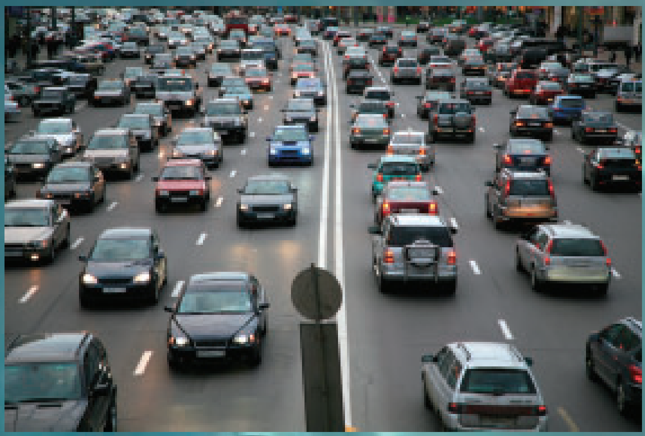


Year	Ontario's CO_2 Emissions (kilotonnes/year)
1995	175 000
1996	182 000
1997	186 000
1998	187 000
1999	191 000
2000	201 000
2001	193 000
2002	199 000
2003	203 000
2004	199 000
2005	201 000

CO_2 emissions are measured in kilotonnes (kt); 1 kt = 1000 tonnes (t) and 1 t = 1000 kg.

Chapter

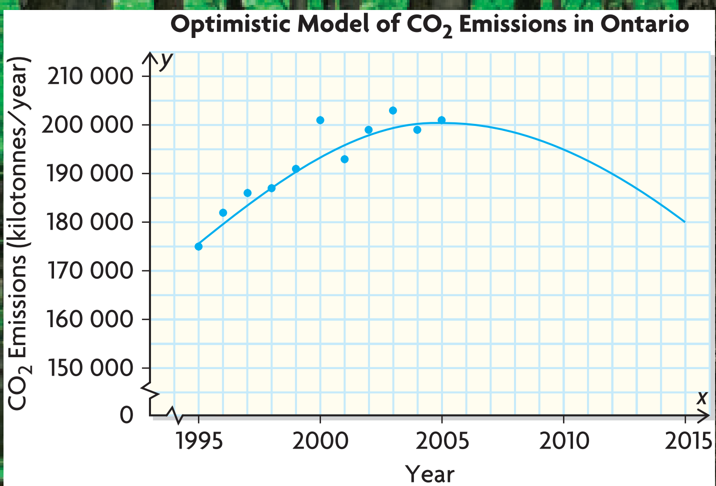
6

Quadratic Equations

GOALS

You will be able to

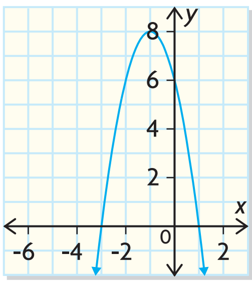
- Solve quadratic equations graphically, by factoring, and by using the quadratic formula
- Write a quadratic relation in vertex form by completing the square
- Solve and model problems involving quadratic relations in standard, factored, and vertex forms



? Recent attention to the environment has raised awareness about the effects of carbon dioxide in the atmosphere. Many countries are developing strategies to reduce their CO_2 emissions.

How can you use a quadratic model to predict when Ontario's CO_2 emissions might drop below 1995 levels?

6 Getting Started



WORDS YOU NEED to Know

1. State the vertex, equation of the axis of symmetry, and zeros of the parabola at the left.

2. Match each form with the correct equation.

a) standard form	**i)** $y = -2(x + 3)(x - 1)$
b) factored form	**ii)** $y = -2(x + 1)^2 + 8$
c) vertex form	**iii)** $y = -2x^2 - 4x + 6$

SKILLS AND CONCEPTS You Need

Graphing Quadratic Relations

Study *Aid*

- For more help and practice, see Lessons 5.6, 3.3, and 5.3.

Different strategies can be used to graph a quadratic relation. The strategy you use might depend on the form of the relation.

EXAMPLE

Describe a strategy you could use to graph each quadratic relation.

a) $y = x^2 + 4x - 1$ **b)** $y = -2(x + 3)(x - 5)$ **c)** $y = 2(x - 3)^2 - 4$

Solution

a) The equation is in standard form.
- Partially factor the equation to locate two ordered pairs with the same y-coordinate.
- Determine the x-coordinate of the vertex by calculating the mean of the x-coordinates of the points you determined above.
- Substitute the x-coordinate of the vertex into the equation to determine the y-coordinate of the vertex.
- Substitute two other values of x into the equation to determine two more points on the parabola.
- Use symmetry to determine the points on the parabola that are directly across from the two additional points you determined.
- Plot the vertex and points, then sketch the parabola.

b) The equation is in factored form.
- Locate the zeros by setting each factor to zero and solving each equation.
- Determine the x-coordinate of the vertex by calculating the mean of the x-coordinates of the zeros that you determined above.
- Substitute the x-coordinate of the vertex into the equation to determine the y-coordinate of the vertex.
- Plot the vertex and zeros, then sketch the parabola.

c) The equation is in vertex form.
- Locate the vertex and the axis of symmetry.
- Determine the y-intercept by letting x equal 0.
- Use symmetry to determine the point on the parabola that is directly across from the y-intercept.
- Plot the vertex and points, then sketch the parabola.

3. Graph each quadratic relation.

a) $y = (x + 4)^2 - 3$ **c)** $y = (x + 5)(x - 7)$ **e)** $y = 2x^2 + x - 1$

b) $y = -3(x - 3)^2 - 1$ **d)** $y = \frac{1}{2}(x - 4)(x - 7)$ **f)** $y = -3x^2 - 5x$

Factoring Quadratic Expressions

You can use a variety of strategies to factor a quadratic expression.

Study Aid
- For more help and practice, see Lessons 4.2 to 4.6.

EXAMPLE

Factor. Use an area diagram for part a). Use decomposition for part b).

a) $x^2 - 7x - 18$ **b)** $4x^2 + 8x - 5$

Solution

a) $x^2 - 7x - 18$

	x	-9
x	x^2	$-9x$
2	$2x$	-18

This is a trinomial where $a = 1$ and there are no common factors. Look for two binomials that each start with x. To determine the factors, find two numbers whose product is -18 and whose sum is -7. The numbers are -9 and 2.

$x^2 - 7x - 18 = (x - 9)(x + 2)$

b) $4x^2 + 8x - 5$

This is a trinomial where $a \neq 1$ and there are no common factors. Look for two numbers whose sum is 8 and whose product is $(4)(-5) = -20$. The numbers are -2 and 10. Use these to decompose the middle term.

$= 4x^2 - 2x + 10x - 5$

$= \underline{4x^2 - 2x} + \underline{10x - 5}$

$= 2x(2x - 1) + 5(2x - 1)$

Group the terms in pairs, and divide out the common factors.

$= (2x - 1)(2x + 5)$

Divide out the common binomial as a common factor.

4. Factor each expression, if possible.

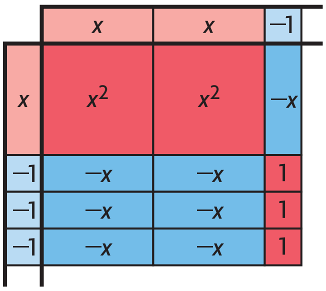
a) $x^2 + 8x + 12$ **c)** $x^2 + 7x - 30$ **e)** $-6x^2 - 7x + 24$

b) $x^2 - 5x + 6$ **d)** $9x^2 - 30x + 25$ **f)** $2x^2 - x - 5$

Study Aid

- For help, see the Review of Essential Skills and Knowledge Appendix.

Question	Appendix
5	A-9



PRACTICE

5. Solve each equation.

a) $4x + 8 = 0$ **c)** $-2x + 12 = 0$

b) $5x - 3 = 0$ **d)** $12x + 7 = 0$

6. Expand and simplify.

a) $(3x - 5)(x - 4)$ **c)** $(2x + 3)(4x - 5)$ **e)** $(3a + 7)(3a + 7)$

b) $(n + 1)(n - 1)$ **d)** $(7 - 3p)(2p + 5)$ **f)** $(6x - 5)^2$

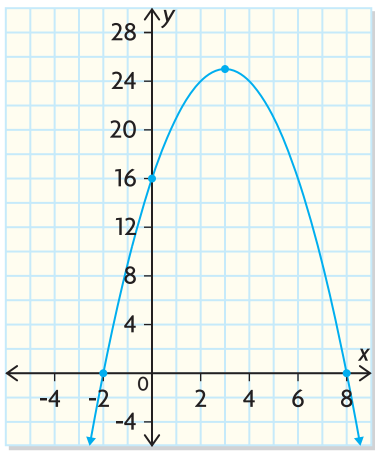
7. The algebra tiles at the left show $2x^2 - 7x + 3$ and its factors. Determine the factors for each expression. Use algebra tiles or area diagrams, if you wish.

a) $x^2 + 4x + 3$ **c)** $3x^2 - 5x - 2$ **e)** $2x^2 + 12x$

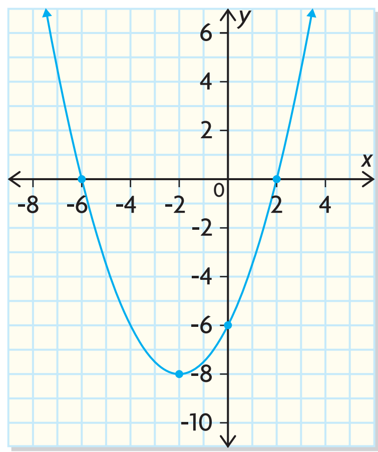
b) $x^2 - 8x + 16$ **d)** $4x^2 - 9$ **f)** $9x^2 - 6x + 1$

8. For each quadratic relation, determine the zeros, the y-intercept, the equation of the axis of symmetry, the vertex, and the equation in standard form.

a)



b)



9. For each quadratic relation, determine the y-intercept, the equation of the axis of symmetry, and the vertex.

a) $y = (x - 4)(x + 6)$ **b)** $y = -4(x - 3)^2 - 5$

10. Do you agree or disagree with each statement? Provide examples to support your answers.

a) Every quadratic expression can be written as the product of two linear factors.

b) Every quadratic relation has a maximum value or a minimum value.

c) The graph of a quadratic relation always has two x-intercepts.

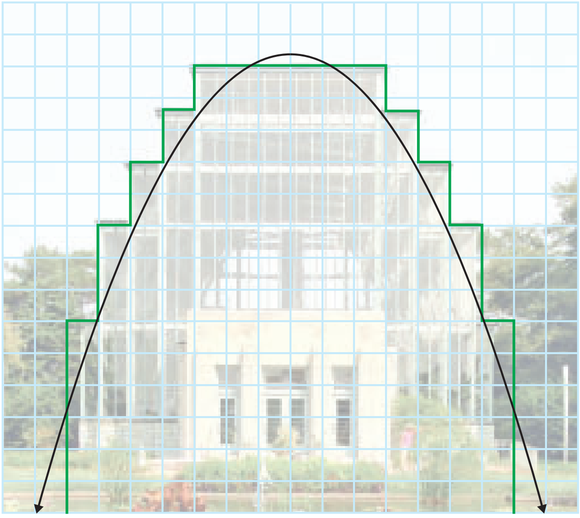
APPLYING What You Know

YOU WILL NEED

- grid paper
- ruler

The Jewel Box

This building is called the Jewel Box. It is a large greenhouse in St. Louis, Missouri. Its design is based on a parabola that passes through the corners of the roof line.



? What quadratic relations can be used to model this parabola?

A. Trace the parabola from the photo at the right onto grid paper. Decide where to draw the x- and y-axes.

B. Create an algebraic model for the parabola in vertex form.

C. Create an algebraic model for the parabola in factored form.

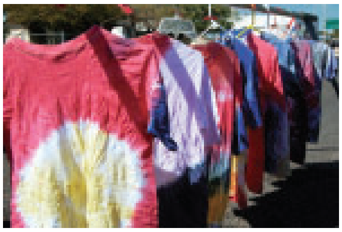
D. How are the two models you created for parts B and C the same? How are they different?

E. Write both of your models in standard form. Do all three models represent the same parabola? Explain.

F. Which form of the quadratic relation do you prefer to model the shape of the roof line of the Jewel Box? Justify your answer.

6.1 Solving Quadratic Equations

YOU WILL NEED
- grid paper
- ruler
- graphing calculator



GOAL

Use graphical and algebraic strategies to solve quadratic equations.

INVESTIGATE the Math

Andy and Susie run a custom T-shirt business. From past experience, they know that they can model their expected profit, in dollars, with the relation $P = -x^2 + 120x - 2000$, where x is the number of T-shirts they sell. Andy wants to sell enough T-shirts to earn $1200. Susie wants to sell just enough T-shirts to break even because she wants to close the business.

? How can Andy and Susie determine the number of T-shirts they must sell to achieve their goals?

quadratic equation
an equation that contains at least one term whose highest degree is 2; for example, $x^2 + x - 2 = 0$

A. Why can you use the **quadratic equation** $-x^2 + 120x - 2000 = 0$ to determine the number of T-shirts that must be sold to achieve Susie's goal?

B. Factor the left side of the equation in part A. Use the factors to determine the number of T-shirts that must be sold to achieve Susie's goal.

C. Use your factors for part B to predict what the graph of the profit relation will look like. Sketch the graph, based on your prediction.

D. Graph the profit relation using a graphing calculator. Was your prediction for part C correct?

E. What quadratic equation can you use to describe Andy's goal of making a profit of $1200?

root
a solution; a number that can be substituted for the variable to make the equation a true statement; for example, $x = 1$ is a root of $x^2 + x - 2 = 0$, since $1^2 + 1 - 2 = 0$

F. How can you use your graph for part D to determine the **roots** of your equation for part E?

G. How many T-shirts must be sold to achieve Andy's goal?

Reflecting

H. Why did factoring $-x^2 + 120x - 2000$ help you determine the break-even points?

I. Are the roots of the equation $-x^2 + 120x - 2000 = 0$ also zeros or x-intercepts of the relation $y = -x^2 + 120x - 2000$? Explain.

J. Why would factoring the left side of $-x^2 + 120x - 2000 = 1200$ not help you determine the number of T-shirts that Andy has to sell?

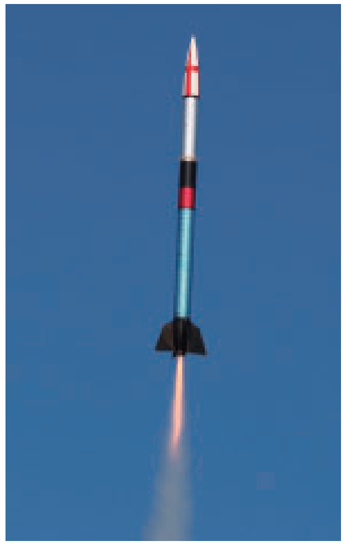
K. Explain why it would help you solve the equation in part J if you were to write it as $-x^2 + 120x - 2000 - 1200 = 0$.

L. To solve $ax + b = c$, you isolate x. Why would you not isolate x^2 to solve $ax^2 + bx + c = 0$?

APPLY the Math

EXAMPLE 1 **Selecting a strategy** to solve a quadratic equation

The user's manual for Arleen's model rocket says that the equation $h = -5t^2 + 40t$ models the approximate height, in metres, of the rocket after t seconds. When will Arleen's rocket reach a height of 60 m?



