9x - 14$

14. A professional cliff diver's height above the water can be modelled by the equation $h = -5t^2 + 20$, where h is the diver's height in metres and t is the elapsed time in seconds.

A

a) Draw a height versus time graph.
b) Determine the height of the cliff that the diver jumped from.
c) Determine when the diver will enter the water.

15. A baseball is thrown from the top of a building and falls to the ground below. The height of the baseball above the ground is approximated by the relation $h = -5t^2 + 10t + 40$, where h is the height above the ground in metres and t is the elapsed time in seconds. Determine the maximum height that is reached by the ball.

16. Factor each expression.

a) $m^2 + 4mn - 5n^2$
b) $x^2 + 12xy + 35y^2$
c) $a^2 + ab - 12b^2$
d) $c^2 - 12cd - 85d^2$
e) $r^2 + 13rs + 12s^2$
f) $18p^2 - 9pq + q^2$

17. Paul says that if you can factor $x^2 - bx - c$, you can factor $x^2 + bx - c$. Do you agree? Explain.

T

18. Create a mind map that shows the connections between $x^2 + bx + c$ and its factors.

Extending

19. Factor each expression.

a) $x^4 + 6x^2 - 27$
b) $a^4 + 10a^2 + 9$
c) $-4m^4 + 16m^2n^2 + 20n^4$
d) $(a - b)^2 - 15(a - b) + 26$

20. Factor, and then simplify. Assume that the denominator is never zero.

a) $\dfrac{x^2 - 6x + 8}{x - 4}$
b) $\dfrac{a^2 - 3a - 28}{a + 4}$
c) $\dfrac{x^2 + x - 30}{x - 5}$
d) $\dfrac{2x^2 - 24x + 64}{2x - 16}$

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 4.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 to 6.

Q: How do you determine the greatest common factor of the terms in a polynomial?

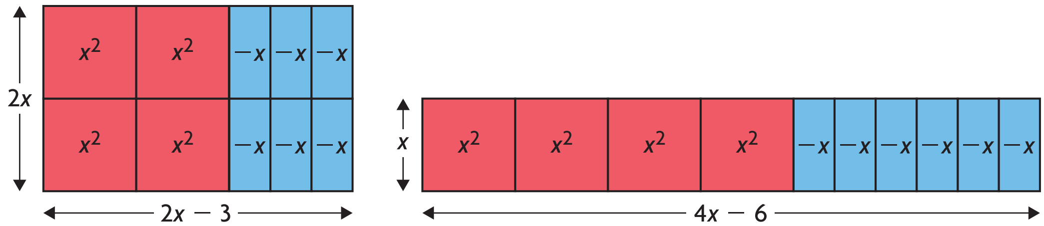
A1: Sometimes you can represent the terms with algebra tiles and arrange the tiles into rectangles. The arrangement that has the greatest possible width shows the greatest common factor.

EXAMPLE

Factor $4x^2 - 6x$.

Solution

The greatest common factor of the terms in $4x^2 - 6x$ is $2x$, since this is the greatest width possible in a rectangular arrangement of tiles.



Once you divide out the common factor, the remaining terms represent the other dimension of the rectangle.

$4x^2 - 6x = 2x(2x - 3)$

A2: You can determine the greatest common factor of the coefficients and the greatest common factor of the variables, and then multiply the GCFs together. Sometimes you may need to group terms since the GCF can be a monomial or a binomial.

EXAMPLE

Factor $5xa - 5xb + 2ya - 2yb$.

Solution

When you group terms,
$\underline{5xa - 5xb} + \underline{2ya - 2yb} = 5x(a - b) + 2y(a - b)$,
and the GCF is $(a - b)$.

Divide out the common factor.
$5xa - 5xb + 2ya - 2yb = (a - b)(5x + 2y)$

Study Aid

- See Lesson 4.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 7 to 13.

Q: How can you factor a quadratic expression of the form $x^2 + bx + c$?

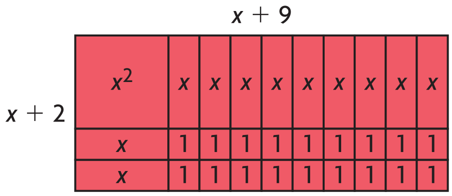
A1: You can form a rectangular area model using tiles. The length and width of the rectangle are the factors.

EXAMPLE

Factor $x^2 + 11x + 18$.

Solution

Algebra tiles can be arranged to form this rectangle.



This rectangle has a width of $x + 2$ and a length of $x + 9$.
These are the factors of $x^2 + 11x + 18$.

$x^2 + 11x + 18 = (x + 9)(x + 2)$

A2: You can look for values whose sum is b and whose product is c, and then use these values to factor the expression.

EXAMPLE

Factor $x^2 - 3x - 40$.

Solution

Pairs of numbers whose product is -40 are

- 40 and -1
- 20 and -2
- 10 and -4
- 8 and -5
- -40 and 1
- -20 and 2
- -10 and 4
- -8 and 5

The only pair whose sum is -3 is -8 and 5.

$x^2 - 3x - 40 = (x - 8)(x + 5)$

Expanding verifies this result.

$(x - 8)(x + 5)$
$= x^2 + 5x - 8x - 40$
$= x^2 - 3x - 40$

PRACTICE Questions

Lesson 4.1

1. State the GCF for each pair of terms.

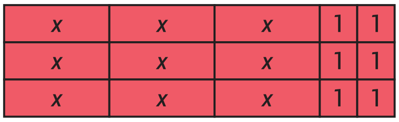
a) 24 and 60
b) x^3 and x^2
c) $10y$ and $5y^2$
d) $-8a^2b$ and $-12ab^2$
e) c^4d^2 and c^3d
f) $27m^4n^2$ and $36m^2n^3$

2. Determine the missing factor.

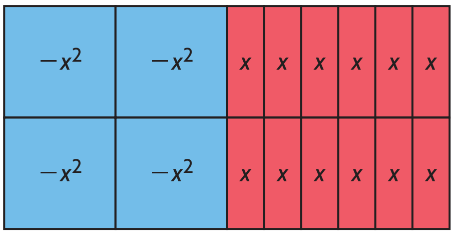
a) $7x - 28y = (\blacksquare)(x - 4y)$
b) $6x - 9y = (3)(\blacksquare)$
c) $24a^2 + 12b^2 = (\blacksquare)(2a^2 + b^2)$
d) $x^2 - xy^3 = (x)(\blacksquare)$
e) $-15x^2 + 6y^2 = (\blacksquare)(5x^2 - 2y^2)$
f) $a^4b^3 - a^3b^2 = (a^3b^2)(\blacksquare)$

3. The tiles in each model represent an algebraic expression. Identify the expression and the greatest common factor of its terms.

a)



b)



4. Factor each expression.

a) $7z + 35$
b) $-28x^2 + 4x^3$
c) $5m^2 - 10mn + 5$
d) $x^2y^4 - xy^2 + x^3y$

5. A parabola is defined by the equation $y = 5x^2 - 15x$. Explain how you would determine the coordinates of the vertex of the parabola, without using a table of values or graphing technology.

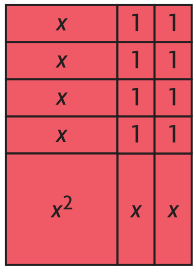
6. Factor each expression.

a) $3x(5y - 2) + 5(5y - 2)$
b) $4a(b + 6) - 3(b + 6)$
c) $6xt - 2xy - 3t + y$
d) $4ab + 4ac - b^2 - bc$

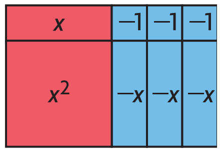
Lesson 4.3

7. Each model represents a quadratic expression. Identify the expression and its factors.

a)



b)



8. Determine the value of each symbol.

a) $x^2 + \blacklozenge x + 12 = (x + 3)(x + \blacksquare)$
b) $x^2 + \blacksquare x + \blacklozenge = (x + 3)(x + 3)$
c) $x^2 - 12x + \blacksquare = (x - \blacklozenge)(x - \blacklozenge)$
d) $x^2 - 7x + \blacklozenge = (x - 3)(x - \blacksquare)$

9. Factor.

a) $x^2 + 8x - 33$
b) $n^2 + 7n - 18$
c) $b^2 - 10b - 11$
d) $x^2 - 14x + 45$
e) $c^2 + 5c - 14$
f) $y^2 - 17y + 72$

10. Factor.

a) $3a^2 - 3a - 36$
b) $x^3 - 6x^2 - 16x$
c) $2x^2 + 14x - 120$
d) $4b^2 - 36b + 72$
e) $-d^3 + d^2 + 30d$
f) $xy^3 + 2xy^2 + xy$

11. Deanna throws a rock from the top of a cliff into the air. The height of the rock above the base of the cliff is modelled by the equation $h = -5t^2 + 10t + 75$, where h is the height of the rock in metres and t is the time in seconds.

a) How high is the cliff?
b) When does the rock reach its maximum height?
c) What is the rock's maximum height?