Amir's Solution: Selecting a factoring strategy

$$-5t^2 + 40t = 60$$

I substituted 60 for h because I wanted to calculate the time for the height 60 m.

$$-5t^2 + 40t - 60 = 0$$

I subtracted 60 from both sides of the equation to make the right side equal zero. I did this so that I could determine the zeros of the corresponding relation.

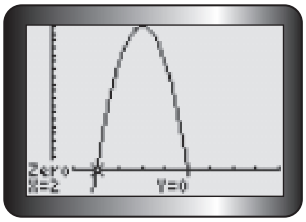
$$-5(t^2 - 8t + 12) = 0$$
$$-5(t - 2)(t - 6) = 0$$
$$t - 2 = 0 \quad \text{or} \quad t - 6 = 0$$
$$t = 2 \qquad\qquad t = 6$$

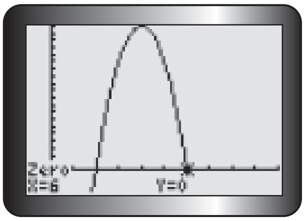
I divided out the common factor of -5. Then I factored the trinomial. The trinomial will equal zero if either factor equals zero. I set each factor equal to zero and solved both equations. This gave me the zeros of the parabola.

The rocket is 60 m above the ground at 2 s on the way up and 6 s on the way down.

Tech | **Support**

For help locating the zeros of a relation using a TI 83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.



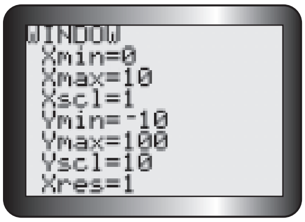


I verified my solutions by graphing $y = -5x^2 + 40x - 60$, and then locating the zeros.

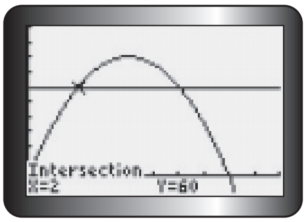
My solutions were correct.

Alex's Solution: Selecting a graphing strategy

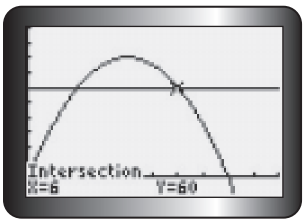
Using a graphing calculator, I entered the height equation in Y1. I substituted the variable *y* for *h* and the variable *x* for *t*. I entered 60 in Y2 to determine when the rocket will reach 60 m.



I estimated that the rocket will travel about 100 m and be in the air for about 10 s, so I used these window settings.



I used the Intersect operation to locate the intersection points of the two graphs.



Tech | **Support**

For help determining points of intersection using a TI-83/84 graphing calculator, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.

The rocket is 60 m off the ground after 2 s and after 6 s.

EXAMPLE 2 Selecting a factoring strategy to solve a quadratic equation

Determine the roots of $6x^2 - 11x - 10 = 0$.

Annette's Solution

$6x^2 - 11x - 10 = 0$

Product $= -60$ Sum $= -11$

$(1)(-60)$ $1 + (-60) = -59$ ✘

$(2)(-30)$ $2 + (-30) = -28$ ✘

$(3)(-20)$ $3 + (-20) = -17$ ✘

$(4)(-15)$ $4 + (-15) = -11$ ✔

Since the trinomial in the equation contains no common factors and is one where $a \neq 1$, I used decomposition. I looked for two numbers whose sum is -11 and whose product is $(6)(-10) = -60$.

$$6x^2 - 15x + 4x - 10 = 0$$
$$3x(2x - 5) + 2(2x - 5) = 0$$
$$(2x - 5)(3x + 2) = 0$$

Since the numbers were -15 and 4, I used these to decompose the middle term. I factored the first two terms and then the last two terms. Then, I divided out the common factor of $2x - 5$.

$2x - 5 = 0$ or $3x + 2 = 0$

$2x = 5$ $3x = -2$

$x = \frac{5}{2}$ $x = -\frac{2}{3}$

I set each factor equal to zero and solved each equation.

The roots of $6x^2 - 11x - 10 = 0$ are $x = 2\frac{1}{2}$ and $x = -\frac{2}{3}$.

EXAMPLE 3 Reasoning about how to solve a quadratic equation

Determine all the values of x that satisfy the equation $x^2 + 4 = 3x(x - 5)$. If necessary, round your answers to two decimal places.

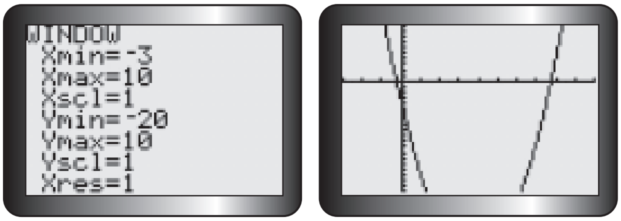
Karl's Solution

$$x^2 + 4 = 3x(x - 5)$$
$$x^2 + 4 = 3x^2 - 15x$$

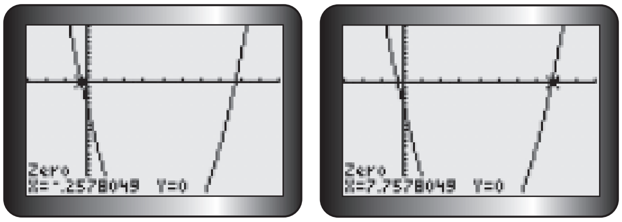
I decided to write an equivalent equation in the form $ax^2 + bx + c = 0$, which I could solve by graphing or factoring. I expanded the expression on the right side of the equation.

$$0 = 3x^2 - x^2 - 15x - 4$$
$$0 = 2x^2 - 15x - 4$$

I used inverse operations to make the left side of the equation equal to zero. I couldn't factor the right side of the equation, so I decided to use a graph.



I graphed $y = 2x^2 - 15x - 4$ using these window settings. From the graph, I could see that one x-intercept was between -1 and 0 and the other x-intercept was between 7 and 8.



Using the Zero operation of the calculator, I estimated that the x-intercepts were about -0.258 and 7.758.

Tech Support

For help determining the zeros of a relation using a TI-83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.

The solutions are $x \doteq -0.26$ and $x \doteq 7.76$.

I rounded the solutions to two decimal places. These are reasonable estimates, since the solutions are not exact.

EXAMPLE 4 Reflecting on the reasonableness of a solution

A ball is thrown from the top of a seaside cliff. Its height, h, in metres, above the sea after t seconds can be modelled by $h = -5t^2 + 21t + 120$. How long will the ball take to fall 20 m below its initial height?

Jacqueline's Solution

$h = -5t^2 + 21t + 120$

$h = -5(0)^2 + 21(0) + 120$

$h = 120$

I let $t = 0$ to determine the initial height of the ball.

The cliff is 120 m high, so the ball starts 120 m above the sea.

$120 - 20 = 100$

The initial height of the ball was 120 m. When the ball had fallen 20 m below its initial height, it was 100 m above the sea.

Let $h = 100$.

$100 = -5t^2 + 21t + 120$

$0 = -5t^2 + 21t + 120 - 100$

$0 = -5t^2 + 21t + 20$

I substituted 100 for h in the relation. I wrote the equation in the form $0 = ax^2 + bx + c$ so that I could solve it by graphing or factoring.

I subtracted 100 from both sides of the equation to make the left side equal to 0.

$$0 = 5t^2 - 21t - 20$$
$$0 = 5t^2 - 25t + 4t - 20$$
$$0 = 5t(t - 5) + 4(t - 5)$$
$$0 = (5t + 4)(t - 5)$$

I multiplied all the terms, on both sides of the equation, by -1 because I wanted $5t^2$ to be positive. I factored the right side of the equation using decomposition.

$$5t + 4 = 0 \quad \text{or} \quad t - 5 = 0$$
$$5t = -4 \qquad\qquad t = 5$$
$$t = -\frac{4}{5}$$

I set each factor equal to zero and solved for t.

The ball will take 5 s to fall 20 m below its initial height.

Since the ball was thrown at $t = 0$, I knew that the solution $t = -\frac{4}{5}$ didn't make sense. I used the solution $t = 5$ since this did make sense.

In Summary

Key Ideas

- A quadratic equation is any equation that contains a polynomial in one variable whose degree is 2; for example, $x^2 + 6x + 9 = 0$.
- All quadratic equations can be expressed in the form $ax^2 + bx + c = 0$ using algebraic strategies. In this form, the equation can be solved by
  - factoring the quadratic expression, setting each factor equal to zero, and solving the resulting equations
    or
  - graphing the corresponding relation $y = ax^2 + bx + c$ and determining the zeros, or x-intercepts

Need to Know

- Roots and solutions have the same meaning. These are all values that satisfy an equation.
- Some quadratic equations can be solved by factoring. Other quadratic equations must be solved by using a graph.
- If you use factoring to solve a quadratic equation, write the equation in the form $ax^2 + bx + c = 0$ before you try to factor.
- To solve $ax^2 + bx + c = d$ using a graph, graph $y = ax^2 + bx + c$ and $y = d$ on the same axes. The solutions to the equation are the x-coordinates of the points where the parabola and the horizontal line intersect.

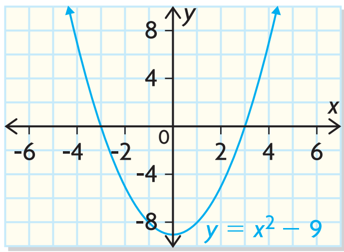
CHECK Your Understanding

1. The solutions to each equation are the x-intercepts of the corresponding quadratic relation. State the quadratic relation.

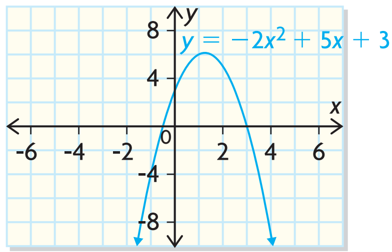
a) $x^2 - 4x + 4 = 0$ **b)** $2x^2 - 9x = 5$

2. Use the graph of each quadratic relation to determine the roots to each quadratic equation, where $y = 0$.

a)



b)



3. Solve each equation.

a) $x(x + 4) = 0$

b) $(x + 10)(x + 8) = 0$

c) $(x - 5)^2 = 0$

d) $(3x + 8)(x - 4) = 0$

e) $x^2 + 5x + 6 = 0$

f) $x^2 - 2x = 8$

PRACTISING

4. Determine whether the given value is a root of the equation.

a) $x = 2; x^2 + x - 6 = 0$

b) $x = 4; x^2 + 7x - 8 = 0$

c) $x = -\frac{1}{2}; 2x^2 + 11x + 5 = 0$

d) $x = \frac{3}{2}; 8x^2 + 10x - 3 = 0$

e) $x = -5; x^2 - 4x - 5 = 0$

f) $x = 2; 3x^2 - 2x - 8 = 0$

5. Solve each equation by factoring. Use an equivalent equation, if necessary.

a) $x^2 + 2x - 15 = 0$

b) $x^2 + 5x - 24 = 0$

c) $x^2 + 4x + 4 = 0$

d) $x^2 - 5x = 0$

e) $x^2 - 6x = 16$

f) $x^2 + 12 = 7x$

6. Solve by factoring. Verify your solutions.

a) $3x^2 - 5x - 2 = 0$

b) $2x^2 + 3x - 2 = 0$

c) $3x^2 - 4x - 15 = 0$

d) $6x^2 - x - 2 = 0$

e) $4x^2 - 4x = 3$

f) $9x^2 + 1 = 6x$

7. Simplify and then solve each equation.

K **a)** $x(x + 1) = 12$

b) $2x(x + 4) = x + 4$

c) $3x(x + 2) = 2x^2 - (4 - x)$

d) $3x(x + 6) + 50 = 2x^2 + 3(x - 2)$

e) $(x + 2)^2 + x = 2(3x + 5)$

f) $(2x + 1)^2 = x + 2$

8. Determine the roots of each equation.

a) $x^2 + 4x - 32 = 0$

b) $x^2 + 11x + 30 = 0$

c) $5x^2 - 28x - 12 = 0$

d) $x^2 + 5x = 14$

e) $4x^2 + 25 = 20x$

f) $3x^2 + 16x - 7 = 5$

9. Solve each equation. Round your answers to two decimal places.

a) $x^2 + 5x - 2 = 0$

b) $4x^2 - 8x + 3 = 0$

c) $x^2 + 1 = 4 - 2x^2$

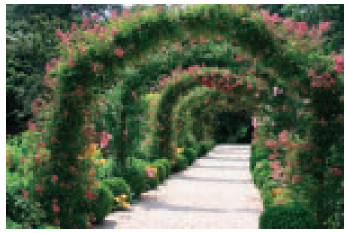
d) $x(x + 5) = 2x + 7$

e) $3x^2 + 5x - 3 = x^2 + 4x + 1$

f) $(x + 3)^2 - 2x = 15$

10. Conor has a summer lawn-mowing business. Based on experience, Conor knows that $P = -5x^2 + 200x - 1500$ models his profit, P, in dollars, where x is the amount, in dollars, charged per lawn.
    a) How much does he need to charge if he wants to break even?
    b) How much does he need to charge if he wants to have a profit of \$500?

11. Stacey maintains the gardens in the city parks. In the summer, she plans to build a walkway through the rose garden. The area of the walkway, A, in square metres, is given by $A = 160x + 4x^2$, where x is the width of the walkway in metres. If the area of the walkway must be 900 m^2, determine the width.



Environment *Connection*

By photosynthesis, green plants remove carbon dioxide from the air and produce oxygen.

12. Patrick owns an apartment building. He knows that the money he earns in a month depends on the rent he charges. This relationship can be modelled by $E = \frac{1}{50}R(1650 - R)$, where E is Patrick's monthly earnings, in dollars, and R is the amount of rent, in dollars, he charges each tenant.
    a) How much will he earn if he sets the rent at \$900?
    b) If Patrick wants to earn at least \$13 000, between what two values should he set the rent?

13. **T** Determine the points of intersection of the line $y = -2x + 7$ and the parabola $y = 2x^2 + 3x - 5$.

14. **A** While hiking along the top of a cliff, Harlan knocked a pebble over the edge. The height, h, in metres, of the pebble above the ground after t seconds is modelled by $h = -5t^2 - 4t + 120$.
    a) How long will the pebble take to hit the ground?
    b) For how long is the height of the pebble greater than 95 m?

15. **C** Is it possible to solve a quadratic equation that is not factorable over the set of integers? Explain.

16. **a)** Describe when and why you would rewrite a quadratic equation to solve it. In your answer, include $x^2 - 2x = 15$, rewritten as $x^2 - 2x - 15 = 0$.
    b) Explain how the relation $y = x^2 - 2x - 15$ can be used to solve $x^2 - 2x - 15 = 0$.

Extending

17. Solve the equations $x^4 - 9x^2 + 20 = 0$ and $x^3 - 9x^2 + 20x = 0$ by first solving the equation $x^2 - 9x + 20 = 0$.

18. Will all quadratic equations always have two solutions? Explain how you know and support your claim with examples.

6.2 Exploring the Creation of Perfect Squares

YOU WILL NEED

- algebra tiles

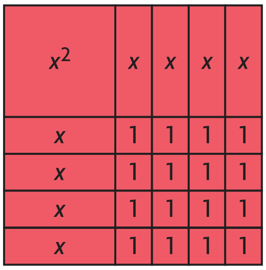
GOAL

Recognize the relationship between the coefficients and constants of perfect-square trinomials.