12. When factoring a quadratic expression of the form $x^2 + bx + c$, why does it make more sense to consider the value of c before the value of b? Explain.

13. Use the quadratic relation determined by $y = x^2 + 4x - 21$.

a) Express the relation in factored form.
b) Determine the zeros and the vertex.
c) Sketch its graph.

4.4 Factoring Quadratics: $ax^2 + bx + c$

GOAL

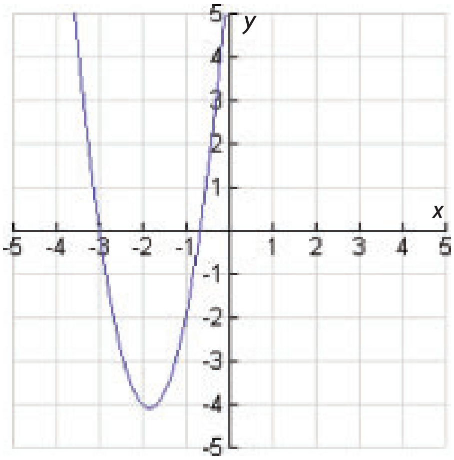
Factor quadratic expressions of the form $ax^2 + bx + c$, where $a \neq 1$.

YOU WILL NEED

- algebra tiles

LEARN ABOUT the Math

Kellie was asked to determine the x-intercepts of $y = 3x^2 + 11x + 6$ algebraically. She created a graph using graphing technology and estimated that the x-intercepts are about $x = -0.6$ and $x = -3$.



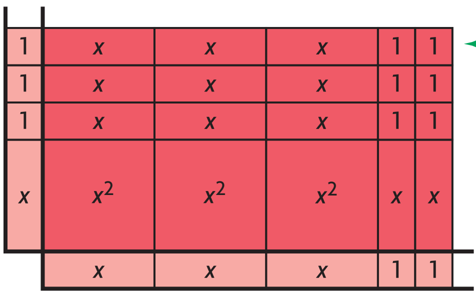
Kellie knows that if she can write the equation in factored form, she can use the factors to determine the x-intercepts. She is unsure about how to proceed because the first term in the expression has a coefficient of 3 and there is no common factor.

? How can you factor $3x^2 + 11x + 6$?

EXAMPLE 1 **Selecting a strategy** to factor a trinomial, where $a \neq 1$

Factor $3x^2 + 11x + 6$, and determine the x-intercepts of $y = 3x^2 + 11x + 6$.

Ellen's Solution: Selecting an algebra tile model



$3x^2 + 11x + 6 = (3x + 2)(x + 3)$

I used tiles to create a rectangular area model of the trinomial.

I placed the tiles along the length and width to read off the factors. The length is $3x + 2$, and the width is $x + 3$.

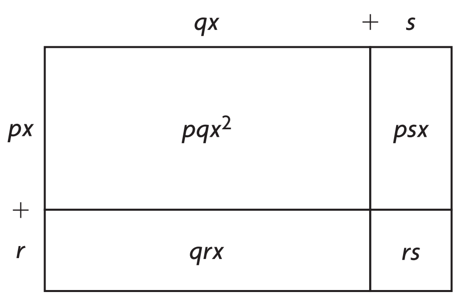
The equation in factored form is
$y = (3x + 2)(x + 3)$.
Let $3x + 2 = 0$ or $x + 3 = 0$.

$$3x = -2 \qquad x = -3$$
$$x = -\frac{2}{3}$$

The x-intercepts occur when $y = 0$. This happens when either factor is equal to 0.

The x-intercepts are $-\frac{2}{3}$ and -3.

Neil's Solution: Selecting an area diagram and a systematic approach



I thought about the general situation, where $(px + r)$ and $(qx + s)$ represent the unknown factors. I created an area model and used it to look for patterns between the coefficients in the factors and the coefficients in the trinomial.

$$(px + r)(qx + s) = pqx^2 + psx + qrx + rs$$
$$= pqx^2 + (ps + qr)x + rs$$

Suppose that
$3x^2 + 11x + 6 = (px + r)(qx + s)$.

I imagined writing two factors for this product. I had to figure out the coefficients and the constants in the factors.

$$(px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$
$$= 3x^2 + 11x + 6$$

I matched the coefficients and the constants.

p	q	r	s	$ps + qr$
3	1	3	2	9
1	3	2	3	9
1	3	3	2	11 ✓

I needed values of p and q that, when multiplied, would give a product of 3. I also needed values of r and s that would give a product of 6.

$p = 1$, $q = 3$, $r = 3$, and $s = 2$
$3x^2 + 11x + 6 = (x + 3)(3x + 2)$

The middle coefficient is 11, so I tried different combinations of $ps + qr$ to get 11.

The equation in factored form is
$y = (x + 3)(3x + 2)$.
Let $x + 3 = 0$ or $3x + 2 = 0$.
$x = -3 \qquad 3x = -2$
$x = -\frac{2}{3}$

The x-intercepts are -3 and $-\frac{2}{3}$.

The x-intercepts occur when $y = 0$. This happens when either factor is equal to zero.

Astrid's Solution: Selecting a decomposition strategy

$3x^2 + 11x + 6$
$= (px + r)(qx + s)$

I imagined writing two factors for this product. I had to figure out the coefficients and the constants in the factors.

$(px + r)(qx + s)$
$= pxqx + pxs + rqx + rs$
$= pqx^2 + (qr + ps)x + rs$

I multiplied the binomials. I noticed that I would get the product of all four missing values if I multiplied the coefficient of x^2 (pq) and the constant (rs).

ps and qr, the two values that are added to get the coefficient of the middle term, are both factors of $pqrs$.

$3x^2 + 11x + 6$
$= 3x^2 + ?x + ?x + 6$

If I added the product of two of these values (ps) to the product of the other two (qr), I would get the coefficient of x.

$3 \times 6 = 18$
The factors of 18 are 1, 2, 3, 6, 9, and 18.

$11 = 9 + 2$

I needed to **decompose** the 11 from $11x$ into two parts. Each part had to be a factor of 18, because $3 \times 6 = 18$.

decompose
break a number or an expression into the parts that make it up

$3x^2 + 9x + 2x + 6$
$= 3x^2 + 9x + 2x + 6$
$= 3x(x + 3) + 2(x + 3)$

$= (x + 3)(3x + 2)$

I divided out the greatest common factors from the first two terms and then from the last two terms.

I factored out the binomial common factor.

The equation in factored form is
$y = (x + 3)(3x + 2)$.

Let $x + 3 = 0$ or $3x + 2 = 0$.

$$x = -3 \qquad 3x = -2$$
$$x = -\frac{2}{3}$$

The x-intercepts occur when $y = 0$. This happens when either factor is equal to zero.

The x-intercepts are -3 and $-\frac{2}{3}$.

Reflecting

A. Explain how Ellen's algebra tile arrangement shows the factors of the expression.

B. How is Neil's strategy similar to the strategy used to factor trinomials of the form $x^2 + bx + c$? How is it different?

C. How would Astrid's decomposition change if she had been factoring $3x^2 + 22x + 24$ instead?

D. Which factoring strategy do you prefer? Explain why.

APPLY the Math

EXAMPLE 2 **Selecting a systematic strategy** to factor a trinomial, where $a \neq 1$

Factor $4x^2 - 8x - 5$.

Katie's Solution

$$4x^2 - 8x - 5 = (px + r)(qx + s)$$
$$= pqx^2 + (ps + qr)x + rs$$
$$pq = 4 \quad \text{and} \quad rs = -5$$

I wrote the quadratic as the product of two binomials with unknown coefficients and constants. Then I listed all the possible pairs of values for pq and rs.

p	q
1	4
4	1
2	2

r	s
1	−5
−5	1

$$pqx^2 + (ps + qr)x + rs = 4x^2 - 8x - 5$$
$$ps + qr = -8$$

I had to choose values that would make $ps + qr = -8$.

$(px + r)(qx + s) = (2x + 1)(2x - 5)$

So, $4x^2 - 8x - 5 = (2x + 1)(2x - 5)$.