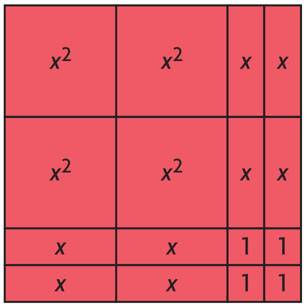
EXPLORE the Math

Quadratic expressions like $x^2 + 8x + 16$ and $4x^2 + 8x + 4$ are perfect-square trinomials. Quadratic expressions like $4x^2 + 8x + 3$ are not.

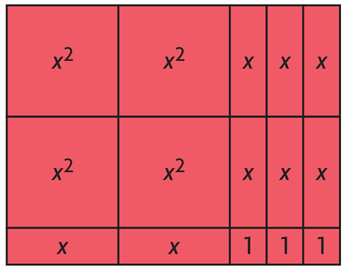
$x^2 + 8x + 16$



$4x^2 + 8x + 4$



$4x^2 + 8x + 3$



? How can you decide what value for c makes expressions of the form $ax^2 + abx + c$, $a \neq 0$, perfect-square trinomials?

A. Factor $x^2 + 8x + 16$ and $4x^2 + 8x + 4$ completely. Explain why these expressions are called perfect-square trinomials.

B. Using algebra tiles, create an arrangement that helps you determine the constant term c that must be added to create perfect-square trinomials. Verify by factoring each new trinomial you created.

i) $x^2 + 2x + c$ **iii)** $x^2 + 6x + c$ **v)** $x^2 + 10x + c$
ii) $x^2 + 4x + c$ **iv)** $x^2 + 8x + c$ **vi)** $x^2 + 12x + c$

C. For each trinomial you created in part B, compare the coefficient of x and the constant term you added. Explain how these numbers are related.

D. How are the expressions below different from those in part B?

i) $x^2 - 4x + c$ **iii)** $x^2 - 6x + c$ **v)** $x^2 - 2x + c$
ii) $x^2 - 8x + c$ **iv)** $x^2 - 12x + c$ **vi)** $x^2 - 10x + c$

E. Using algebra tiles or an area diagram, determine the constant term c that must be added to each of the expressions in part D to create perfect-square trinomials. Verify by factoring each new trinomial.

F. For each trinomial you created for part E, compare the coefficient of x and the constant term you added. Does the relationship you discovered in part C still apply?

G. Each expression below contains a common factor. Factor the expression and then determine the constant term c that must be added to each expression to make it a multiple of a perfect-square trinomial. Verify by factoring each new trinomial.

i) $2x^2 + 4x + c$ **iii)** $3x^2 - 6x + c$ **v)** $5x^2 + 25x + c$
ii) $3x^2 - 12x + c$ **iv)** $-x^2 + 4x + c$ **vi)** $6x^2 + 54x + c$

Reflecting

H. How can you predict the value of c that will make $x^2 + bx + c$ a perfect-square trinomial?

I. How can you predict the value of c that will make $ax^2 + abx + c$ a perfect-square trinomial?

In Summary

Key Idea

- If $(x + b)^2 = x^2 + 2bx + b^2$, then $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$. So, in all perfect-square trinomials, the constant term is half the coefficient of the x term squared.

Need to Know

- To create a perfect square that includes $x^2 + bx$ and no other terms with a variable, add $\left(\frac{b}{2}\right)^2$.
- To create a perfect square that includes $ax^2 + abx$ and no other terms with a variable, factor out a and then create a perfect square that includes $x^2 + bx$. This results in adding $a\left(\frac{b}{2}\right)^2$.

FURTHER *Your Understanding*

1. Determine the value of c that will create a perfect-square trinomial. Verify by factoring the trinomial you created.

a) $x^2 + 8x + c$ **c)** $x^2 + 40x + c$ **e)** $x^2 - 5x + c$
b) $x^2 - 14x + c$ **d)** $x^2 + 20x + c$ **f)** $x^2 + x + c$

2. Each expression is a multiple of a perfect-square trinomial. Determine the value of c.

a) $3x^2 + 30x + c$ **c)** $-4x^2 - 8x + c$ **e)** $5x^2 - 10x + c$
b) $2x^2 - 12x + c$ **d)** $6x^2 - 60x + c$ **f)** $7x^2 + 42x + c$

Curious Math

Does 65 Equal 64?

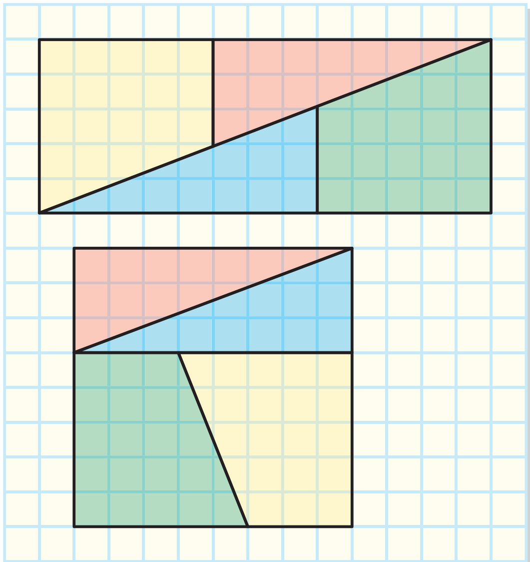
The steps at the right seem to prove that 65 equals 64.

1. Copy the steps. Explain how each step is obtained from the step above it.
2. Can you find any problems with any of the steps?

Let $a = 1$ and $b = 1$.
So $a = b$.

$$\begin{aligned} a \times a &= a \times b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a + b)(a - b) &= b(a - b) \\ a + b &= b \\ 2 &= 1 \\ 2 + 63 &= 1 + 63 \\ 65 &= 64 \end{aligned}$$

The two diagrams below also seem to prove that 65 equals 64.



3. How do the colours make the rectangle and the square appear to have the same area?
4. Determine the area of each figure.
5. Use your answers for steps 3 and 4 to explain why these two figures appear to prove that 65 equals 64.
6. These two proofs are called fallacious proofs because they contain an error. How would mathematics and our daily lives be affected if either of these proofs were true?
7. Some fallacious proofs are very complex. Try to create or research another fallacious proof that you can explain to a classmate.

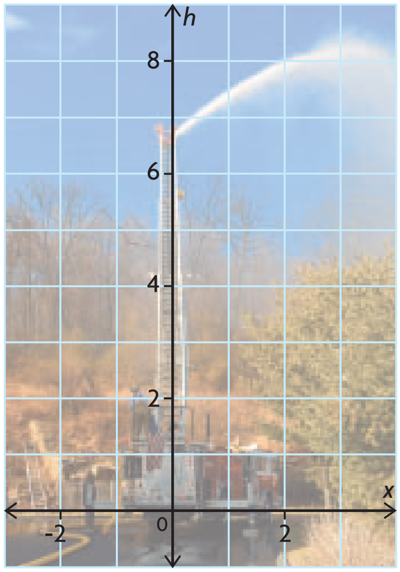
6.3 Completing the Square

GOAL

Write the equation of a parabola in vertex form by completing the square.

YOU WILL NEED

- algebra tiles (optional)
- grid paper
- ruler



LEARN ABOUT the Math

The automated hose on an aerial ladder sprays water on a forest fire. The height of the water, h, in metres, can be modelled by the relation $h = -2.25x^2 + 4.5x + 6.75$, where x is the horizontal distance, in metres, of the water from the nozzle of the hose.

? How high did the water spray from the hose?

Career Connection

As well as fighting fires, firefighters are trained to respond to medical and accident emergencies.

EXAMPLE 1 **Selecting a strategy** to solve a problem

Write the height relation $h = -2.25x^2 + 4.5x + 6.75$ in vertex form by **completing the square** to determine the maximum height.

completing the square

a process used to rewrite a quadratic relation that is in standard form, $y = ax^2 + bx + c$, in its equivalent vertex form, $y = a(x - h)^2 + k$

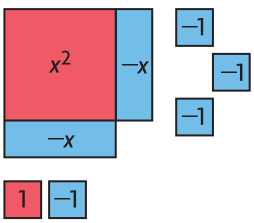
Joan's Solution: Selecting algebra tiles to complete the square

$y = ax^2 + bx + c$
$y = a(x - h)^2 + k$

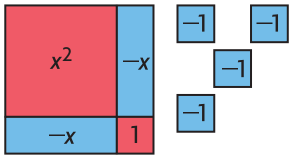
I knew that the vertex and maximum value can be determined from an equation in vertex form. I also knew that the value of a is the same in both standard form and vertex form.

$h = -2.25x^2 + 4.5x + 6.75$
$h = -2.25(x^2 - 2x - 3)$

I factored out -2.25 to get a trinomial with integer coefficients that I might be able to factor.



Because I wanted the equation in vertex form, I tried to make $x^2 - 2x - 3$ into a perfect square using tiles. I needed 1 positive unit tile in the corner to create a perfect square. I had 3 negative unit tiles, so I added 1 zero pair.



$h = -2.25[(x - 1)^2 - 4]$

I arranged the tiles to make the perfect square $x^2 - 2x + 1$. I had four negative unit tiles left over. This showed that $x^2 - 2x - 3 = (x - 1)^2 - 4$.

$h = -2.25[(x - 1)^2 - 4]$
$h = -2.25(x - 1)^2 - (-2.25)(4)$
$h = -2.25(x - 1)^2 + 9$

To write the equation in vertex form, I multiplied by -2.25 using the distributive property.

The water sprayed to a maximum height of 9 m above the ground.

The vertex of the parabola is (1, 9), and $a < 0$. So, the y-coordinate of the vertex gives the maximum height of the water.

Arianna's Solution: Selecting an algebraic strategy to complete the square

$h = -2.25x^2 + 4.5x + 6.75$
$h = -2.25(x^2 - 2x) + 6.75$

Since $a < 0$, the parabola opens downward and the maximum height is the y-coordinate of the vertex. I had to write the equation in vertex form.
To do so, I needed to create a perfect-square trinomial that used the variable x. I started by factoring out the coefficient of x^2 from the first two terms, since a perfect square can be created using the x^2 and x terms.

$-\frac{2}{2} = -1$ and $(-1)^2 = 1$,
so $x^2 - 2x + 1 = (x - 1)^2$

To create a perfect square, the constant term had to be the square of half the coefficient of the x term.

$h = -2.25(x^2 - 2x + 1 - 1) + 6.75$

I knew that if I added 1 in the brackets, I would have to subtract 1 so that I did not change the equation.

$$h = -2.25\left[(x^2 - 2x + 1) - 1\right] + 6.75$$
$$h = -2.25\left[(x - 1)^2 - 1\right] + 6.75$$

I factored the perfect square.

$$h = -2.25(x - 1)^2 - (-2.25)(1) + 6.75$$
$$h = -2.25(x - 1)^2 + 9$$

I multiplied by −2.25 and collected like terms.

The vertex is (1, 9), so the water sprayed to a maximum height of 9 m above the ground.

Reflecting

A. Why did both Joan and Arianna factor out –2.25 first?

B. Whose strategy do you prefer? Why?

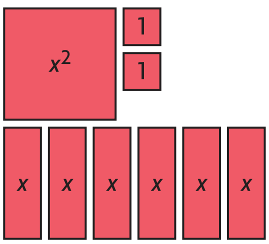
C. Explain how both strategies involve completing a square.

APPLY the Math

EXAMPLE 2 **Connecting** a model to the algebraic process of completing the square

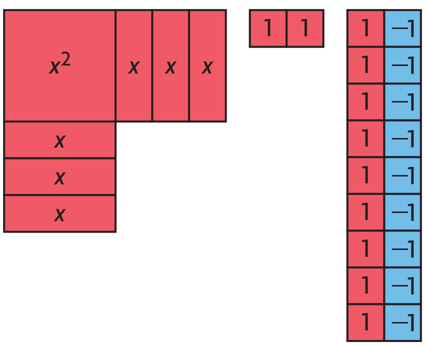
Write $y = x^2 + 6x + 2$ in vertex form, and then graph the relation.

Anya's Solution



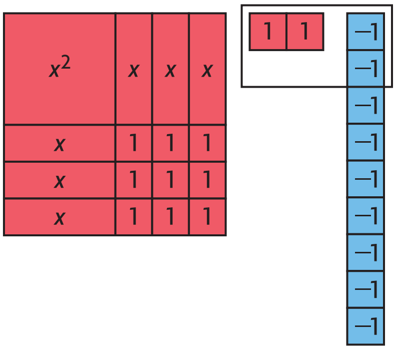
$$y = x^2 + 6x + 2$$

To write the relation in vertex form, I decided to complete the square using algebra tiles. Since there was only one x^2 tile, I had to make only one square.



$$y = x^2 + 6x + 9 - 9 + 2$$

I tried to form a square from the tiles, but I didn't have enough unit tiles. I needed 9 positive unit tiles to complete the square. To keep everything balanced, I added 9 zero pairs.



I completed the square. I had $(x + 3)$ for its length, with 2 positive unit tiles and 9 negative unit tiles left over. Since I could form 2 zero pairs, I had 7 negative unit tiles left over.

$y = (x^2 + 6x + 9) - 9 + 2$
$y = (x + 3)^2 - 9 + 2$
$y = (x + 3)^2 - 7$

$y = x^2 + 6x + 2$ in vertex form is
$y = (x + 3)^2 - 7.$

From the algebra tile model, I was able to write the relation in vertex form.

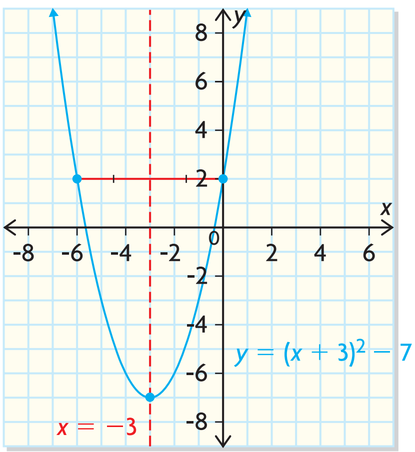
The vertex is $(-3, -7)$.
Since $a > 0$, the parabola opens upward.
The equation of the axis of symmetry is $x = -3$.

Using the vertex form of the equation, I determined the vertex, the direction of opening, and the equation of the axis of symmetry.

$y = (0 + 3)^2 - 7$
$y = 9 - 7$
$y = 2$

I let $x = 0$ to determine the y-intercept.

The y-intercept is 2.



I plotted the vertex and the y-intercept. I used symmetry to determine that $(-6, 2)$ is also a point on the parabola.

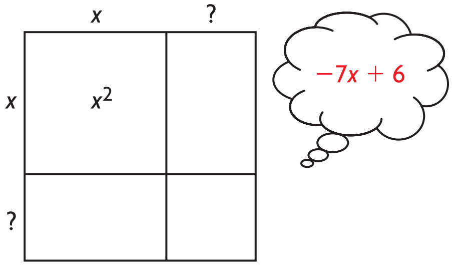
EXAMPLE 3 **Solving a problem** with an area diagram to complete the square

Cassidy's diving platform is 6 ft above the water. One of her dives can be modelled by the equation $d = x^2 - 7x + 6$, where d is her position relative to the surface of the water and x is her horizontal distance from the platform. Both distances are measured in feet. How deep did Cassidy go before coming back up to the surface?

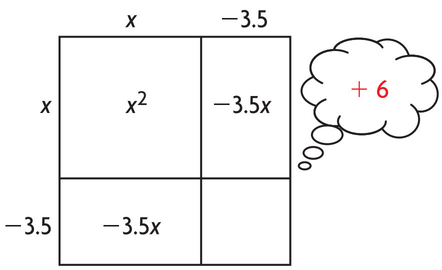
Sefu's Solution: Using an area diagram

$d = x^2 - 7x + 6$

Since the relation is quadratic and $a > 0$, Cassidy's deepest point will be at the vertex of a parabola that opens upward. I decided to complete the square.