The values $p = 2$, $q = 2$, $r = 1$, and $s = -5$ work because
- pq is $(2)(2) = 4$
- rs is $(1)(-5) = -5$
- $ps + qr$ is $(2)(-5) + (2)(1) = -8$

$$(2x + 1)(2x - 5) = 4x^2 - 10x + 2x - 5$$
$$= 4x^2 - 8x - 5$$

I checked by multiplying.

EXAMPLE 3 **Selecting a decomposition strategy** to factor a trinomial

Factor $12x^2 - 25x + 12$.

Braedon's Solution

$$12x^2 - 25x + 12$$
$$= 12x^2 - 16x - 9x + 12$$

I looked for two numbers whose sum is -25 and whose product is $(12)(12) = 144$. I knew that both numbers must be negative, since the sum is negative and the product is positive. The numbers are -16 and -9. I used these numbers to decompose the middle term.

$$= \underline{12x^2 - 16x} \; \underline{- 9x + 12}$$
$$= 4x(3x - 4) - 3(3x - 4)$$
$$= (3x - 4)(4x - 3)$$

I factored the first two terms and then the last two terms. Then I divided out the common factor of $3x - 4$.

EXAMPLE 4 **Selecting a guess-and-test strategy** to factor a trinomial

Factor $7x^2 + 19x - 6$.

Dylan's Solution

$7x^2 + 19x - 6$

	$7x$	?
x	$7x^2$	?
?	?	-6

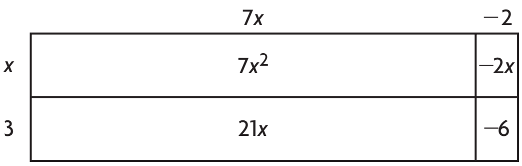
I thought of the product of the factors as the dimensions of a rectangle with the area $7x^2 + 19x - 6$.

The only factors of $7x^2$ are $7x$ and x. The factors of -6 are -6 and 1, -2 and 3, 6 and -1, and 2 and -3. I had to determine which factors of $7x^2$ and -6 would add to $19x$.

	$7x$	-6
x	$7x^2$	$-6x$
1	$7x$	-6

$(7x - 6)(x + 1) = 7x^2 + x - 6$ wrong factors

I used trial and error to determine the values in place of the question marks. Then I checked by multiplying.



I repeated this process until I found the combination that worked.

$(7x - 2)(x + 3) = 7x^2 + 19x - 6$ worked

$7x^2 + 19x - 6 = (7x - 2)(x + 3)$

In Summary

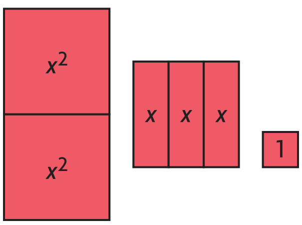
Key Idea

- If the quadratic expression $ax^2 + bx + c$ (where $a \neq 1$) can be factored, then the factors have the form $(px + r)(qx + s)$, where $pq = a$, $rs = c$, and $ps + rq = b$.

Need to Know

- If the quadratic expression $ax^2 + bx + c$ (where $a \neq 1$) can be factored, then the factors can be found by
  - forming a rectangle using algebra tiles
  - using the algebraic model $(px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$ systematically
  - using decomposition
  - using guess and test
- A trinomial of the form $ax^2 + bx + c$ (where $a \neq 1$) can be factored if there are two integers whose product is ac and whose sum is b.

CHECK Your Understanding



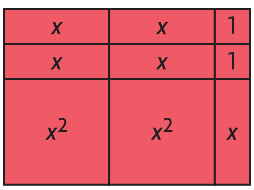
1. a) Write the trinomial that is represented by the algebra tiles at the left.

b) Sketch what the tiles would look like if they were arranged in a rectangle.

c) Use your sketch to determine the factors of the trinomial.

2. Each of the following four diagrams represents a trinomial. Identify the trinomial and its factors.

a)



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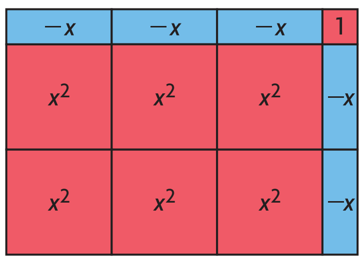

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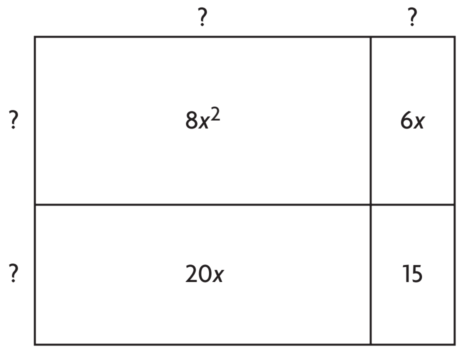

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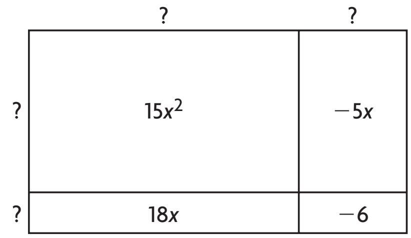


3. Determine the missing factor.

a) $2c^2 + 7c - 4 = (c + 4)(\blacksquare)$

b) $4z^2 - 9z - 9 = (\blacksquare)(z - 3)$

c) $6y^2 - y - 1 = (3y + 1)(\blacksquare)$

d) $6p^2 + 7p - 3 = (\blacksquare)(2p + 3)$

PRACTISING

4. Determine the value of each symbol.

a) $5x^2 + \blacklozenge x + 3 = (x + 3)(5x + \blacksquare)$

b) $2x^2 - \blacksquare x - \blacklozenge = (2x + 3)(x - 2)$

c) $12x^2 - 7x + \blacksquare = (3x - \blacklozenge)(4x - \blacklozenge)$

d) $14x^2 - 29x + \blacklozenge = (2x - 3)(7x - \blacksquare)$

5. Factor each expression.

a) $2x^2 + x - 6$

b) $3n^2 - 11n - 4$

c) $10a^2 + 3a - 1$

d) $4x^2 - 16x + 15$

e) $2c^2 + 5c - 12$

f) $6x^2 + 5x + 1$

6. Factor.

a) $6x^2 - 13x + 6$

b) $10m^2 + m - 3$

c) $2a^2 - 11a + 12$

d) $4x^2 - 20x + 25$

e) $5d^2 + 8 - 14d$

f) $6n^2 - 20 + 26n$

7. Factor.

a) $15x^2 + 4x - 4$

b) $18m^2 - 3m - 10$

c) $16a^2 - 50a + 36$

d) $35x^2 - 27x - 18$

e) $63n^2 + 126n + 48$

f) $24d^2 + 35 - 62d$

8. Write three different quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, that have $(3x - 4)$ as a factor.

9. The area of a rectangle is given by each of the following trinomials.
K Determine expressions for the length and width of the rectangle.

a) $A = 6x^2 + 17x - 3$

b) $A = 8x^2 - 26x + 15$

10. Identify possible integers, k, that allow each quadratic trinomial
T to be factored.

a) $kx^2 + 5x + 2$ **b)** $9x^2 + kx - 5$ **c)** $12x^2 - 20x + k$

11. Factor each expression.

a) $6x^2 + 34x - 12$
b) $18v^2 + 33v - 30$
c) $48c^2 - 160c + 100$
d) $5b^3 - 17b^2 + 6b$
e) $-6x - 51xy + 27xy^2$
f) $-7a^2 - 29a + 30$

12. Determine whether each polynomial has $(k + 5)$ as one of its factors.

a) $k^2 + 9k - 52$
b) $4k^3 + 32k^2 + 60k$
c) $6k^2 + 23k + 7$
d) $10 + 19k - 15k^2$
e) $7k^2 + 29k - 30$
f) $10k^2 + 65k + 75$

13. Examine each quadratic relation below.

i) Express the relation in factored form.
ii) Determine the zeros.
iii) Determine the coordinates of the vertex.
iv) Sketch the graph of the relation.

a) $y = 2x^2 - 9x + 4$ **b)** $y = -2x^2 + 7x + 15$

14. A computer software company models the profit on its latest video
A game using the relation $P = -4x^2 + 20x - 9$, where x is the number of games produced in hundred thousands and P is the profit in millions of dollars.

a) What are the break-even points for the company?
b) What is the maximum profit that the company can earn?
c) How many games must the company produce to earn the maximum profit?