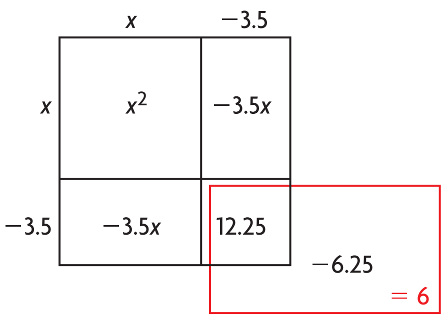


To complete the square, I drew a square area diagram. I knew that the length and width would have to be the same and that they both would be x plus a positive or negative constant.



Since the middle term $-7x$ had to be split equally between the two sides of the square, I found the constant by dividing the coefficient of the middle term by 2.

$$\frac{-7}{2} = -3.5$$



The constant term in the original equation is 6, but when I multiplied the -3.5s together to create the perfect square, I got 12.25, so I had to subtract 6.25.

$d = x^2 - 7x + 6$

$d = (x - 3.5)^2 - 6.25$

I used the dimensions of my square along with the extra -6.25 to write the equivalent relation in vertex form.

Cassidy dove to a depth of 6.25 ft before turning back toward the surface.

The vertex is $(3.5, -6.25)$ and since $a > 0$, the y-coordinate of the vertex is her lowest point.

EXAMPLE 4 | **Solving a problem** by determining the maximum value

Christopher threw a football. Its height, h, in metres, after t seconds can be modelled by $h = -4.9t^2 + 11.76t + 1.4$. What was the maximum height of the football, and when did it reach this height?

Macy's Solution

$h = -4.9t^2 + 11.76t + 1.4$
$h = -4.9(t^2 - 2.4t) + 1.4$

Since the relation is quadratic, the maximum value occurs at the vertex. To determine this value, I had to write the equation in vertex form. I started by factoring out −4.9 from the first two terms.

$\frac{-2.4}{2} = -1.2$ and $(-1.2)^2 = 1.44$

To determine the constant I had to add to $t^2 - 2.4t$ to create a perfect square, I divided the coefficient of t by 2. Then I squared my result.

$h = -4.9(t^2 - 2.4t + 1.44 - 1.44) + 1.4$
$h = -4.9[(t^2 - 2.4t + 1.44) - 1.44] + 1.4$
$h = -4.9[(t - 1.2)^2 - 1.44] + 1.4$

I completed the square by adding and subtracting 1.44, so the value of the expression value did not change. I grouped the three terms that formed the perfect square. Then I factored.

$h = -4.9(t - 1.2)^2 + 7.056 + 1.4$
$h = -4.9(t - 1.2)^2 + 8.456$
The vertex is (1.2, 8.456).

I multiplied by −4.9 using the distributive property. Then I added the constant terms.

The football reached a maximum height of 8.456 m after 1.2 s.

Since $a < 0$, the y-coordinate of the vertex is the maximum value. The x-coordinate is the time when the maximum value occurred.

In Summary

Key Idea

- A quadratic relation in standard form, $y = ax^2 + bx + c$, can be rewritten in its equivalent vertex form, $y = a(x - h)^2 + k$, by creating a perfect square within the expression and then factoring it. This technique is called completing the square.

Need to Know

- When completing the square, factor out the coefficient of x^2 from the terms that contain variables. Then divide the coefficient of the x term by 2 and square the result. This tells you what must be added and subtracted to create an equivalent expression that contains a perfect square.
- Completing the square can be used to determine the vertex of a quadratic relation in standard form.

CHECK Your Understanding

1. Copy and replace each symbol to complete the square.

a) $y = x^2 + 12x + 5$
$y = x^2 + 12x + ■ - ■ + 5$
$y = (x^2 + 12x + ■) - ■ + 5$
$y = (x + ◆)^2 - ●$

b) $y = 4x^2 + 24x - 15$
$y = 4(x^2 + ■x) - 15$
$y = 4(x^2 + 6x + ◆ - ◆) - 15$
$y = 4[(x^2 + 6x + ◆) - ◆] - 15$
$y = 4(x + ●)^2 - ⬟ - 15$
$y = 4(x + ●)^2 - ★$

2. Write each relation in vertex form by completing the square.

a) $y = x^2 + 8x$ **b)** $y = x^2 - 12x - 3$ **c)** $y = x^2 + 8x + 6$

3. Complete the square to state the coordinates of the vertex of each relation.

a) $y = 2x^2 + 8x$ **b)** $y = -5x^2 - 20x + 6$ **c)** $y = 4x^2 - 10x + 1$

PRACTISING

4. Consider the relation $y = -2x^2 + 12x - 11$.

a) Complete the square to write the relation in vertex form.
b) Graph the relation.

5. Determine the maximum or minimum value of each relation by completing the square.

a) $y = x^2 + 14x$
b) $y = 8x^2 - 96x + 15$
c) $y = -12x^2 + 96x + 6$
d) $y = -10x^2 + 20x - 5$
e) $y = -4.9x^2 - 19.6x + 0.5$
f) $y = 2.8x^2 - 33.6x + 3.1$

6. Complete the square to express each relation in vertex form.
K Then graph the relation.

a) $y = x^2 + 10x + 20$
b) $y = -x^2 + 6x - 1$
c) $y = 2x^2 + 4x - 2$
d) $y = -0.5x^2 - 3x + 4$

7. Complete the square to express each relation in vertex form. Then describe the transformations that must be applied to the graph of $y = x^2$ to graph the relation.

a) $y = x^2 - 8x + 4$
b) $y = x^2 + 12x + 36$
c) $y = 4x^2 + 16x + 36$
d) $y = -3x^2 + 12x - 6$
e) $y = 0.5x^2 - 4x - 8$
f) $y = 2x^2 - x + 3$

8. Joan kicked a soccer ball. The height of the ball, h, in metres, can be
A modelled by $h = -1.2x^2 + 6x$, where x is the horizontal distance, in metres, from where she kicked the ball.

a) What was the initial height of the ball when she kicked it? How do you know?
b) Complete the square to write the relation in vertex form.
c) State the vertex of the relation.
d) What does each coordinate of the vertex represent in this situation?
e) How far did Joan kick the ball?

Health *Connection*

An active lifestyle contributes to good physical and mental health.

9. Carly has just opened her own nail salon. Based on experience, she knows that her daily profit, P, in dollars, can be modelled by the relation $P = -15x^2 + 240x - 640$, where x is the number of clients per day. How many clients should she book each day to maximize her profit?

10. The cost, C, in dollars, to hire landscapers to weed and seed a local park can be modelled by $C = 6x^2 - 60x + 900$, where x is the number of landscapers hired to do the work. How many landscapers should be hired to minimize the cost?

$y = -2x^2 + 16x - 7$

$y = -2(x^2 + 8x) - 7$

$y = -2(x^2 + 8x + 64 - 64) - 7$

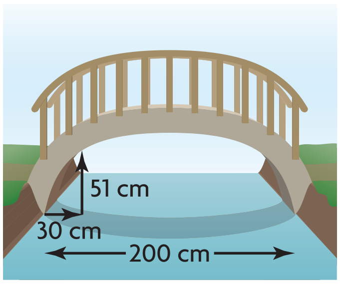
$y = -2(x + 8)^2 - 64 - 7$

$y = -2(x + 8)^2 - 73$

Therefore, the vertex is $(73, -8)$.

11. Neilles determined the vertex of a relation by completing the square, as shown at the left. When he checked his answer at the back of his textbook, it did not match the answer given. Identify each mistake that he made, explain why it is a mistake, and provide the correct solution.

12. **T** Bob wants to cut a wire that is 60 cm long into two pieces. Then he wants to make each piece into a square. Determine how the wire should be cut so that the total area of the two squares is as small as possible.



13. Kayli wants to build a parabolic bridge over a stream in her backyard as shown at the left. The bridge must span a width of 200 cm. It must be at least 51 cm high where it is 30 cm from the bank on each side. How high will her bridge be?

14. **C**
   a) Determine the vertex of the quadratic relation $y = 2x^2 - 4x + 5$ by completing the square.
   b) How does changing the value of the constant term in the relation in part a) affect the coordinates of the vertex?

15. The main character in a video game, Tammy, must swing on a vine to cross a river. If she grabs the vine at a point that is too low and swings within 80 cm of the surface of the river, a crocodile will come out of the river and catch her. From where she is standing on the riverbank, Tammy can reach a point on the vine where her height above the river, h, is modelled by the relation $h = 12x^2 - 76.8x + 198$, where x is the horizontal distance of her swing from her starting point. Should Tammy jump? Justify your answer.

16. Explain how to determine the vertex of $y = x^2 - 2x - 35$ using three different strategies. Which strategy do you prefer? Explain your choice.

Extending

17. Celeste has just started her own dog-grooming business. On the first day, she groomed four dogs for a profit of \$26.80. On the second day, she groomed 15 dogs for a profit of \$416.20. She thinks that she will maximize her profit if she grooms 11 dogs per day. Assuming that her profit can be modelled by a quadratic relation, calculate her maximum profit.

18. Complete the square to determine the vertex of $y = x^2 + bx + c$.

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How are the roots of a quadratic equation and the zeros of a quadratic relation related?

A: The roots of the equation $ax^2 + bx + c = 0$ are the zeros, or x-intercepts, of the relation $y = ax^2 + bx + c$.

Q: What strategies can you use to solve a quadratic equation?

A1: If the equation is in the form $ax^2 + bx + c = 0$, you can graph the relation $y = ax^2 + bx + c$ and locate the zeros on your graph. If the trinomial is factorable, you can factor it, set each factor equal to zero, and solve the equations.

Study *Aid*

- See Lesson 6.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1 to 5.

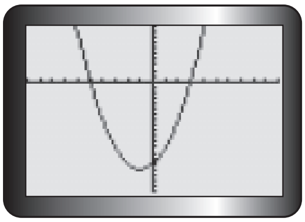
EXAMPLE

Solve $x^2 + 2x - 15 = 0$.

Solution

By Graphing Technology or

Graph using a scale $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.



$x = -5$ and $x = 3$

By Factoring

$x^2 + 2x - 15 = 0$

$(x - 3)(x + 5) = 0$

$x - 3 = 0$ or $x + 5 = 0$

$x = 3$ $\quad$ $x = -5$

$x = 3$ and $x = -5$

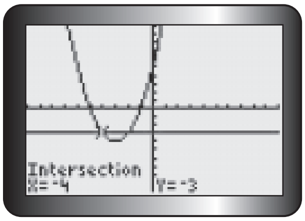
A2: If the equation is in the form $ax^2 + bx + c = d$, you can graph $y = ax^2 + bx + c$ and $y = d$ and determine the points of intersection. Alternatively, you can rearrange the equation so that one side is equal to zero. Then you can graph or factor the resulting equation to solve it.

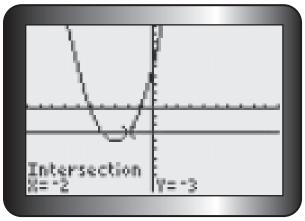
EXAMPLE

Solve $x^2 + 6x + 5 = -3$.

Solution

$y = x^2 + 6x + 5$
$y = -3$

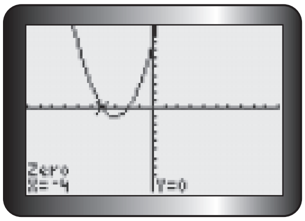


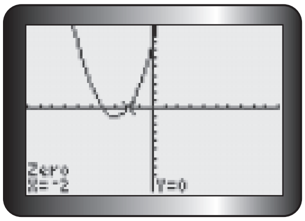


$x = -4$ and $x = -2$

or

$x^2 + 6x + 5 = -3$
$x^2 + 6x + 8 = 0$





$x = -4$ and $x = -2$

or

$x^2 + 6x + 5 = -3$
$x^2 + 6x + 8 = 0$
$(x + 4)(x + 2) = 0$
$x + 4 = 0$ or $x + 2 = 0$
$x = -4$ $x = -2$
$x = -4$ and $x = -2$

Study Aid

- See Lesson 6.3, Examples 1 to 4.
- Try Mid-Chapter Review Questions 7 to 10.

Q: How can you change a quadratic relation from standard form to vertex form?

A: To write a quadratic relation in vertex from, complete the square as shown below.

EXAMPLE

Write the equation in vertex form.
$y = 2x^2 - 9x + 2.5$

Solution

- When the coefficient of x^2 is a number other than 1, factor it from the x^2 and x terms. This will leave a binomial inside the brackets.

$y = 2x^2 - 9x + 2.5$
$y = 2(x^2 - 4.5x) + 2.5$

- To complete the square for the binomial, add and subtract the square of half the coefficient of the x term.

$y = 2(x^2 - 4.5x + 2.25^2 - 2.25^2) + 2.5$
$y = 2(x^2 - 4.5x + 5.0625 - 5.0625) + 2.5$

- Group together the three terms that form the perfect square, and factor it.

$y = 2[(x^2 - 4.5x + 5.0625) - 5.0625] + 2.5$
$y = 2[(x - 2.25)^2 - 5.0625] + 2.5$

- Use the distributive property to multiply. Then combine the constants.

$y = 2[(x - 2.25)^2 - (5.0625)] + 2.5$
$y = 2(x - 2.25)^2 - 2(5.0625) + 2.5$
$y = 2(x - 2.25)^2 - 10.125 + 2.5$
$y = 2(x - 2.25)^2 - 7.625$

PRACTICE Questions

Lesson 6.1

1. Solve each quadratic equation.

a) $x^2 + 4x = 12$
b) $x^2 + 8x + 9 = 0$
c) $x^2 - 9x = -4$
d) $-3x^2 - 2x + 3 = 0$
e) $2x^2 - 5x + 10 = 15$
f) $\frac{1}{2}x^2 + 10x - 2 = -10$

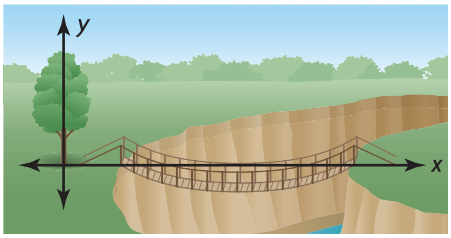
2. Determine the roots.

a) $x^2 + 6x - 16 = 0$
b) $2x^2 + x - 3 = 0$
c) $x^2 + 3x - 10 = 0$
d) $6x^2 + 7x - 5 = 0$
e) $-3x^2 - 9x + 12 = 0$
f) $\frac{1}{2}x^2 + 6x + 16 = 0$

3. Solve using any strategy.

a) $x^2 + 12x + 45 = 10$
b) $2x^2 + 7x + 5 = 9$
c) $x(6x - 1) = 12$
d) $x(x + 3) - 20 = 5(x + 3)$

4. Kari drew this sketch of a small suspension bridge over a gorge near her home.



She determined that the bridge can be modelled by the relation $y = 0.1x^2 - 1.2x + 2$. How wide is the gorge, if 1 unit on her graph represents 1 m?

5. If a ball were thrown on Mars, its height, h, in metres, might be modelled by the relation $h = -1.9t^2 + 18t + 1$, where t is the time in seconds since the ball was thrown.

a) Determine when the ball would be 20 m or higher above Mars' surface.
b) Determine when the ball would hit the surface.