15. Factor each expression.

a) $8x^2 - 13xy + 5y^2$
b) $5a^2 - 17ab + 6b^2$
c) $-12s^2 - sr + 35r^2$
d) $16c^4 + 64c^2 + 39$
e) $14v^6 - 39v^3 + 27$
f) $c^3d^3 + 2c^2d^2 - 8cd$

16. Create a flow chart that would help you decide which strategy
C you should use to factor a given polynomial.

Extending

17. Factor.

a) $6(a + b)^2 + 11(a + b) + 3$
b) $5(x - y)^2 - 7(x - y) - 6$
c) $8(x + 1)^2 - 14(x + 1) + 3$
d) $12(a - 2)^4 + 52(a - 2)^2 - 40$

18. Can a quadratic expression of the form $ax^2 + bx + c$ always be factored if $b^2 - 4ac$ is a **perfect square**? Explain.

4.5 Factoring Quadratics: Special Cases

GOAL

Factor perfect-square trinomials and differences of squares.

YOU WILL NEED

- algebra tiles

LEARN ABOUT the Math

Nadia claims that the equation $y = 4x^2 + 12x + 9$ will always generate a value that is a perfect square, if x represents any natural number.

? How can you show that Nadia's claim is correct?

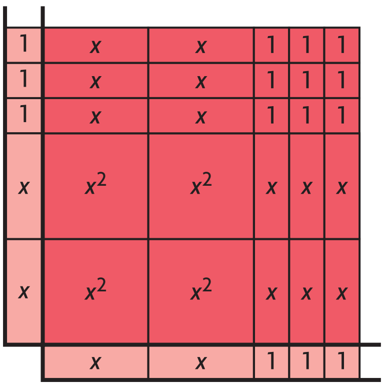
EXAMPLE 1 | Connecting an expression to its factors

Show that Nadia's claim is correct.

Parma's Solution: Selecting an algebra tile model

x	y	Perfect Square?
1	25	Yes: $25 = 5 \times 5$
3	81	Yes: $81 = 9 \times 9$
9	441	Yes: $441 = 21 \times 21$

First I substituted some numbers for x into the equation. Each time, the result was a perfect square. Nadia's claim seems to be correct.



I decided to see how algebra tiles could be arranged. The only arrangement of tiles that seemed to work was a square.

I lined up tiles along two edges to determine the side length of the square.

Each side is $(2x + 3)$ long.

$4x^2 + 12x + 9 = (2x + 3)^2$

$y = 4x^2 + 12x + 9$
is the same as $y = (2x + 3)^2$.

The equation $y = 4x^2 + 12x + 9$ will always result in a perfect square.

The equation factors and the binomial factors are identical. Any number that is substituted for x gets squared. This ensures that the result will always be a perfect square.

Jarrod's Solution: Reasoning logically to factor

$y = 4x^2 + 12x + 9$
$4 = 2^2$ and $9 = 3^2$

I noticed that both 9 and 4 in the equation are perfect squares.

Does $4x^2 + 12x + 9 = (2x + 3)^2$?

I tested to see if the trinomial is a perfect square, with identical factors, by expanding $(2x + 3)(2x + 3)$.

$$\begin{aligned}(2x + 3)^2 &= (2x + 3)(2x + 3)\\ &= 4x^2 + 6x + 6x + 9\\ &= 4x^2 + 12x + 9 \quad \text{worked}\end{aligned}$$
$4x^2 + 12x + 9 = (2x + 3)^2$

This worked, since I got the correct coefficient of x.

$y = 4x^2 + 12x + 9$
is the same as $y = (2x + 3)^2$.

The equation $y = 4x^2 + 12x + 9$ will always generate a perfect square.

The equation has identical factors. Any number that is substituted for x gets squared. This ensures that the result will always be a perfect square.

Reflecting

A. Why is the name "perfect-square trinomial" suitable for a polynomial like $4x^2 + 12x + 9$?

B. Why did Jarrod use the square root of 4 and the square root of 9 as values in the factors?

C. Nigel said that a quadratic expression cannot be a perfect square unless the coefficient of x is even. How do Parma's and Jarrod's solutions show that Nigel is correct?

APPLY the Math

EXAMPLE 2 — **Connecting** decomposition with factoring a perfect square

Factor $25x^2 - 40x + 16$.

Andy's Solution

For the expression $25x^2 - 40x + 16$, $-40 = (-20) + (-20)$ and $(-20)(-20) = 400$

I decomposed -40, the coefficient of x, as the sum of two numbers whose product is $25 \times 16 = 400$.

$25x^2 - 20x - 20x + 16$

I wrote the x term in decomposed form.

$25x^2 - 20x - 20x + 16$
$= 5x(5x - 4) - 4(5x - 4)$

I divided out the GCF $5x$ from the first two terms. I divided out the GCF -4 from the last two terms.

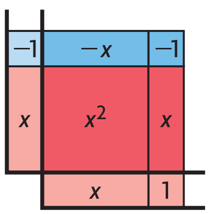
$= (5x - 4)(5x - 4)$
$= (5x - 4)^2$

Then I divided out the binomial common factor.

EXAMPLE 3 — **Representing** a difference of squares as a square tile model

Factor $x^2 - 1$.

Lori's Solution



$x^2 - 1 = (x + 1)(x - 1)$

When using tiles to factor, the unit tiles are always diagonal from the x^2 tiles. I arranged my tiles this way, but the rectangle was not complete. I added one vertical x tile and one horizontal $-x$ tile to make a square. This is the same as adding zero to the expression, since $x + (-x) = 0$.

The dimensions of the square are $(x + 1)$ and $(x - 1)$.

EXAMPLE 4 **Reasoning** logically to factor a difference of squares

Factor each expression.

a) $x^2 - 64$ **b)** $81x^4 - 25y^2$

Natalie's Solution

a) $x^2 - 64$, where $\sqrt{x^2} = x$ and $\sqrt{64} = 8$

This is a binomial. Both terms are perfect squares, and they are separated by a subtraction sign. There is no *x* term.

$x^2 - 64 = (x + 8)(x - 8)$

I know, from expanding, that this type of expression has factors that are binomials. The terms of the binomials contain the square roots of the terms of the original expression. One binomial contains the sum of the square roots, and the other binomial contains the difference.

$(x + 8)(x - 8)$
$= x^2 - 8x + 8x - 64$
$= x^2 - 64$

I checked by multiplying. When expanded, the *x* term equals zero.

b) $81x^4 - 25y^2$, where $\sqrt{81x^4} = 9x^2$ and $\sqrt{25y^2} = 5y$

This is a difference of squares. Both terms are perfect squares, and they are separated by a subtraction sign.

$81x^4 - 25y^2 = (9x^2 + 5y)(9x^2 - 5y)$

Both factors are binomials. The terms of the binomials contain the square roots of the terms of the original expression. One binomial contains the sum of the square roots. The other binomial contains the difference.

$(9x^2 + 5y)(9x^2 - 5y)$
$= 81x^4 - 45x^2y + 45x^2y - 25y^2$
$= 81x^4 - 25y^2$

I checked by multiplying.

In Summary

Key Ideas

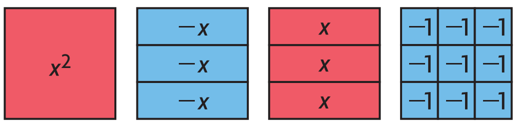
- A polynomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect-square trinomial:
  - $a^2 + 2ab + b^2$ can be factored as $(a + b)^2$.
  - $a^2 - 2ab + b^2$ can be factored as $(a - b)^2$.
- A polynomial of the form $a^2 - b^2$ is a difference of squares and can be factored as $(a + b)(a - b)$.

Need to Know

- A perfect-square trinomial and a difference of squares can be factored by
  - forming a square using algebra tiles
  - using decomposition
  - using logical reasoning

CHECK *Your Understanding*

1. a) Write the quadratic expression that is represented by these algebra tiles.

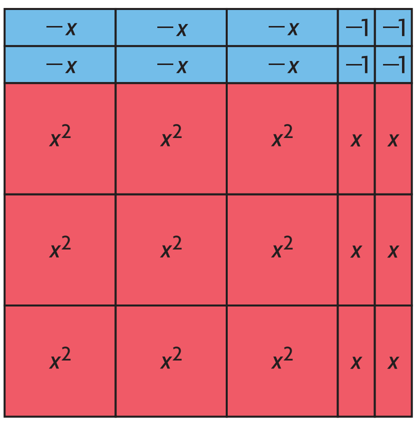


b) Sketch what the tiles would look like if they were arranged in a rectangle.