Lesson 6.2

6. Determine the value of c needed to create a perfect-square trinomial.

a) $x^2 + 8x + c$
b) $x^2 - 10x + c$
c) $x^2 + 5x + c$
d) $x^2 - 7x + c$
e) $-4x^2 + 24x + c$
f) $2x^2 - 18x + c$

Lesson 6.3

7. Write each relation in vertex form by completing the square.

a) $y = x^2 + 6x - 3$
b) $y = x^2 - 4x + 5$
c) $y = 2x^2 + 16x + 30$
d) $y = -3x^2 - 18x - 17$
e) $y = 2x^2 + 10x + 8$
f) $y = -3x^2 + 9x - 2$

8. Consider the relation $y = -4x^2 + 40x - 91$.

a) Complete the square to write the equation in vertex form.
b) Determine the vertex and the equation of the axis of symmetry.
c) Graph the relation.

9. Martha bakes and sells her own organic dog treats for \$15/kg. For every \$1 price increase, she will lose sales. Her revenue, R, in dollars, can be modelled by $R = -10x^2 + 100x + 3750$, where x is the number of \$1 increases. What selling price will maximize her revenue?

10. For his costume party, Byron hung a spider from a spring that was attached to the ceiling at one end. Fern hit the spider so that it began to bounce up and down. The height of the spider above the ground, h, in centimetres, during one bounce can be modelled by $h = 10t^2 - 40t + 240$, where t seconds is the time since the spider was hit. When was the spider closest to the ground during this bounce?

6.4 The Quadratic Formula

YOU WILL NEED

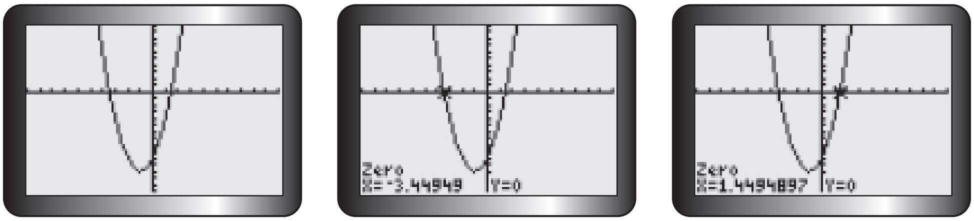
- graphing calculator

GOAL

Understand the development of the quadratic formula, and use the quadratic formula to solve quadratic equations.

LEARN ABOUT the Math

Devlin says that he cannot solve the equation $2x^2 + 4x - 10 = 0$ by factoring because his graphing calculator shows him that the zeros of the relation $y = 2x^2 + 4x - 10$ are not integers.



He wonders if there is a way to solve quadratic equations that cannot be factored over the set of integers.

? How can quadratic equations be solved without factoring or using a graph?

EXAMPLE 1 **Selecting a strategy** to solve a quadratic equation

Solve $2x^2 + 4x - 10 = 0$.

Kyle's Solution: Solving a quadratic equation using the vertex form

$$2x^2 + 4x - 10 = 0$$

Since the equation contains an x^2 term as well as an x term, I knew that I couldn't isolate x like I do for linear equations. But the vertex form of a quadratic equation does contain a single x term. I wondered whether I could isolate x if I wrote the expression on the left in this form. I decided to complete the square.

$$2(x^2 + 2x) - 10 = 0$$

$$\frac{2}{2} = 1 \text{ and } 1^2 = 1$$

$$2(x^2 + 2x + 1 - 1) - 10 = 0$$

I factored 2 from the x^2 and x terms. I divided the coefficient of x by 2. Then I squared my result to determine what I needed to add and subtract to create a perfect square within the expression on the left side.

$$2[(x^2 + 2x + 1) - 1] - 10 = 0$$
$$2[(x + 1)^2 - 1] - 10 = 0$$
$$2(x + 1)^2 - 2 - 10 = 0$$
$$2(x + 1)^2 - 12 = 0$$

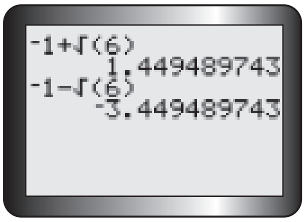
I grouped together the three terms that formed the perfect-square trinomial and then factored. Finally, I multiplied and combined the constants.

$$2(x + 1)^2 = 12$$
$$\frac{2(x + 1)^2}{2} = \frac{12}{2}$$
$$(x + 1)^2 = 6$$
$$\sqrt{(x + 1)^2} = \pm\sqrt{6}$$
$$x + 1 = \pm\sqrt{6}$$
$$x = -1 \pm \sqrt{6}$$

I isolated $(x + 1)^2$ using inverse operations. Since $(x + 1)$ is squared, I took the square roots of both sides. I remembered that there are two square roots for every number: a positive one and a negative one. Then I solved for x.

$x = -1 + \sqrt{6}$ and $x = -1 - \sqrt{6}$ are the exact solutions.

I got two answers. This makes sense because these roots are the x-intercepts of the graph of $y = 2x^2 + 4x - 10$.



I decided to compare the roots I calculated with the x-intercepts of Devlin's graph by writing these numbers as decimals. My roots and Devlin's x-intercepts were the same.

The roots of $2x^2 + 4x - 10 = 0$ are approximately $x = 1.4$ and $x = -3.4$.

Liz's Solution: Solving a quadratic equation by developing a formula

$$ax^2 + bx + c = 0$$

I thought I could solve for x if I could write the standard form of an equation in vertex form, since this form is the only one that has a single x term. I realized that I would have to work with letters instead of numbers, but I reasoned that the process would be the same.

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

I decided to complete the square, so I factored a from the x^2 and x terms.

$$\frac{b}{a} \div 2 = \frac{b}{a} \times \frac{1}{2} = \frac{b}{2a} \text{ and } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

To determine what I needed to add and subtract to create a perfect square within the expression, I divided the coefficient of x by 2. Then I squared my result.

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$
$$a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2}\right] + c = 0$$
$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c = 0$$
$$a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c = 0$$
$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

I added and subtracted $\frac{b^2}{4a^2}$ to the binomial inside the brackets. I grouped together the terms that formed the perfect-square trinomial and then factored. Finally, I multiplied by a and simplified.

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$
$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

To solve for x, I used inverse operations. I divided both sides by a. I took the square root of both sides. Since the right side represents a number, I had to determine both the positive and negative square roots.

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

I subtracted $\frac{b}{2a}$ from both sides and simplified the expression.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

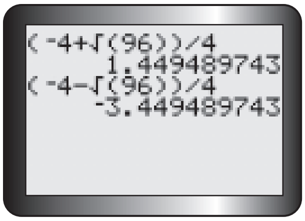
I reasoned that my formula could be used for any quadratic equation. The $\pm$ in the numerator means that there could be two solutions: one when you add the square root of $b^2 - 4ac$ to $-b$ before dividing by $2a$, and another when you subtract.

$$2x^2 + 4x - 10 = 0$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-10)}}{2(2)}$$
$$x = \frac{-4 \pm \sqrt{16 + 80}}{4}$$
$$x = \frac{-4 \pm \sqrt{96}}{4}$$

To verify my formula, I checked that the roots were the same as Devlin's x-intercepts. I substituted $a = 2$, $b = 4$, and $c = -10$ into my formula.

The roots of $2x^2 + 4x - 10 = 0$ are

$x = \frac{-4 + \sqrt{96}}{4}$ and $x = \frac{-4 - \sqrt{96}}{4}$.



I calculated my roots as decimals so that I could compare them with Devlin's x-intercepts. They were the same.

Reflecting

A. Why does it make sense that a quadratic equation may have two solutions?

B. How are Kyle's solution and Liz's solution the same? How are they different?

C. Why does it make sense that a, b, and c are part of the **quadratic formula**?

quadratic formula

a formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0$; the quadratic formula is written using the coefficients and the constant in the equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

APPLY the Math

EXAMPLE 2 **Selecting a tool** to verify the roots of a quadratic equation

Solve $5x^2 - 4x - 3 = 0$. Round your solutions to two decimal places. Verify your solutions using a graphing calculator.

Maddy's Solution

$$5x^2 - 4x - 3 = 0$$
$$a = 5,\ b = -4,\ c = -3$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

I noticed that the trinomial in this equation is not factorable over the set of integers, so I decided to use the quadratic formula. I identified the values of a, b, and c.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-3)}}{2(5)}$$

I substituted the values for a, b, and c and simplified.

$$x = \frac{4 \pm \sqrt{16 + 60}}{10}$$
$$x = \frac{4 \pm \sqrt{76}}{10}$$
$$x \doteq \frac{4 \pm 8.718}{10}$$

I calculated $\sqrt{76}$ and rounded to 3 decimal places.

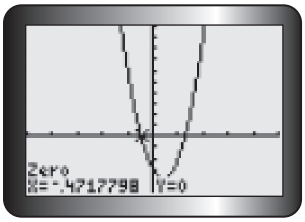
$$x \doteq \frac{4 - 8.718}{10} \text{ or } x \doteq \frac{4 + 8.718}{10}$$

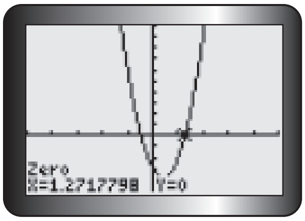
$$x = -0.4718 \qquad x = 1.2718$$

I knew that the ± in the formula meant that I would have two different solutions. I wrote the two solutions separately.

$$x \doteq -0.47 \text{ and } x \doteq 1.27$$

I rounded to two decimal places.





I entered the relation $y = 5x^2 - 4x - 3$ into my graphing calculator and used the Zero operation to verify my solutions.

The zeros of the relation agreed with the roots I had calculated using the formula.

Tech Support

For help using a TI-83/84 graphing calculator to determine the zeros of a relation, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.

EXAMPLE 3 **Reasoning** about solving quadratic equations

Solve each equation. Round your solutions to two decimal places.

a) $2x^2 - 10 = 8$ **b)** $3x(5x - 4) + 2x = x^2 - 4(x - 3)$

Graham's Solution

a) $2x^2 - 10 = 8$

I noticed that this quadratic equation did not contain an x term. I reasoned that I could solve the equation if I isolated the x^2 term.

$$2x^2 = 8 + 10$$
$$2x^2 = 18$$

I added 10 to both sides.

$$\frac{2x^2}{2} = \frac{18}{2}$$

I divided both sides by 2.

$$x^2 = 9$$
$$\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

I took the square root of both sides.

$$x = 3 \text{ and } x = -3$$

b) $3x(5x - 4) + 2x = x^2 - 4(x - 3)$

I simplified the equation by multiplying, using the distributive property.

$$15x^2 - 12x + 2x = x^2 - 4x + 12$$
$$14x^2 - 6x - 12 = 0$$

I rearranged the equation so that the right side was 0. I decided to use the quadratic formula since I didn't quickly see the factors.

$a = 14, b = -6, c = -12$

I identified the values of *a*, *b*, and *c*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(14)(-12)}}{2(14)}$$
$$x = \frac{6 \pm \sqrt{708}}{28}$$
$$x \doteq -0.74 \text{ and } x \doteq 1.16$$

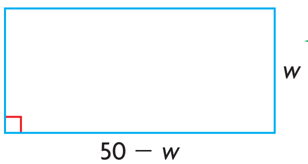
I substituted the values of *a*, *b*, and *c* into the quadratic formula and evaluated. I rounded the solutions to two decimal places.

EXAMPLE 4 Solving a problem using a quadratic model

A rectangular field is going to be completely enclosed by 100 m of fencing. Create a quadratic relation that shows how the area of the field will depend on its width. Then determine the dimensions of the field that will result in an area of 575 m^2. Round your answers to the nearest hundredth of a metre.

Bruce's Solution

Let the width of the field be w metres.



I started with a diagram to help me organize my thinking. I decided to represent the width of the field by *w*.

Since the perimeter will be 100 m, the length will have to be $\frac{100 - 2w}{2} = 50 - w$.

$$A = lw$$
$$= w(50 - w)$$
$$= 50w - w^2$$
$$575 = 50w - w^2$$

I wrote a quadratic relation for the area by multiplying the length and the width. Then I set the area equal to 575.

$$0 = -w^2 + 50w - 575$$

I rearranged the equation so that the left side was 0.

$a = -1, b = 50, c = -575$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

I decided to use the quadratic formula, so I identified the values of *a*, *b*, and *c*.

$$w = \frac{-50 \pm \sqrt{50^2 - 4(-1)(-575)}}{2(-1)}$$

I substituted these values into the formula and evaluated. I rounded the solutions to two decimal places.

$$w = \frac{-50 \pm \sqrt{2500 - 2300}}{-2}$$

$$w = \frac{-50 \pm \sqrt{200}}{-2}$$

$w \doteq 17.93$ and $w \doteq 32.07$

The field is either 17.93 m or 32.07 m wide.

$50 - 17.93 = 32.07$

If $w = 17.93$ m, the field is 32.07 m long.

$50 - 32.07 = 17.93$

If $w = 32.07$ m, the field is 17.93 m long.

I used the two possible values of w to determine the values for the length. It made sense that the roots could be either the length or the width.

In Summary

Key Idea

- The roots of a quadratic equation of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Need to Know

- The quadratic formula was developed by completing the square to solve $ax^2 + bx + c = 0$.
- The quadratic formula provides a way to calculate the roots of a quadratic equation without graphing or factoring.
- The solutions to the equation $ax^2 + bx + c = 0$ correspond to the zeros, or x-intercepts, of the relation $y = ax^2 + bx + c$.
- Quadratic equations that do not contain an x term can be solved by isolating the x^2 term.
- Quadratic equations of the form $a(x - h)^2 + k = 0$ can be solved by isolating the x term.

CHECK Your Understanding

1. State the values of a, b, and c that you would substitute into the quadratic formula to solve each equation. Rearrange the equation, if necessary.

a) $x^2 + 5x - 2 = 0$

b) $4x^2 - 3 = 0$

c) $x^2 + 6x = 0$

d) $2x(x - 5) = x^2 + 1$

2. **i)** Solve each equation by factoring.
ii) Solve each equation using the quadratic formula.
iii) State which strategy you prefer for each equation, and explain why.

a) $x^2 + 18x - 63 = 0$ **b)** $8x^2 - 10x - 3 = 0$

3. Solve each equation.

a) $2x^2 = 50$
b) $x^2 - 1 = 0$
c) $3x^2 - 2 = 10$
d) $x(x - 2) = 36 - 2x$

PRACTISING

4. Determine the roots of each equation. Round the roots to two decimal places, if necessary.

a) $(x + 1)^2 - 16 = 0$
b) $-2(x + 5)^2 + 2 = 0$
c) $-3(x - 7)^2 + 3 = 0$
d) $4(x - 2)^2 - 5 = 0$
e) $-6(x + 3)^2 + 12 = 0$
f) $0.25(x - 4)^2 - 4 = 0$

5. Solve each equation using the quadratic formula.

a) $6x^2 - x - 15 = 0$
b) $4x^2 - 20x + 25 = 0$
c) $x^2 - 16 = 0$
d) $5x^2 - 11x = 0$
e) $x^2 + 9x + 20 = 0$
f) $12x^2 - 40 = 17x$

6. Could you have solved the equations in question 5 using a different strategy? Explain.

7. C If you can solve a quadratic equation by factoring it over the set of integers, what would be true about the roots you could determine using the quadratic formula? Explain.