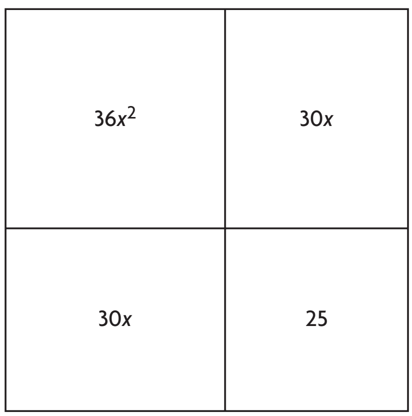
c) Use your sketch to determine the factors of the trinomial.

2. Each model represents a quadratic expression. Identify the polynomial and its factors.

a)



b)



3. Determine the missing factor.

a) $x^2 - 100 = (x + 10)(■)$
b) $n^2 + 10n + 25 = (■)(n + 5)$
c) $81a^2 - 16 = (■)(9a - 4)$
d) $20x^2 - 5 = (■)(2x - 1)(2x + 1)$
e) $25m^2 - 70m + 49 = (■)^2$
f) $18x^2 - 48x + 32 = 2(■)^2$

PRACTISING

4. Determine the value of each symbol.

a) $4x^2 + ◆x + 25 = (2x + ■)^2$
b) $25x^2 - ◆ = (■x + 3)(■x - 3)$
c) $16x^2 - ◆x + 81 = (4x - ■)(4x - ■)$
d) $◆x^2 - 64 = (3x - ■)(3x + ■)$

5. Factor each expression.

a) $x^2 - 25$
b) $y^2 - 81$
c) $a^2 - 36$
d) $4c^2 - 49$
e) $9x^2 - 4$
f) $25d^2 - 144$

6. Factor.

a) $x^2 + 10x + 25$
b) $b^2 + 8b + 16$
c) $m^2 - 4m + 4$
d) $4c^2 - 44c + 121$
e) $16p^2 + 72p + 81$
f) $25z^2 - 30z + 9$

7. Factor.

K **a)** $49a^2 + 56a + 16$
b) $4x^2 - 25$
c) $-50x^2 - 40x - 8$
d) $4a^2 - 256$
e) $225 - 16x^2$
f) $(x + 1)^2 + 2(x + 1) + 1$

8. You can use the pattern for the difference of squares to help you do
A mental calculations. Show how you can do this for each expression.

a) $64^2 - 60^2$
b) $18^2 - 12^2$

9. Explain how you know that $8x^2 - 18x + 9$ cannot be factored as

C **a)** a perfect square
b) a difference of squares

10. Factor each expression.

a) $x^4 - 12x^2 + 36$
b) $a^4 - 16$
c) $49x^2 - 100$
d) $12x^2 - 60x + 75$
e) $x^4 - 24x^2 + 144$
f) $289x^6 - 81$

11. Factor.

a) $x^2 - 16xy + 64y^2$
b) $36x^2 - 25y^2$
c) $16x^2 - 72xy + 81y^2$
d) $1 - 9a^2b^4$
e) $-18x^2 + 24xy - 8y^2$
f) $50x^3 - 8xy^2$

12. Factor each expression. Explain the strategy you used.

T **a)** $x^2 - c^2 - 8x + 16$
b) $4c^2 - a^2 - 6ab - 9b^2$

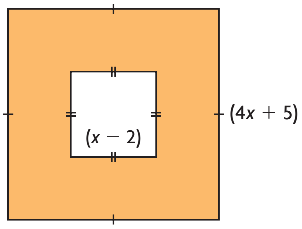
13. Examine each quadratic relation below.

i) Express the relation in factored form.
ii) Determine the zeros.
iii) Determine the coordinates of the vertex.
iv) Sketch the graph of the relation.

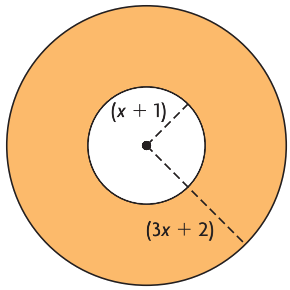
a) $y = -x^2 + 16x - 64$ **b)** $y = 4x^2 - 1$

14. Determine a simplified expression for the area of each shaded region.

a)

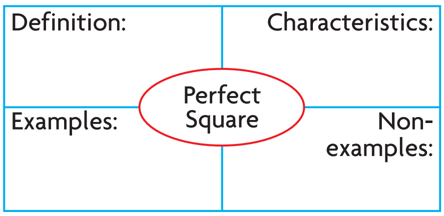


b)



15. Copy and complete the following charts to show what you know about each type of polynomial.

a)



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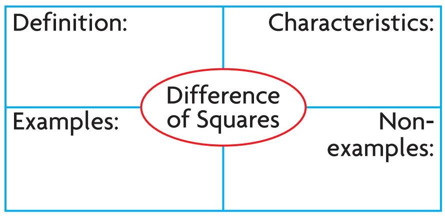


Extending

16. a) Multiply $(a + b)(a^2 - ab + b^2)$.

b) Compare your product for part a) with the factors of the original expression. Identify the pattern you see.

c) Use the pattern you identified for part b) to factor each expression.

i) $x^3 + 8$ **iii)** $8x^3 + 1$
ii) $x^3 + 27$ **iv)** $27x^3 + 8$

17. a) Multiply $(a - b)(a^2 + ab + b^2)$.

b) Compare your product for part a) with the factors of the original expression. Identify the pattern you see.

c) Use the pattern you identified for part b) to factor each expression.

i) $x^3 - 27$ **iii)** $8x^3 - 125$
ii) $x^3 - 64$ **iv)** $64x^3 - 27$

Curious Math

Creating Composite Numbers

Jamie claims that if you multiply any natural number greater than 1 by itself four times and then add 4, the result will always be a composite number.

Savita made a chart and tested some numbers to see if Jamie's claim could be true.

Starting Number	Calculation	Result	Prime or Composite?
2	$2 \times 2 \times 2 \times 2 + 4$	20	composite: $20 = 4 \times 5$
3	$3 \times 3 \times 3 \times 3 + 4$	85	composite: $85 = 5 \times 17$
5	$5 \times 5 \times 5 \times 5 + 4$	629	composite: $629 = 17 \times 37$
7	$7 \times 7 \times 7 \times 7 + 4$	2405	composite: $2405 = 5 \times 481$

1. Choose another starting number, and test this number to see if Jamie's claim holds.
2. Based on your observations, do you think Jamie's calculation will always result in a composite number? Explain.
3. Use n to represent any natural number, where $n > 1$. Create an algebraic expression that represents any result of Jamie's calculation.
4. Add the terms $4n^2$ and $-4n^2$ to the expression you created in step 3.
5. Factor the expression you created for step 4.
6. Explain why the factors you found for step 5 prove that Jamie's claim is true.

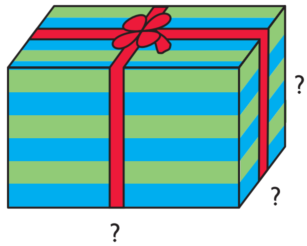
4.6 Reasoning about Factoring Polynomials

GOAL

Use reasoning to factor a variety of polynomials.

LEARN ABOUT the Math

The trinomial $8x^3 - 6x^2 - 5x$ represents the volume of a rectangular prism.



? What algebraic expressions represent the dimensions of this prism?

EXAMPLE 1 **Solving a problem** using a factoring strategy

Determine the dimensions of the rectangular prism.

Rob's Solution

$$V = l \times w \times h$$

The volume of a rectangular prism is calculated by multiplying together its length, width, and height.

$$V = 8x^3 - 6x^2 - 5x$$
$$= x(8x^2 - 6x - 5)$$

I tried to factor the expression. Each term contained the common factor of x, so I divided this out. The other factor was a trinomial, where $a \neq 1$.

$$= x(8x^2 - 10x + 4x - 5)$$

To factor the trinomial, I used decomposition. I looked for two numbers whose sum is -6 and whose product is $(8)(-5) = -40$. I knew that one number must be negative and the other number must be positive, since the sum and the product are both negative. The numbers are -10 and 4. I used these numbers to decompose the middle term.

$$= x(8x^2 - 10x + 4x - 5)$$
$$= x[2x(4x - 5) + 1(4x - 5)]$$
$$= x(4x - 5)(2x + 1)$$

I factored the first two terms of the expression in brackets and then the last two terms. I divided out the common factor of $4x - 5$.

$$= x(4x - 5)(2x + 1)$$

Possible dimensions of the rectangular prism are length $= 4x - 5$, width $= 2x + 1$, and height $= x$.

I can't be sure which dimension is which, but I know that the volume expression is correct when these terms are multiplied together.

Reflecting

A. Why did Rob decide to use a factoring strategy?

B. Why did he start by dividing out the common factor from the expression?

C. Why can he not be sure which factors correspond to which dimensions of the prism?

APPLY the Math

EXAMPLE 2 **Reasoning to factor polynomials**

Factor each expression.

a) $x^2 + x - 132$ **b)** $16x^2 - 88x + 121$ **c)** $-18x^4 + 32x^2$

Sunny's Solution

a) $x^2 + x - 132$
$= (x + 12)(x - 11)$

This expression has three terms. It's a trinomial, and there are no common factors.

The factors must be two binomials that both start with x. To determine the factors, I need two numbers whose product is -132 and whose sum is 1. The numbers are 12 and -11.

b) $16x^2 - 88x + 121$
$= (4x - 11)(4x - 11)$
$= (4x - 11)^2$

This expression has three terms. It's a trinomial, and there are no common factors.

The first and last terms are perfect squares. When I doubled the product of their square roots, I got the middle term.

This is a perfect-square trinomial.

c) $-18x^4 + 32x^2$

This expression has two terms and a greatest common factor of $-2x^2$. I divided out the GCF.

$= -2x^2(9x^2 - 16)$
$= -2x^2(3x - 4)(3x + 4)$

The factor $9x^2 - 16$ is a difference of squares. The factors of $9x^2 - 16$ are binomials that contain the square roots of each term.

EXAMPLE 3 Selecting a grouping strategy to factor

Factor $x^5y + x^2y^3 - x^3y^3 - y^5$.

Monique's Solution

$$x^5y + x^2y^3 - x^3y^3 - y^5$$
$$= y(x^5 + x^2y^2 - x^3y^2 - y^4)$$

This expression has four terms, so I can't use strategies that work with trinomials. I divided out the greatest common factor of y.

$$= y(\underline{x^5 + x^2y^2}\ \underline{- x^3y^2 - y^4})$$
$$= y[x^2(x^3 + y^2) - y^2(x^3 + y^2)]$$

I grouped the first two terms and the last two terms in the second factor, because I saw that the first grouping had a common factor of x^2 and the second grouping had a common factor of y^2. I divided out the common factor from each pair.

$$= y[(x^3 + y^2)(x^2 - y^2)]$$
$$= y(x^3 + y^2)(x - y)(x + y)$$

Then I factored out $x^3 + y^2$, since it's a common factor.

Finally, I factored $x^2 - y^2$ using the pattern for the difference of squares.

In Summary

Key Idea

- The strategy that you use to factor an algebraic expression depends on the number of terms and the type of terms in the expression.

Need to Know

- You can use the following checklist to decide how to factor an algebraic expression:
  - Divide out all the common monomial factors.
  - If the expression has two terms, check for a difference of squares: $a^2 - b^2 = (a + b)(a - b)$
  - If the expression has three terms, check for a perfect square: $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$
  - If the expression has three terms and is of the form $x^2 + bx + c$, look for factors of the form $(x + r)(x + s)$, where $c = r \times s$ and $b = r + s$.
  - If the expression has three terms and is of the form $ax^2 + bx + c$, $a \neq 1$, look for factors of the form $(px + r)(qx + s)$, where $c = r \times s$, $a = p \times q$, and $b = ps + qr$.
  - If the expression has four or more terms, try a grouping strategy.

CHECK Your Understanding

1. Identify the type of algebraic expression and the factoring strategies you would use to factor the expression.

a) $6xy + 12x^2y^2 - 4x^3y^3$

b) $20x^2 + 11x - 3$

c) $3x^2 + 3xa - 2x - 2a$

d) $49y^2 - 9$

e) $3x^2 - 3x - 90$

f) $x^2 - 13x + 42$

2. Factor each expression in question 1.

PRACTISING

3. a) Create a factorable algebraic expression for each situation.

i) a perfect-square trinomial

ii) a trinomial of the form $ax^2 + bx + c$, where $a = 1$

iii) a difference of squares

iv) a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$

v) an expression that contains a monomial common factor

vi) an expression that contains a binomial common factor

b) Exchange the expressions you created with a partner. Factor your partner's expressions, and discuss the solutions.

4. Determine the value of each symbol.

a) $-10a^3 + 15a^2 = -5a(\blacklozenge a^2 - \blacksquare a)$

b) $x^2 - \blacklozenge x - 63 = (x + 7)(x - \blacksquare)$

c) $25x^2 - \blacklozenge = (\blacksquare x - 7)(\blacksquare x + \bullet)$

d) $6x^2 + \bullet x - 10 = (2x + \blacksquare)(\blacklozenge x - 2)$

5. Recall the question about working backwards on the opening page of this chapter. Work backwards to determine the value of each symbol:

$$(\blacktriangle x + \bullet)(\blacksquare x + \blacklozenge) = 3x^2 + 11x + 10$$

6. Factor each expression.

K **a)** $16x^2 - 25$

b) $-6b^2a - 9b^3 + 15b^2$

c) $c^2 - 12c + 35$

d) $49d^2 + 14d + 1$

e) $12x^2 + 4x - 21$

f) $2wz + 6w - 5z - 15$

7. Factor.

a) $10x^2 + 3x - 1$

b) $144a^4 - 121$

c) $24ac - 8c + 21a - 7$

d) $x^3 - 11x^2 + 18x$

e) $18x^2 + 60x + 50$

f) $x^2y - 4y$

8. Factor.

a) $2s^2 + 4s - 6$

b) $14 - 5w - w^2$

c) $z^4 - 13z^2 + 36$

d) $8s^2 - 50r^2$

e) $36 - 84g + 49g^2$

f) $x^2 + 10x + 25 - 16y^2$

9. Explain why each expression is not factored fully.

C a) $x^4 - 1 = (x^2 - 1)(x^2 + 1)$

b) $x^2y - 9xy + 20y = y(x^2 - 9x + 20)$

c) $15x^2 + 6xy - 5x - 2y = 3x(5x + 2y) - (5x + 2y)$

d) $48a^4c^3 - 3b^4c^3 = 3c^3(16a^4 - b^4)$

10. Factor.

a) $xy - ty + xs - ts$

b) $-25x^2 + 16y^2$

c) $a^4 - 13a^2 + 36$

d) $x^2 - y^2 - 2y - 1$

e) $2(a + b)^2 + 5(a + b) + 3$

f) $6x^3 - 63x - 13x^2$

11. The area of a rectangle is given by the relation $A = 8x^2 + 18x + 7$.

a) Determine expressions for the possible dimensions of this rectangle.

b) Determine the dimensions and area of this rectangle if $x = 3$ cm.

12. The area of a circle is given by the relation $A = \pi x^2 + 10\pi x + 25\pi$.

A a) Determine an expression for the radius of this circle.

b) Determine the radius and area of this circle if $x = 10$ cm.

13. The volume of a rectangular prism is given by the relation $V = 2x^3 + 14x^2 + 24x$.

a) Determine expressions for the possible dimensions of this prism.

b) Determine the dimensions and volume of this prism if $x = 5$ cm.

14. Decide whether each polynomial has $(x + y)$ as one of its factors.
T Justify your decision.

a) $xy + 3x^2y - 4xy + 6x^2y$

b) $x^5 - x^3y^2 + x^3 - xy^2$

c) $x^2y + 6y - 9xy + x^2y + xy$

d) $x^3 + 5x^2 + 6x + x^2y + 5xy + 6y$

15. Create a graphic organizer that would help you decide which strategy you should use to factor a given algebraic expression.

Extending

16. Factor each expression, if possible.

a) $\frac{x^2}{9} - \frac{1}{4}$

b) $100 - (a - 5)^2$

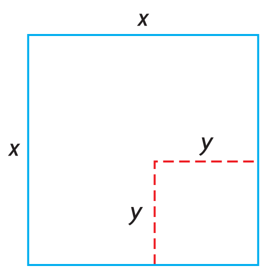
c) $4x^2 + y^2$

d) $\frac{25a^2}{64} - \frac{9b^2}{49}$

e) $(x + 3)^2 - (y - 3)^2$

f) $4(c - 5)^4 + 12(c - 5)^2 + 9$

17. A square with sides that measure x units is drawn. Then a square with sides that measure y units is removed. Use the diagram to explain why $x^2 - y^2 = (x + y)(x - y)$.



4 Chapter Review

Study Aid

- See Lesson 4.4, Examples 1 to 4.
- Try Chapter Review Questions 8 to 11.