8. K Determine the roots of each equation. Round the roots to two decimal places.

a) $x^2 - 4x - 1 = 0$
b) $5x^2 - 6x - 2 = 0$
c) $3w^2 + 8w + 2 = 0$
d) $2x^2 - x - 3 = 0$
e) $m^2 - 5m + 3 = 0$
f) $-3x^2 + 12x - 7 = 0$

9. Solve each equation. Round your solutions to two decimal places.

a) $2x^2 - 5x = 3(x + 4)$
b) $(x + 4)^2 = 2(x + 5)$
c) $x(x + 3) = 5 - x^2$
d) $3x(x + 4) = (4x - 1)^2$
e) $(x - 2)(2x + 3) = x + 1$
f) $(x - 3)^2 + 5 = 3(x + 1)$

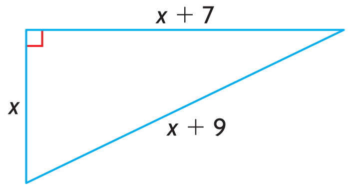
10. Solve each equation. Round your solutions to two decimal places.

a) $2x^2 + 5x - 14 = 0$
b) $3x^2 + 7.5x = 21$
c) $3x(0.4x + 1) = 8.4$
d) $0.2x^2 = -0.5x + 1.4$

11. **a)** What do you notice about your solutions in question 10?
b) How could you have predicted this before using the quadratic formula?

12. Algebraically determine the points of intersection of the parabolas
T $y = 2x^2 + 5x - 8$ and $y = -3x^2 + 8x - 1$.

13. Calculate the value of x.

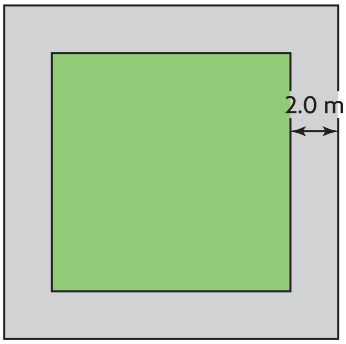


14. A trained stunt diver is diving off a platform that is 15 m high into
A a pool of water that is 45 cm deep. The height, h, in metres, of the stunt diver above the water is modelled by $h = -4.9t^2 + 1.2t + 15$, where t is the time in seconds after starting the dive.

a) How long is the stunt diver above 15 m?

b) How long is the stunt diver in the air?

15. A rectangle is 5 cm longer than it is wide. The diagonal of the rectangle is 18 cm. Determine the dimensions of the rectangle.



16. A square lawn is surrounded by a concrete walkway that is 2.0 m wide, as shown at the left. If the area of the walkway equals the area of the lawn, what are the dimensions of the lawn? Express the dimensions to the nearest tenth of a metre.

17. Use a chart like the one below to compare the advantages of solving a quadratic equation by factoring and by using the quadratic formula. Provide an example of an equation that you would solve using each strategy.

Strategy	Advantages	Example
Factoring		
Quadratic Formula		

Extending

18. Determine the points of intersection of the line $y = 2x + 5$ and the circle $x^2 + y^2 = 36$.

19. Determine a quadratic equation, in standard form, that has each pair of roots.

a) $x = -3$ and $x = 5$ **b)** $x = \frac{2 \pm \sqrt{5}}{3}$

20. Three sides of a right triangle are consecutive even numbers, when measured in centimetres. Calculate the length of each side.

6.5 Interpreting Quadratic Equation Roots

GOAL

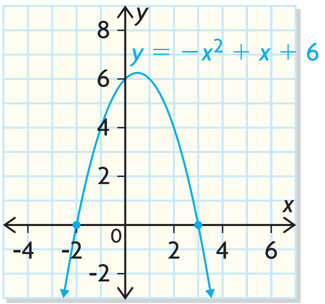
Determine the number of roots of a quadratic equation, and relate these roots to the corresponding relation.

YOU WILL NEED

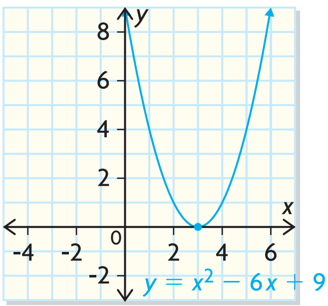
- graphing calculator
- grid paper
- ruler

INVESTIGATE the Math

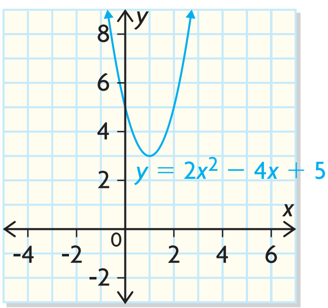
Quadratic relations may have two, one, or no *x*-intercepts.



This graph shows that the quadratic equation $-x^2 + x + 6 = 0$ has two solutions, $x = -2$ and $x = 3$.



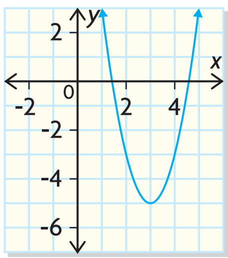
This graph shows that the quadratic equation $x^2 - 6x + 9 = 0$ has one solution, $x = 3$.



This graph shows that the quadratic equation $2x^2 - 4x + 5 = 0$ has no solutions.

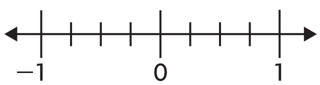
? How can you determine the number of solutions to a quadratic equation without solving it?

A. Copy and complete this table using a graphing calculator. The first line has been completed for you.

Quadratic Relation	Sketch of Graph	Quadratic Equation Used to Determine *x*-intercepts	Roots of the Equation
$y = 2x^2 - 12x + 13$		$2x^2 - 12x + 13 = 0$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(13)}}{2(2)}$ $x = \frac{12 \pm \sqrt{40}}{4}$ $x \doteq 1.42$ or $x \doteq 4.58$
$y = -2x^2 - 4x - 2$			
$y = -3x^2 + 9x + 12$			
$y = x^2 - 6x + 13$			
$y = -2x^2 - 4x - 5$			
$y = x^2 + 6x + 9$			

real numbers

the set of numbers that corresponds to each point on the number line shown; fractions, decimals, integers, and numbers like $\sqrt{2}$ are all real numbers



discriminant

the expression $b^2 - 4ac$ in the quadratic formula

B. How is the number of **real number** solutions related to the value of the **discriminant**?

Reflecting

C. Why are there no real number solutions when the discriminant is negative?

D. Why is there one real number solution when the discriminant is zero?

E. Why are there two real number solutions when the discriminant is positive?

APPLY the Math

EXAMPLE 1 Connecting the real roots to the *x*-intercepts

Without solving, determine the number of real roots of each equation and describe the graph of the corresponding relation.

a) $3x^2 + 4x + 5 = 0$

b) $-2x^2 + 7x + 1 = 0$

c) $9x^2 - 12x + 4 = 0$

Steve's Solution

a) $3x^2 + 4x + 5 = 0$

$D = b^2 - 4ac$
$= 4^2 - 4(3)(5)$
$= 16 - 60$
$= -44$

I substituted the values of *a*, *b*, and *c* into the discriminant, which I called *D*.

There are no real roots.

Since the discriminant is negative, there are no real roots.

The graph of $y = 3x^2 + 4x + 5$ has no *x*-intercepts. Since $a > 0$, the parabola opens upward and its vertex is above the *x*-axis.

b) $-2x^2 + 7x + 1 = 0$

$D = b^2 - 4ac$
$= 7^2 - 4(-2)(1)$
$= 49 + 8$
$= 57$

I substituted the values of *a*, *b*, and *c* into the discriminant.

There are two real roots.

Since the discriminant is positive, there are two real roots.

The graph of $y = -2x^2 + 7x + 1$ has two *x*-intercepts. Since $a < 0$, the parabola opens downward and its vertex is above the *x*-axis.

c) $9x^2 - 12x + 4 = 0$

$D = b^2 - 4ac$
$= (-12)^2 - 4(9)(4)$
$= 144 - 144$
$= 0$

I substituted the values of *a*, *b*, and *c* into the discriminant.

There is one real root.

Since the discriminant is zero, there is one real root.

The graph of $y = 9x^2 - 12x + 4$ has one *x*-intercept. Since $a > 0$, the parabola opens upward and its vertex is on the *x*-axis.

EXAMPLE 2 **Selecting a strategy** to determine the number of zeros

Determine the number of zeros for $y = -2x^2 + 16x - 35$.

John's Solution: Completing the square

$$y = -2x^2 + 16x - 35$$
$$y = -2(x^2 - 8x) - 35$$
$$y = -2(x^2 - 8x + 16 - 16) - 35$$
$$y = -2(x - 4)^2 + 32 - 35$$
$$y = -2(x - 4)^2 - 3$$

I completed the square to determine the vertex of the parabola. The vertex is at $(4, -3)$. Since $a < 0$, the parabola opens downward.

There are no zeros.

Since the vertex is below the x-axis and the parabola opens downward, I knew that it could never cross the x-axis.

Erin's Solution: Using the quadratic formula

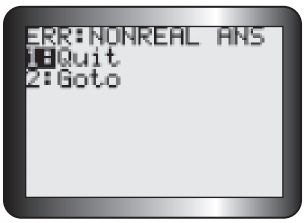
$$y = -2x^2 + 16x - 35$$
$$0 = -2x^2 + 16x - 35$$

The zeros occur when $y = 0$, so I substituted $y = 0$ into the relation. This resulted in a quadratic equation.

$$x = \frac{-16 \pm \sqrt{16^2 - 4(-2)(-35)}}{2(-2)}$$
$$x = \frac{-16 \pm \sqrt{-24}}{-4}$$

I substituted $a = -2$, $b = 16$, and $c = -35$ into the quadratic formula.

I tried to calculate the square root of -24, but my calculator displayed this error message.



There are no real number solutions to the equation, so the relation has no x-intercepts.

My calculator displayed an error message because there is no real number that can be multiplied by itself to give a negative result.

Cathy's Solution: Using the discriminant

$y = -2x^2 + 16x - 35$

$D = 16^2 - 4(-2)(-35)$

$= 256 - 280$

$= -24$

I substituted $a = -2$, $b = 16$, and $c = -35$ into the discriminant and evaluated.

The relation has no zeros.

Since the discriminant is negative, there are no zeros.

In Summary

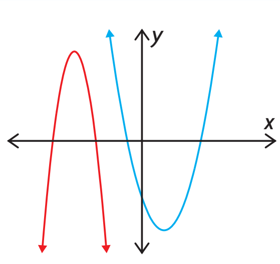
Key Idea

- You can use the quadratic formula to determine whether a quadratic equation has two, one, or no real solutions, without solving the equation.

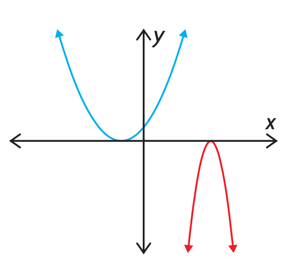
Need to Know

- The value of the expression $D = b^2 - 4ac$ gives the number of real solutions to a quadratic equation and the number of zeros in the graph of its corresponding relation.
  - If $D > 0$, there are two distinct real roots, or zeros.
  - If $D = 0$, there is one real root, or zero.
  - If $D < 0$, there are no real roots, or zeros.
- The direction of opening of a graph and the position of the vertex determines whether the graph has two, one, or no zeros and indicates whether the corresponding equation has two, one, or no real roots.

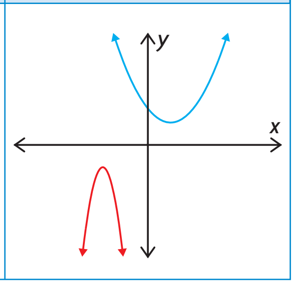
Two Real Roots



One Real Root



No Real Roots



CHECK Your Understanding

1. **a)** Determine the roots of $x^2 - 6x + 5 = 0$ by using the quadratic formula and by factoring.
   b) What do your results for part a) tell you about the graph of $y = x^2 - 6x + 5$?
   c) Verify your answer for part b) using the discriminant.

2. Determine the number of real solutions that each equation has, without solving the equation. Explain your reasoning.
   a) $(x - 1)^2 + 3 = 0$
   b) $-2(x - 5)^2 + 8 = 0$
   c) $5(x + 3)^2 = 0$

PRACTISING

3. Use the discriminant to determine the number of real solutions that each equation has.
   a) $x^2 + 3x - 5 = 0$
   b) $6x^2 + 5x + 12 = 0$
   c) $-x^2 + 8x = 12$
   d) $-2x^2 + 8x - 8 = 0$
   e) $3x^2 + 2x = 5x + 12$
   f) $-17x - 9 = 4x^2 - 5x$

4. State the number of times that each relation passes through the x-axis. Justify your answer.
   a) $y = 3(x + 2)^2 - 5$
   b) $y = -2(x + 5)^2 - 8$
   c) $y = 2(x - 7)^2$
   d) $y = 5(x - 12)^2 + 81$
   e) $y = -4.9x^2 + 5$
   f) $y = -6x^2$

5. Without graphing, determine the number of zeros that each relation has.
   K a) $y = 3(x + 2)^2 + 5$
   b) $y = -2x^2 + 3x - 7$
   c) $y = x(x - 7)$
   d) $y = 5(x + 2)(x + 2)$
   e) $y = 3x^2 + 6x - 8$
   f) $y = -6x^2 + 9$

6. A Emma sells her handmade jewellery at a local market. She has always sold her silver toe rings for \$10 each, but she is thinking about raising the price. Emma knows that her weekly revenue, r, in dollars, is modelled by $r = 250 + 5n - 2n^2$, where n is the amount that she increases the price. Is it possible for Emma to earn \$500 in revenue? Explain.

7. In some places, a suspension bridge is the only passage over a river. The height of one such bridge, h, in metres, above the riverbed can be modelled by $h = 0.005x^2 + 24$.
   a) How many zeros do you expect the relation to have? Why?
   b) If the area was flooded, how high could the water level rise before the bridge was no longer safe to use?

8. The height of a super ball, h, in metres, can be modelled by $h = -4.9t^2 + 10.78t + 1.071$, where t is the time in seconds since the ball was thrown.
   a) How many zeros do you expect this relation to have? Why?
   b) Verify your answer for part a) algebraically.
   c) How many times do you think the ball will pass through a height of 5 m? 7 m? 9 m?
   d) Verify your answers for part c) algebraically.

9. Determine whether the vertex of each parabola lies above, below, or on the *x*-axis. Explain how you know.

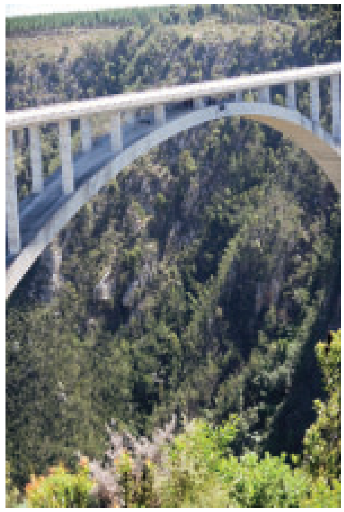
a) $h = 2t^2 - 4t + 1.5$
b) $h = 0.5t^2 - 2t + 0.5$
c) $h = 5t^2 - 30t + 45$
d) $h = 0.5t^2 - 4t + 7.75$

10. For what value(s) of k does the equation $y = 5x^2 + 6x + k$ have each number of roots?

a) two real roots **b)** one real root **c)** no real roots

11. Meg went bungee jumping from the Bloukrans River bridge in South Africa last summer. During the free fall on her first jump, her height above the water, h, in metres, was modelled by $h = -5t^2 + t + 216$, where t is the time in seconds since she jumped.

a) How high above the water is the platform from which she jumped?
b) Show that if her hair just touches the water on her first jump, the corresponding quadratic equation has two solutions. Explain what the solutions mean.