FREQUENTLY ASKED Questions

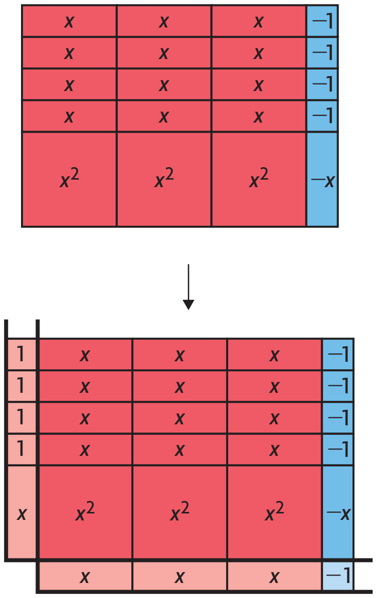
Q: How can you factor a quadratic expression of the form $ax^2 + bx + c$, where $a \neq 1$?

A: Some polynomials of this form can be factored, but others cannot. Try to factor the expression using one of the methods below. If none of these methods work, the expression cannot be factored.

EXAMPLE

Factor $3x^2 + 11x - 4$.

Solution



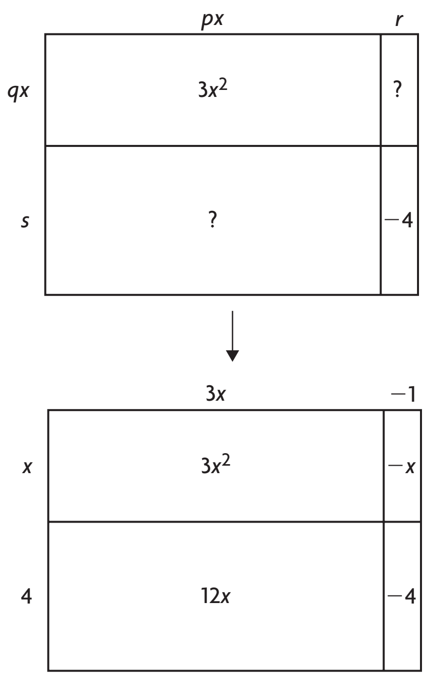
Method 1

Arrange algebra tiles to form a rectangle. The rectangle for $3x^2 + 11x - 4$ was not complete, so $+x$ and $-x$ were added to complete it. The length and width are the factors.

The algebra tiles show that $3x^2 + 11x - 4 = (3x - 1)(x + 4)$.

You can check by multiplying.

$$\begin{aligned}(3x - 1)(x + 4) &= 3x^2 + 12x - x - 4 \\ &= 3x^2 + 11x - 4\end{aligned}$$



Method 2

Use guess and test with an area diagram.

If there are factors for $3x^2 + 11x - 4$, you know that they are of the form $(px + r)(qx - s)$ since the constant is negative. You also know that $p \times q = 3$, so the values of p and q can be 3 and 1, or -3 and -1. As well, you know that $r \times s = -4$, so the values of r and s can be 2 and -2, 4 and -1, or -1 and 4.

Try different combinations of p, q, r, and s until you find the combination that gives $ps + qr = 11$.

The area diagram shows that $3x^2 + 11x - 4 = (3x - 1)(x + 4)$.

Method 3

Use decomposition. Decompose $+11$ as the sum of two numbers that multiply to -12 since $-4 \times 3 = -12$. Use -1 and 12.

$$3x^2 + 11x - 4$$
$$= \underline{3x^2 - 1x} \; \underline{+ 12x - 4}$$
$$= x(3x - 1) + 4(3x - 1)$$
$$= (3x - 1)(x + 4)$$

Q: How can you recognize a perfect-square trinomial or a difference of squares, and how do you factor it?

A: An expression is a perfect-square trinomial when the coefficient of x^2 is a perfect square, the constant is a perfect square, and the middle coefficient is twice the product of the two square roots. For example, $25x^2 + 60x + 36$ is a perfect square whose factors are $(5x + 6)$ and $(5x + 6)$.

An expression is a difference of squares when it consists of two perfect square terms and one of these terms is subtracted from the other. For example, $100x^2 - 81$ is a difference of squares whose factors are $(10x - 9)$ and $(10x + 9)$.

Study Aid

- See Lesson 4.5, Examples 1 to 4.
- Try Chapter Review Questions 12 to14.

Q: How do you decide which strategy you should use to factor an algebraic expression?

A: The strategy you use depends on the type of expression you are given and the number of terms it contains. You can follow this checklist to help you decide which strategy to use:

- Divide out all the common monomial factors.
- If the expression has two terms, check for a difference of squares: $a^2 - b^2 = (a + b)(a - b)$
- If the expression has three terms, check for a perfect square: $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$
- If the expression has three terms and is of the form $x^2 + bx + c$, look for factors of the form $(x + r)(x + s)$, where $c = r \times s$ and $b = r + s$.
- If the expression has three terms and is of the form $ax^2 + bx + c$, $a \neq 1$, look for factors of the form $(px + r)(qx + s)$, where $c = r \times s$, $a = p \times q$, and $b = ps + qr$.
- If the expression has four or more terms, try a grouping strategy.

Study Aid

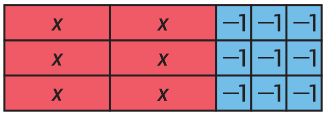
- See Lesson 4.6, Examples 1 to 3.
- Try Chapter Review Questions 15 to 19.

PRACTICE Questions

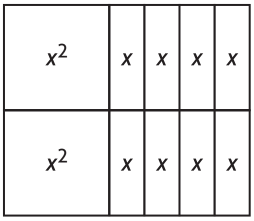
Lesson 4.1

1. Each model represents an algebraic expression. Identify the expression and its factors.

a)



b)



2. Factor each expression.

a) $20x^2 - 4x$

b) $3n^2 - 6n + 15$

c) $-2x^3 + 6x^2 + 4x$

d) $6a(3 - 7a) - 5(3 - 7a)$

3. The area of a rectangle is given by the relation $A = 16x^2 - 24$.

a) Determine possible dimensions of this rectangle.

b) Is there more than one possibility? Explain.

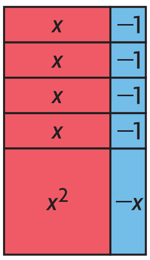
4. a) Write three polynomials whose terms have a greatest common factor of $4x^3y$.

b) Factor each polynomial you wrote for part a).

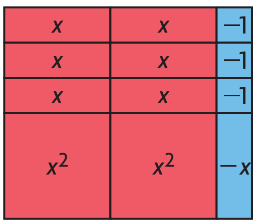
Lesson 4.2

5. Identify each expression that is modelled below, and state its factors.

a)



b)



Lesson 4.3

6. Factor each expression.

a) $x^2 + 16x + 63$

b) $x^2 - 7x - 60$

c) $x^2 + 6x - 27$

d) $5x^2 - 5x - 100$

7. Examine the relation $y = x^2 + 7x + 12$.

a) Write the relation in factored form.

b) Determine the coordinates of the x-intercepts.

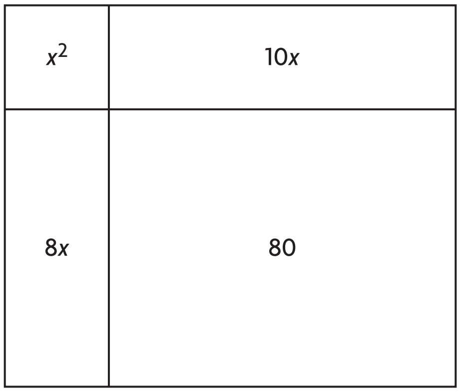
c) Determine the coordinates of the vertex.

d) State the minimum value of the relation and where the minimum value occurs.

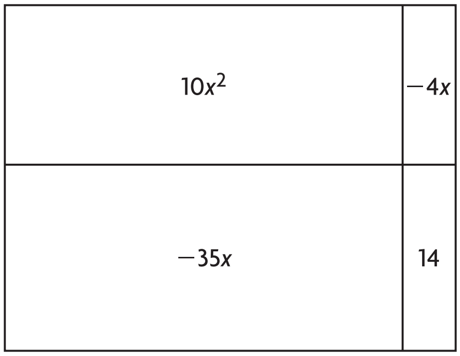
Lesson 4.4

8. Identify each expression that is modelled below, and state its factors.

a)



b)



9. Explain the strategy you would use to factor each trinomial.

a) $15x^2 - 4x - 4$

b) $20x^2 + 3x - 2$

c) $7a^2 + 6a - 16$

d) $20y^2 - 17y - 10$

10. Factor each expression.

a) $7x^2 - 19x - 6$

b) $4a^2 + 23a + 15$

c) $12x^2 - 16x + 5$

d) $6n^2 - 11ny - 10y^2$

11. Erica and Asif sell newly designed digital watches. The profit on the watches they sell is determined by the relation $P = -2n^2 + 120n - 1000$, where n is the number of watches sold and P is the profit in dollars.

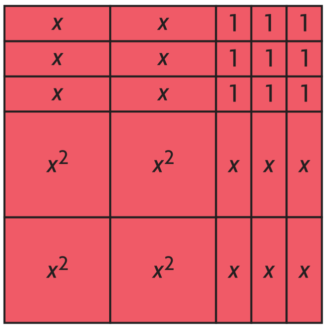
a) What are the break-even points for Erica and Asif?

b) What is the maximum profit that Erica and Asif can earn?

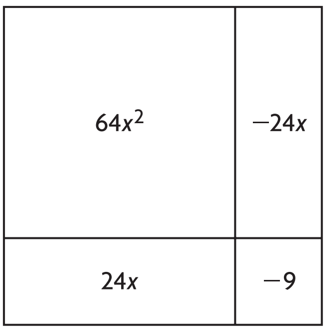
Lesson 4.5

12. Identify each expression that is modelled below, and state its factors.

a)



b)



13. Factor each expression.

a) $144x^2 - 25$

b) $36a^2 + 12a + 1$

c) $18x^5 - 512xy^2$

d) $4(x - 2)^2 - 20(x - 2) + 25$

e) $(x + 5)^2 - y^2$

f) $x^2 - 6x + 9 - 4y^2$

14. The polynomial $x^2 - 25$ can be factored. Can the polynomial $x^2 + 25$ be factored? Explain.

Lesson 4.6

15. How is expanding an algebraic expression related to factoring an algebraic expression? Use an example in your explanation.

16. Factor each expression.

a) $7x^2 - 26x - 8$

b) $64a^6 - 25$

c) $18ac - 12a - 15c + 10$

d) $4x^2y - 44xy + 72y$

e) $20x^2 + 61x + 45$

f) $z^4 - 13z^2 + 40$

17. Factor.

a) $2s^2 + 3s - 5$

b) $15 - 2w - w^2$

c) $z^4 - 4z^2 - 32$

d) $16s^2 - 121r^2$

e) $9 - 30g + 25g^2$

f) $x^2 + 16x + 64 - 25y^2$

18. A packaging company creates different-sized cardboard boxes. The volume of a box is given by $V = 18x^3 - 2x + 45x^2 - 5$.

a) Determine expressions for the possible dimensions of these boxes.

b) Determine the dimensions and volume of a box if $x = 2$ cm.

19. Determine the coordinates of the vertex of each relation.

a) $y = x^2 - 10x + 24$

b) $y = 2x^2 - 24x + 72$

c) $y = -5x^2 + 500$

d) $y = 2x^2 - 7x - 4$

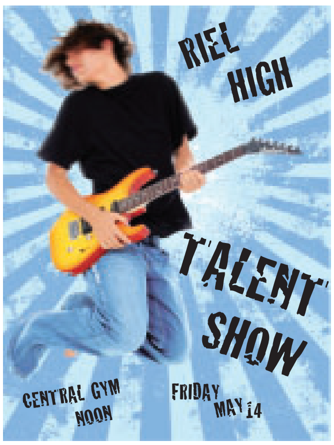
e) $y = 4x^2 + 16x$

f) $y = x^2 + 10x + 25$

4 Chapter Self-Test

Process Checklist

- ✔ Question 4: Did you **reflect** on your thinking to decide which strategy you prefer?
- ✔ Questions 5 and 6: Did you **select strategies** that are appropriate for the expressions?
- ✔ Question 9: Did you **connect** factoring with the factored form of a quadratic relation from Chapter 3?



1. Determine the value of each symbol.

a) $x^2 - ◆x - 56 = (x + 7)(x - ■)$

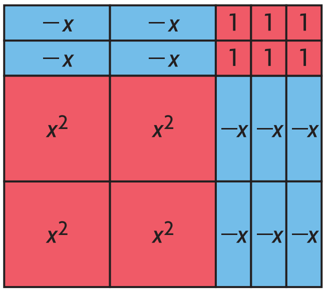
b) $16x^2 - ◆ = (■x - 3)(■x + ●)$

c) $12x^2 + ●x + 5 = (4x + ■)(◆x + 5)$

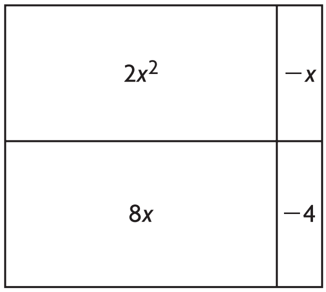
d) $25x^2 + ●x + 49 = (■x + ◆)^2$

2. Identify each trinomial that is modelled below, and state its factors.

a)



b)



3. Factor each expression.

a) $20x^5 - 30x^3$

b) $-8yc^3 + 4y^2c - 6yc$

c) $2a(3b + 5) + 7(3b + 5)$

d) $2st + 6s + 5t + 15$

4. a) Factor $25x^2 - 30x + 9$ using two different strategies.

b) Which strategy do you prefer? Explain why.

5. Factor each expression.

a) $x^2 + 4x - 77$

b) $a^2 - 3a - 10$

c) $3x^2 - 12x + 12$

d) $m^3 + 3m^2 - 4m$

6. Factor.

a) $6x^2 - x - 2$

b) $8n^2 + 8n - 6$

c) $9x^2 + 12x + 4$

d) $6ax^2 + 5ax - 4a$

7. A graphic arts company creates posters with areas that are given by the equation $A = 2x^2 + 11x + 12$.

a) Write expressions for possible dimensions of the posters.

b) Write expressions for the dimensions of a poster whose width is doubled and whose length is increased by 2. Write the new area as a simplified polynomial.

c) Write expressions for possible dimensions of a poster whose area is given by the expression $18x^2 + 99x + 108$.

8. Factor each expression.

a) $225x^2 - 4$

b) $9a^2 - 48a + 64$

c) $x^6 - 4y^2$

d) $(3 + n)^2 - 10(3 + n) + 25$

9. A parabola is defined by the equation $y = 2x^2 - 11x + 5$. Explain how you can determine the vertex of the parabola without using graphing technology.

4 Chapter Task

The Factoring Challenge

How well can you play this game to show what you know about factoring polynomials?

Number of players: 3 per group
Materials needed for each player: 10 blank cards, a recording sheet, a calculator (optional)

Rules

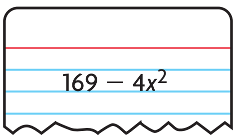
1. On your own, create 10 polynomials that satisfy the following conditions:
   - Two must be quadratic binomials, but only one is factorable.
   - Six must be quadratic trinomials, but only four are factorable.
   - Two must have at least four terms, but only one is factorable.

   Write each of your 10 polynomials on a separate card.
2. Form a group with two other students, and combine all your cards. Place the cards face down so that the polynomials are not visible. Decide on the order in which you will play. The goal of the game is to accumulate the most points.
3. On your turn, choose a card and factor the polynomial if possible. If you are correct, you get 5 points; if you are incorrect, you get 0 points. If you say that the polynomial cannot be factored but it can, you get 0 points.

   If the polynomial is not factorable, you must try to change one or two coefficients or the constant to make it factorable. Your points are determined by the number of changes you make. For example, suppose that you turn over $2x^2 - 5x + 8$. If you change it to $2x^2 - 10x + 8$, you get 1 point; if you change it to $2x^2 - 6x + 4$, you get 2 points. If you create another unfactorable polynomial, you get 0 points.
4. Take turns playing the game. You must record each polynomial you turn over, its factors (if possible), and the points you get. If you turn over a polynomial that cannot be factored, record the new polynomial you create, its factors, and the points you get.

Recording Sheet for Shirley

Polynomial	Factors	Points Earned
$169 - 4x^2$	$(13 + 2x)(13 - 2x)$	5



5. Each player gets 10 turns. The player with the most points wins.
6. As a class, discuss any polynomials that were difficult to factor.

Task Checklist

- ✔ Did you create a variety of polynomials that met the given conditions?
- ✔ Did you verify that the correct number of polynomials were factorable?
- ✔ Did you submit your recording sheet?