Safety *Connection*

Bungee jumping is an activity associated with a high degree of risk. This activity should only be performed under the direction of trained professionals.

12. In the relation $y = 4x^2 + 24x - 5$, for which values of y will the corresponding equation have no solutions?

13. **T** A tangent is a line that touches a circle at exactly one point. For what values of k will the line $y = x + k$ be tangent to the circle $x^2 + y^2 = 25$?

14. **C** Sasha claims that the discriminant of a quadratic relation will never be negative if the relation can be written in the form $y = a(x - r)(x - s)$. Do you agree or disagree? Explain.

15. a) Create a quadratic relation, in vertex form, that has two zeros. Then write your relation in standard form. Use the discriminant to verify that it has two zeros.
b) Create a quadratic relation, in vertex form, that has one zero. Then write your relation in standard form. Use the discriminant to verify that it has one zero.
c) Create a quadratic relation, in vertex form, that has no zeros. Then write your relation in standard form. Use the discriminant to verify that it has one zero.

Extending

16. a) Write three quadratic equations in factored form.
b) Expand and simplify your equations.
c) Determine the discriminant for each equation.
d) Explain how you can use the discriminant to determine whether an equation is factorable.

17. Determine the number of points of intersection of the relations $y = (x + 3)^2$ and $y = -2x^2 - 5$.

6.6 Solving Problems Using Quadratic Models

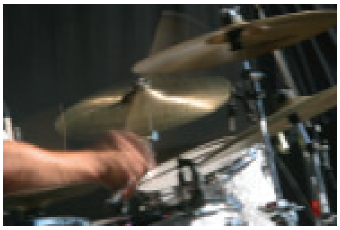
YOU WILL NEED

- grid paper
- ruler
- graphing calculator

GOAL

Solve problems that can be modelled by quadratic relations using a variety of tools and strategies.

LEARN ABOUT the Math



The volunteers at a food bank are arranging a concert to raise money. They have to pay a set fee to the musicians, plus an additional fee to the concert hall for each person attending the concert. The relation $P = -n^2 + 580n - 48\,000$ models the profit, P, in dollars, for the concert, where n is the number of tickets sold.

? How can you determine the number of tickets that must be sold to break even and to maximize the profit?

EXAMPLE 1 **Selecting a strategy** to analyze a quadratic relation

Calculate the number of tickets they must sell to break even. Determine the number of tickets they must sell to maximize the profit.

Jack's Solution: Completing the square

$$P = -n^2 + 580n - 48\,000$$
$$P = -(n^2 - 580n) - 48\,000$$
$$P = -(n^2 - 580n + 84\,100 - 84\,100) - 48\,000$$
$$P = -[(n - 290)^2 - 84\,100] - 48\,000$$
$$P = -(n - 290)^2 + 84\,100 - 48\,000$$
$$P = -(n - 290)^2 + 36\,100$$

I completed the square to write the relation in vertex form so I could determine the maximum profit first.

The volunteers must sell 290 tickets to earn a maximum profit of \$36 100 for the food bank.

Since $a < 0$, the parabola opens downward. The y-coordinate of the vertex (290, 36 100) is the maximum value.

$$0 = -(n - 290)^2 + 36\,100$$

I set $P = 0$ to calculate the break-even points.

$$(n - 290)^2 = 36\,100$$
$$\sqrt{(n - 290)^2} = \pm\sqrt{36\,100}$$
$$n - 290 = \pm 190$$
$$n = 290 + 190 \quad \text{or} \quad n = 290 - 190$$
$$n = 480 \qquad\qquad n = 100$$

I used inverse operations to solve for n.

The volunteers break even if they sell 480 tickets or 100 tickets.

Dineke's Solution: Factoring the relation

$P = -n^2 + 580n - 48\,000$
$P = -(n^2 - 580n + 48\,000)$
$P = -(n - 480)(n - 100)$

I factored the relation to determine the break-even points first.

$0 = -(n - 480)(n - 100)$
$0 = n - 480$ or $0 = n - 100$
$n = 480$ $\quad n = 100$

I knew that the break-even points would occur when the profit equalled zero, so I set $P = 0$. Then I used inverse operations to solve for n.

The volunteers break even if they sell 480 tickets or 100 tickets.

$$n = \frac{480 + 100}{2}$$
$n = 290$

Since the zeros are the same distance from the axis of symmetry, I used them to determine the equation of the axis of symmetry.

$P = -(n - 480)(n - 100)$
$P = -(290 - 480)(290 - 100)$
$P = -(-190)(190)$
$P = 36\,100$

The equation of the axis of symmetry, $n = 290$, gave the n-coordinate of the vertex. I substituted $n = 290$ into the equation to determine the P-coordinate of the vertex.

Therefore, the volunteers will earn a maximum profit of $36 100 for the food bank if they sell 290 tickets.

Reflecting

A. How are Jack's solution and Dineke's solution the same? How are they different?

B. Will both strategies always work? Why or why not?

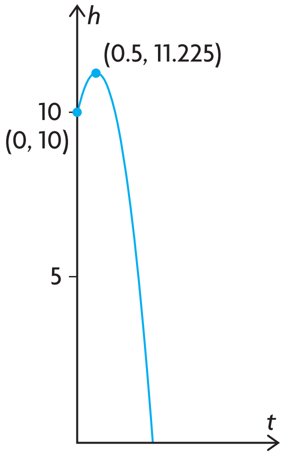
C. Whose strategy would you have used for this problem? Explain your choice.

APPLY the Math

EXAMPLE 2 Solving a problem by creating a quadratic model

Alexandre was practising his 10 m platform dive. Because of gravity, the relation between his height, h, in metres, and the time, t, in seconds, after he dives is quadratic. If Alexandre reached a maximum height of 11.225 m after 0.5 s, how long was he above the water after he dove?

Burns's Solution



I decided to start with a sketch that included the given information. The *y*-intercept is the starting position. The maximum height and time are the coordinates of the vertex, (time, maximum height).

$$
\begin{aligned}
y &= a(x - h)^2 + k \\
10 &= a(0 - 0.5)^2 + 11.225 \\
10 &= a(-0.5)^2 + 11.225 \\
10 - 11.225 &= 0.25a \\
\frac{-1.225}{0.25} &= \frac{0.25a}{0.25} \\
-4.9 &= a \\
h &= -4.9(t - 0.5)^2 + 11.225
\end{aligned}
$$

Since I knew the coordinates of the vertex, I determined a model for the height of the diver in vertex form. Substituting the vertex given for (h, k) and the initial height of the diver for a point (x, y), I used inverse operations to solve for a.

$$0 = -4.9(t - 0.5)^2 + 11.225$$

Since I was determining when Alexandre hit the water, I set $h = 0$.

$$
\begin{aligned}
\frac{-11.225}{-4.9} &= \frac{-4.9(t - 0.5)^2}{-4.9} \\
2.291 &\doteq (t - 0.5)^2 \\
\pm\sqrt{2.291} &= \sqrt{(t - 0.5)^2} \\
\pm 1.514 &\doteq t - 0.5 \\
0.5 \pm 1.514 &\doteq t \\
0.5 + 1.514 \doteq t \quad &\text{or} \quad 0.5 - 1.514 \doteq t \\
2.01 \doteq t \quad & \quad\quad -1.01 \doteq t
\end{aligned}
$$

I used inverse operations to solve for t.

Alexandre was above the water for about 2.0 s after he dove.

The answer -1.01 didn't make sense since time cannot be negative in this situation, so I didn't use it.

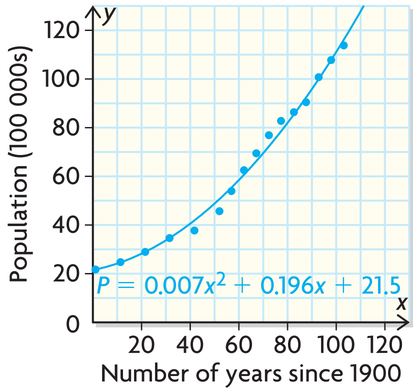
EXAMPLE 3 Reasoning to determine an appropriate solution

Statisticians use various models to make predictions about population growth. Ontario's population (in 100 000s) can be modelled by the relation $P = 0.007x^2 + 0.196x + 21.5$, where x is the number of years since 1900.

a) Using this model, what was Ontario's population in 1925?

b) When will Ontario's population reach 15 million?

Ontario's Population since 1900



Blair's Solution

a)

$x = 1925 - 1900$

$x = 25$

$P = 0.007x^2 + 0.196x + 21.5$

$P = 0.007(25)^2 + 0.196(25) + 21.5$

$P = 30.775$

Using x as the number of years since 1900, I subtracted 1900 from 1925. I substituted my result into the relation to determine the population in 1925.

The population was about 3 077 500 in 1925.

The relation gave the population in 100 000s, so I multiplied my answer by 100 000.

b)

$P = 0.007x^2 + 0.196x + 21.5$

$150 = 0.007x^2 + 0.196x + 21.5$

The population was given in 100 000s, and 15 million = 150 × 100 000. So, I used 150 for P.

$150 - 150 = 0.007x^2 + 0.196x + 21.5 - 150$

$0 = 0.007x^2 + 0.196x - 128.5$

I rearranged my equation so that I could use the quadratic formula to determine its roots.

$a = 0.007,\ b = 0.196,\ c = -128.5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-0.196 \pm \sqrt{0.196^2 - 4(0.007)(-128.5)}}{2(0.007)}$$

$$x = \frac{-0.196 \pm \sqrt{3.636}}{0.014}$$

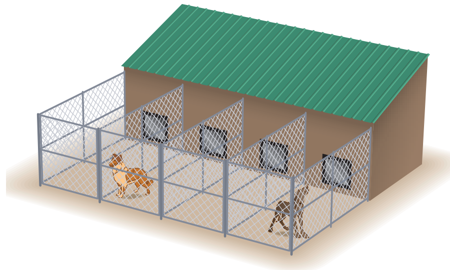
$x \doteq -150.20$ or $x \doteq 122.20$

Ontario's population will be about 15 000 000 in 2022.

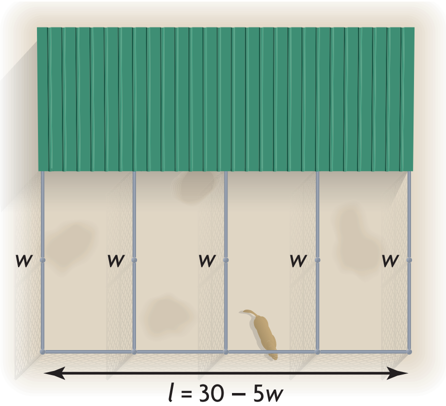
I thought $x = -150.21$ made no sense, since this would mean that the population was 15 000 000 in 1750, which I know is not reasonable. So, I used the other answer and added it to 1900.

EXAMPLE 4 **Solving a problem** by creating a quadratic model

Lila is creating dog runs for her dog kennel. She can afford 30 m of chain-link fence to surround four dog runs. The runs will be attached to a wall, as shown in the diagram. To achieve the maximum area, what dimensions should Lila use for each run and for the combined enclosure?



Mitsu's Solution



I started by sketching the dog runs. I let w represent the width of each run. I let l represent the total length of the enclosure.

To express the length in terms of the width, I subtracted the amount of fencing needed for the 5 fence sides that are perpendicular to the wall from the amount of fencing Lila can afford.

$A = l \times w$
$A = (30 - 5w) \times w$
$A = 30w - 5w^2$
$A = -5w^2 + 30w$
$A = -5(w^2 - 6w)$
$A = -5(w^2 - 6w + 9 - 9)$
$A = -5[(w - 3)^2 - 9]$
$A = -5(w - 3)^2 - (-5)9$
$A = -5(w - 3)^2 + 45$

I wrote an equation for the area and simplified it.

Since I wanted to maximize the area, I completed the square to determine the vertex.

(3, 45) is the vertex, so the maximum area is 45 m^2. It occurs when the width of each run is 3 m.

$l = 30 - 5w$
$= 30 - 5(3)$
$= 15$

I substituted the width into my expression for the length to determine the length of the combined enclosure.

The dimensions of the combined enclosure should be 15 m by 3 m, and the dimensions of each run should be 3 m by 3.75 m.

$15 \div 4 = 3.75$

In Summary

Key Idea

- When solving a problem that involves a quadratic relation, follow these suggestions:
  - Write the relation in the form that is most helpful for the given situation.
  - Use the vertex form to determine the maximum or minimum value of the relation.
  - Use the standard form or factored form to determine the value of x that corresponds to a given y-value of the relation. You may need to use the quadratic formula.

Need to Know

- A problem may have only one realistic solution, even when the quadratic equation that is used to represent the problem has two real solutions. When you solve a quadratic equation, check to make sure that your solutions make sense in the context of the situation.

CHECK Your Understanding

1. For each relation, explain what each coordinate of the vertex represents and what the zeros represent.

a) a relation that models the height, h, of a ball that has been kicked from the ground after time t

b) a relation that models the height, h, of a ball when it is a distance of x metres from where it was thrown from a second-floor balcony

c) a relation that models the profit earned, P, on an item at a given selling price, s

d) a relation that models the cost, C, to create n items using a piece of machinery

e) a relation that models the height, h, of a swing above the ground during one swing, t seconds after the swing begins to move forward

For questions 2 to 17, round all answers to two decimal places, where necessary.

2. A model rocket is shot straight up from the roof of a school. The height, h, in metres, after t seconds can be approximated by $h = 15 + 22t - 5t^2$.

a) What is the height of the school?

b) How long does it take for the rocket to pass a window that is 10 m above the ground?

c) When does the rocket hit the ground?

d) What is the maximum height of the rocket?

PRACTISING

Safety Connection

Smoke detectors provide early warning of fire or smoke. Monitored smoke detectors send a signal to a monitoring station.

3. Water from a fire hose is sprayed on a fire that is coming from a window. The window is 15 m up the side of a wall. The equation $H = -0.011x^2 + 0.99x + 1.6$ models the height of the jet of water, H, and the horizontal distance it can travel from the nozzle, x, both in metres.

a) What is the maximum height that the water can reach?

b) How far back could a firefighter stand, but still have the water reach the window?

4. Brett is jumping on a trampoline in his backyard. Each jump takes about 2 s from beginning to end. He passes his bedroom window, which is 4 m high, 0.4 s into each jump. By modelling Brett's height with a quadratic relation, determine his maximum height.

5. Pauline wants to sell stainless steel water bottles as a school fundraiser. She knows that she will maximize profits, and raise $1024, if she sells the bottles for $28 each. She also knows that she will lose $4160 if she sells the bottles for only $10 each.

a) Write a quadratic relation to model her profit, P, in dollars, if she sells the bottles for x dollars each.

b) What selling price will ensure that she breaks even?

6. **K** A professional stunt performer at a theme park dives off a tower, which is 21 m high, into water below. The performer's height, h, in metres, above the water at t seconds after starting the jump is given by $h = -4.9t^2 + 21$.

a) How long does the performer take to reach the halfway point?

b) How long does the performer take to reach the water?

c) Compare the times for parts a) and b). Explain why the time at the bottom is not twice the time at the halfway point.

7. **A** Harold wants to build five identical pigpens, side by side, on his farm using 30 m of fencing. Determine the dimensions of the enclosure that would give his pigs the largest possible area. Calculate this area.

8. A biologist predicts that the deer population, P, in a certain national park can be modelled by $P = 8x^2 - 112x + 570$, where x is the number of years since 1999.

a) According to this model, how many deer were in the park in 1999?

b) In which year was the deer population a minimum? How many deer were in the park when their population was a minimum?

c) Will the deer population ever reach zero, according to this model?

d) Would you use this model to predict the number of deer in the park in 2020? Explain.

9. The depth underwater, d, in metres, of Daisy the dolphin during a dive can be modelled by $d = 0.1t^2 - 3.5t + 6$, where t is the time in seconds after the dolphin begins her descent toward the water.
a) How long was Daisy underwater?
b) How deep did Daisy dive?

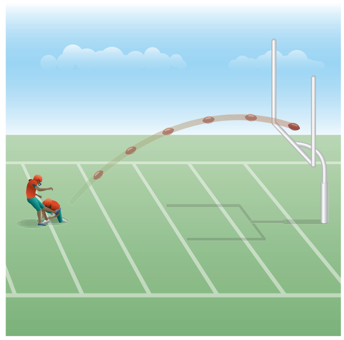
10. The cost, C, in dollars per hour, to run a machine can be modelled by $C = 0.01x^2 - 1.5x + 93.25$, where x is the number of items produced per hour.
a) How many items should be produced each hour to minimize the cost?
b) What production rate will keep the cost below $53?

11. Nick has a beautiful rectangular garden, which measures 3 m by 3 m. He wants to create a uniform border of river rocks around three sides of his garden. If he wants the area of the border and the area of his garden to be equal, how wide should the border be?

12. A ball was thrown from the top of a playground jungle gym, which is 1.5 m high. The ball reached a maximum height of 4.2 m when it was 3 m from where it was thrown. How far from the jungle gym was the ball when it hit the ground?

13. The sum of the squares of three consecutive even integers is 980.
T Determine the integers.

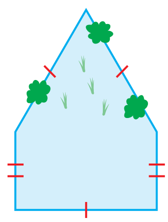
14. Maggie can kick a football 34 m, reaching a maximum height of 16 m.
a) Write an equation to model this situation.
b) To score a field goal, the ball has to pass between the vertical poles and over the horizontal bar, which is 3.3 m above the ground. How far away from the uprights can Maggie be standing so that she has a chance to score a field goal?



15. **a)** Create a problem that you could model using a quadratic relation
C and you could solve using the corresponding quadratic equation.
b) Create a problem that you could model using a quadratic relation and you could solve by determining the coordinates of the vertex.

Extending

16. Mark is designing a pentagonal-shaped play area for a daycare facility. He has 30 m of nylon mesh to enclose the play area. The triangle in the diagram is equilateral. Calculate the dimensions of the rectangle and the triangle, to the nearest tenth of a metre, that will maximize the area he can enclose for the play area.



17. Richie walked 15 m diagonally across a rectangular field. He then returned to his starting position along the outside of the field. The total distance he walked was 36 m. What are the dimensions of the field?

6 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 6.4, Example 1.
- Try Chapter Review Questions 8 to 10.

Q: How can you solve a quadratic equation that is not factorable over the set of integers, without graphing?

A: If the quadratic equation is in the form $ax^2 + bx + c = 0$, you can use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE

Solve $3x^2 - 7x - 5 = 0$. Round to two decimal places.

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a = 3,\ b = -7, \text{ and } c = -5$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-5)}}{2(3)}$$

$$x = \frac{7 \pm \sqrt{49 + 60}}{6}$$

$$x = \frac{7 \pm \sqrt{109}}{6}$$

$$x = \frac{7 + \sqrt{109}}{6} \quad \text{or} \quad x = \frac{7 - \sqrt{109}}{6}$$

$$x \doteq 2.91 \qquad\qquad x \doteq -0.57$$

Study Aid

- See Lesson 6.5, Examples 1 and 2.
- Try Chapter Review Questions 11 and 12.

Q: How can you use part of the quadratic formula to determine the number of real solutions that a quadratic equation has?

A: You can use the discriminant, $D = b^2 - 4ac$. If $D < 0$, there are no real solutions. If $D = 0$, there is one real solution. If $D > 0$, there are two real solutions.

Study Aid

- See Lesson 6.6, Examples 1 and 2.
- Try Chapter Review Questions 13 to 18.

Q: When using a quadratic model, how do you decide whether you should determine the vertex or solve the corresponding equation?

A: If you want to determine a maximum or minimum value, then you should locate the vertex of the relation. If you are given a specific value of y (any number, including 0), then you should solve the corresponding equation.

PRACTICE Questions

Lesson 6.1

1. Solve each equation.
a) $(2x - 5)(3x + 8) = 0$
b) $x^2 + 12x + 32 = 0$
c) $3x^2 - 10x - 8 = 0$
d) $3x^2 - 5x + 5 = 2x^2 + 4x - 3$
e) $2x^2 + 5x - 1 = 0$
f) $5x(x - 1) + 5 = 7 + x(1 - 2x)$

2. The safe stopping distance, in metres, for a boat that is travelling at v kilometres per hour in calm water can be modelled by the relation $d = 0.002(2v^2 + 10v + 3000)$.
a) What is the safe stopping distance if the boat is travelling at 12 km/h?
b) What is the initial speed of the boat if it takes 15 m to stop?

Lesson 6.2

3. Determine the value of c needed to create a perfect-square trinomial.
a) $x^2 + 8x + c$
b) $x^2 - 16x + c$
c) $x^2 + 19x + c$
d) $2x^2 + 12x + c$
e) $-3x^2 + 15x + c$
f) $0.1x^2 - 7x + c$

Lesson 6.3

4. Complete the square to write each quadratic relation in vertex form.
a) $y = x^2 + 8x - 2$
b) $y = x^2 - 20x + 95$
c) $y = -3x^2 + 12x - 2$
d) $y = 0.2x^2 - 0.4x + 1$
e) $y = 2x^2 + 10x - 12$
f) $y = -4.9x^2 - 19.6x + 12$

5. Consider the relation $y = -3x^2 - 12x - 2$.
a) Write the relation in vertex form by completing the square.
b) State the transformations that must be applied to $y = x^2$ to draw the graph of the relation.
c) Graph the relation.

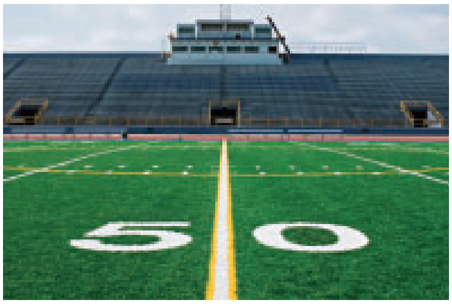
6. A basketball player makes a long pass to another player. The path of the ball can be modelled by $y = -0.2x^2 + 2.4x + 2$, where x is the horizontal distance from the player and y is the height of the ball above the court, both in metres. Determine the maximum height of the ball.

7. Cam has 46 m of fencing to enclose a meditation space on the grounds of his local hospital. He has decided that the meditation space should be rectangular, with fencing on only three sides. What dimensions will give the patients the maximum amount of meditation space?

Lesson 6.4

8. Solve each equation.
a) $3x^2 - 4x - 10 = 0$
b) $-4x^2 + 1 = -15$
c) $x^2 = 6x + 10$
d) $(x - 3)^2 - 4 = 0$
e) $(2x + 5)(3x - 2) = (x + 1)$
f) $1.5x^2 - 6.1x + 1.1 = 0$

9. The height, h, in metres, of a water balloon that is launched across a football stadium can be modelled by $h = -0.1x^2 + 2.4x + 8.1$, where x is the horizontal distance from the launching position, in metres. How far has the balloon travelled when it is 10 m above the ground?



10. A chain is hanging between two posts so that its height above the ground, h, in centimetres, can be determined by $h = 0.0025x^2 - 0.9x + 120$, where x is the horizontal distance from one post, in centimetres. How far from the post is the chain when it is 50 cm from the ground?

Lesson 6.5

11. Without solving, determine the number of solutions that each equation has.
- **a)** $2x^2 - 5x + 1 = 0$
- **b)** $-3.5x^2 - 2.1x - 1 = 0$
- **c)** $x^2 + 5x + 8 = 0$
- **d)** $4x^2 - 15 = 0$
- **e)** $5(x^2 + 2x + 5) = -2(2x - 25)$

12. Without graphing, determine the number of x-intercepts that each relation has.
- **a)** $y = (x - 4)(2x + 9)$
- **b)** $y = -1.8(x - 3)^2 + 2$
- **c)** $y = 2x^2 + 8x + 14$
- **d)** $y = 2x(x - 5) + 7$
- **e)** $y = -1.4x^2 - 4x - 5.4$

Lesson 6.6

13. Skydivers jump out of an airplane at an altitude of 3.5 km. The equation $H = 3500 - 5t^2$ models the altitude, H, in metres, of the skydivers at t seconds after jumping out of the airplane.
- **a)** How far have the skydivers fallen after 10 s?
- **b)** The skydivers open their parachutes at an altitude of 1000 m. How long did they free fall?

14. The arch of the Tyne bridge in England is modelled by $h = -0.008x^2 - 1.296x + 107.5$, where h is the height of the arch above the riverbank and x is the horizontal distance from the riverbank, both in metres. Determine the height of the arch.

15. Tickets to a school dance cost \$5, and the projected attendance is 300 people. For every \$0.50 increase in the ticket price, the dance committee projects that attendance will decrease by 20. What ticket price will generate \$1562.50 in revenue?

16. A room has dimensions of 5 m by 8 m. A rug covers $\frac{3}{4}$ of the floor and leaves a uniform strip of the floor exposed. How wide is the strip?

17. Two integers differ by 12 and the sum of their squares is 1040. Determine the integers.

18. The student council at City High School is thinking about selling T-shirts. To help them decide what to do, they conducted a school-wide survey. Students were asked, "Would you buy a school T-shirt at this price?" The results of the survey are shown.

T-Shirt Price, t (\$)	Students Who Would Buy, N	Revenue, R (\$)
4.00	923	
6.00	752	
8.00	608	
10.00	455	
12.00	287	

- **a)** Use the table to determine the revenue for each possible price.
- **b)** Draw a scatter plot relating the revenue, R, to the T-shirt price, t. Sketch a curve of good fit.
- **c)** Verify that the number of students, N, who would buy a T-shirt for t dollars can be approximated by the relation $N = 1230 - 78t$.
- **d)** Use the equation in part c) to create an algebraic expression for the revenue.
- **e)** The student council needs to bring in revenue of at least \$4750. What price range can they consider?

6 Chapter Self-Test

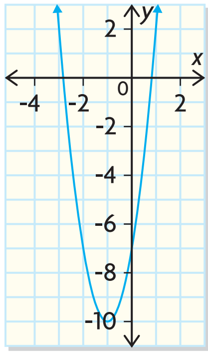
Round all answers to two decimal places where necessary.

1. Use the graph of $y = 3x^2 + 6x - 7$ at the right to estimate the solutions to each equation.

a) $3x^2 + 6x - 7 = 0$

b) $3x^2 + 6x - 7 = -7$

c) $3x^2 + 6x - 9 = 0$



$y = 3x^2 + 6x - 7$

2. Determine the roots of each equation.

a) $x^2 + 5x - 14 = 0$

b) $5x^2 - 9x + 1 = 0$

c) $2x^2 - 8 = 24$

d) $2(x - 1)^2 - 5 = 0$

3. Complete the square to determine the vertex of each parabola.

a) $y = 2x^2 + 12x - 14$

b) $y = 3x^2 - 15x - 24$

4. Can all quadratic relations be written in vertex form by completing the square? Justify your answer.

5. Without solving, determine the number of real roots that each relation has. Justify your answers.

a) $y = 2x^2 - 4x + 7$ **b)** $y = 3(x - 4)(x - 4)$ **c)** $y = (x - 3)^2$

6. April sells specialty teddy bears at various summer festivals. Her profit for a week, P, in dollars, can be modelled by $P = -0.1n^2 + 30n - 1200$, where n is the number of teddy bears she sells during the week.

a) According to this model, could April ever earn a profit of $2000 in one week? Explain.

b) How many teddy bears would she have to sell to break even?

c) How many teddy bears would she have to sell to earn $500?

d) How many teddy bears would she have to sell to maximize her profit?

7. Serge and Francine have 24 m of fencing to enclose a vegetable garden at the back of their house. Determine the dimensions of the largest rectangular garden they could enclose, using the back of their house as one of the sides of the rectangle.

8. Give two reasons why $3x^2 + 6x + 6$ cannot be a perfect square.

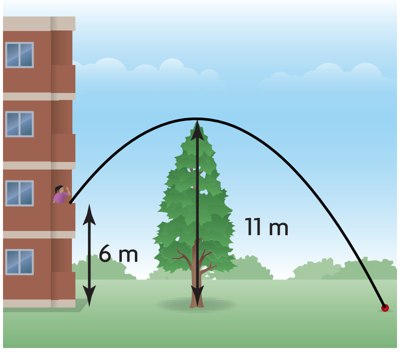
9. A rapid-transit company has 5000 passengers daily, each currently paying a $2.25 fare. For each $0.50 increase, the company estimates that it will lose 150 passengers daily. If the company must be paid at least $15 275 each day to stay in business, what minimum fare must they charge to produce this amount of revenue?

Process | *Checklist*

- ✔ Question 1: Did you compare the algebraic and graphical **representations** to help you estimate?
- ✔ Questions 4 and 5: Did you **communicate** using correct mathematical vocabulary as you justified your answers?
- ✔ Question 6: Did you make **connections** between the quadratic equation and the situation that it is modelling?

6 Chapter Task

Up and Over



On Earth, the quadratic relation $h = -5t^2 + ut + h_0$ can be used to determine the height of an object that has been thrown as it travels through the air, measured from a reference point. In this relation, h is the height of the object in metres, t is the time in seconds since the object was thrown, u is the initial velocity, and h_0 is the initial height.

Myrtle throws a ball upward from a second-floor balcony, 6 m above the ground, with an initial velocity of 2 m/s. In this situation, $u = 2$ and $h_0 = 6$, so the relation that models the height of the ball is $h = -5t^2 + 2t + 6$. Myrtle knows that changing the velocity with which she throws the ball will change the maximum height of the ball. Myrtle wants to know with what velocity she must throw the ball to make it pass over a tree that is 11 m tall.

? What initial velocity will result in a maximum height of 11 m?

A. Suppose that Myrtle just dropped the ball from the balcony, with an initial velocity of 0 m/s. Write a quadratic relation to model this situation.

B. What is the maximum height of the ball in part A?

C. Complete the square of $h = -5t^2 + 2t + 6$ to determine the maximum height of the ball when Myrtle throws the ball with an initial velocity of 2 m/s.

D. Will Myrtle have to increase or decrease the initial velocity with which she throws the ball for it to clear the tree? Explain how you know.

E. Create relations to model the height of the ball when it is thrown from a second-floor balcony with initial velocities of 4 m/s and 6 m/s. Then determine the maximum height of the ball for each relation.

F. Create a scatter plot to show the maximum heights for initial velocities of 0 m/s, 2 m/s, 4 m/s, and 6 m/s. Is this relation quadratic? Explain how you know.

G. Use quadratic regression to determine an algebraic model for your graph for part F.

H. Use the model you created for part G to determine the initial velocity necessary for the ball to clear the tree.

Task *Checklist*

- ✔ Did you show all your calculations?
- ✔ Did you draw and label your graph accurately?
- ✔ Did you answer all the questions reflectively, using complete sentences?
- ✔ Did you explain your thinking clearly?