Functions 11

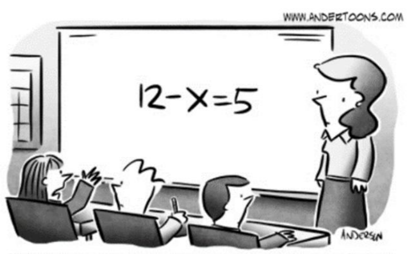
Course Notes

Chapter 2 – Polynomial & Rational Expressions

FACTORED FORM SAVES THE UNIVERSE

We will learn to

- *Determine whether algebraic expressions are equivalent*
- *Add, subtract, multiply, and factor polynomials*
- *Simplify rational expressions*
- *Add, subtract, multiply, and divide rational expressions*
- *Identify restrictions on the domain*



"This time X is 7. Last time it was 4. And the time before that it was 12. Maybe X should take some time to reflect and decide for itself what it really wants to be."

A∞Ω
MATH@TD

Chapter 2 – Polynomial & Rational Expressions

Contents with suggested problems from the Nelson Textbook. These problems are not going to be checked, but you can ask me any questions about them that you like.

Section 2.1

Read Example 3 on page 87 for another view on **equivalence**
Pg 88 – 89 #4bdf, 5 – 7, 10 – 11
For the adventurous: Pg. 90 # 15, 16 (especially 16!)

Section 2.2

Read Examples 2 and 3 on Pgs 92 – 93 **(*In Ex 2, do you prefer Fred's or Atish's solution? Why? In Ex 3 which solution do you prefer? Why? Are both entirely reliable?*)**
Pg 95 #4ace, 5, 6ace, 7 (Hint: *Lee* > *Mathias*…**WHY do I make that claim?**), 9, 11

Section 2.3

Watch the factoring videos
Problems sets will be distributed next day.

Section 2.4

Pg. 112 – 114 #1ac, 2ab, 3bc, 4acf, 5, 10, 15

Section 2.6

Pg. 121 – 123 #1ab, 3, 5, 6abc, 7ab, 8, **Try 13 for a fun challenge**

Section 2.7

Pg 128 – 130 #1bc, 3b, 5c, 6ad, 7acef, 8c, 9ac

Chapter 2 Practice Problems:

Pages 132 – 133 #7abcd, 8ace, 10cd, 12ac, 13b, 14, 15bcf

Not due.

Chapter 2 – Polynomial and Rational Expressions

2.1: Adding and Subtracting Polynomials (so easy it's ridiculous)

Learning Goal: We will learn to determine whether polynomial expressions are equivalent

Definition 2.1.1

A **SINGLE VARIABLE POLYNOMIAL** is a mathematical expression constructed by combining (through addition/subtraction) **POWER FUNCTIONS**. The polynomial is (usually) written in decreasing **order** (**degree**) of the power functions (and no "power" has more than one "term"). **MULTIVARIABLE POLYNOMIALS** also exist, but for ***FUNCTIONS 11 we will only consider polynomials with a single variable***.

Definition 2.1.2

A **POWER FUNCTION** is of the form $y = c \cdot x^n$ where "n" is the power, and "c" is a **real number** called the coefficient.

exponent

π, 2, 0, -1.5

e.g. of power Functions (with their orders):

$7x^5$ order 5 $-\pi x^3$ order 3 $-2x^0$ order 0

is not in polynomial expressions

$3x^{1.5}$ must be an positive integer

e.g.'s of Polynomial Expressions (with their orders):

single variable

$-3x^7 + 8x^5 - 2x^2 + 12$

The degree/order of the expression is 7.

multivariable

$3x^3y^2 + 7x^5y^1 + 2xy$

order 5, order 6, order 2

The order of the expression is 6.

One word that requires a closer look: **TERM**

Definition 2.1.3

In any expression a **TERM** is **CONSTRUCTED BY MULTIPLYING FACTORS TOGETHER**. Terms are separated from each other by addition and subtraction. Polynomials contain "many terms" (in fact "polynomial" literally means "many terms".) **THE TERMS OF A POLYNOMIAL EXPRESSION ARE ALL POWER FUNCTIONS. (We will look at this again!)**

Two power functions are called "**like terms**" if they have the **same order/degree/power** (but they certainly can have different coefficients!).

e.g.: $2x^3$, $7x^5$, $-3x^3$

like terms

$3x^2y^3$, $4x^4y^1$, $8y^3x^2$

like terms

When simplifying and adding/subtracting polynomial expressions we combine like terms by adding/subtracting the coefficients of the like terms.

Example 2.1.1

a) Simplify: $(4x^2 - 5x^3 - 5) + (2x^3 - 4x + 2)$

$= -3x^3 + 4x^2 - 4x - 3$

Take 3 from 9

$\rightarrow 9 - 3 = \underline{\underline{6}}$

b) Subtract $4x^2 - 3x + 1$ from $6x^2 + x$

$= (6x^2 + 1x) - (4x^2 - 3x + 1)$

$1 - -3 = 1 + 3 = 4$

$= 2x^2 + 4x - 1$

Final Note: Polynomial Expressions (or functions too) are considered **equivalent** if they contain **exactly the same terms**. So, you can tell if two expressions are equivalent just by looking at them to see if they contain the same terms…OR we can tell that two mathematical objects are equivalent if they have no difference (eg. The mathematical objects a and b are equivalent if $a - b = 0$)

Success Criteria:

- I can recognize that functions can be equivalent at only a single value but not all values ✓
- I can add and subtract polynomials by collecting like terms

$\rightarrow x^2$ Let $x = 0$

$\rightarrow 3x$

Chapter 2 – Polynomial and Rational Expressions

2.2, 2.3: Multiplying and Factoring Polynomials

↳ homework

Learning Goal: We are learning to simplify polynomials by multiplying. We are reviewing and extending our factoring skills.

2.2: *Multiplying Requires The Distributive Property*

Here we see two "new" concepts (in addition to what you saw last day):

1) **The Distributive Property** (for multiplication **over** addition)

$$a(b+c) = ab + ac$$

2) **The Associative Property** (for multiplication)

$$a(bc) = (ab)c = b(ac)$$

Note: you may also see the **Commutative Property**, but it's not so important to today's stuff

$$ab = ba$$

Example 2.2.1:

Expand and Simplify:

a) $3x(2x^2 - 3x + 2)$

$$= 6x^3 - 9x^2 + 6x$$

b) $(2x-5)(3x+1)$

$$= 6x^2 \underbrace{+ 2x - 15x}_{} - 5$$

$$= 6x^2 - 13x - 5$$

Factoring

Double Distributive

F O I L
First Outside Inside Last

c) $(5-2x)^2$

$= (5-2x)(5-2x)$

$= 25 - 10x - 10x + 4x^2$

$= 4x^2 - 20x + 25$

See Example 3, Pg. 93 about **equivalence**

d) $(x+2)\boxed{(3x-1)(x+5)}$

Associate!!

$= (x+2)(3x^2 + 15x - x - 5)$

$= (x+2)(3x^2 + 14x - 5)$

$= 3x^3 + 14x^2 - 5x + 6x^2 + 28x - 10 = \boxed{3x^3 + 20x^2 + 23x - 10}$

e) $(x-2y+3z)^2$

$= (x-2y+3z)(x-2y+3z)$

$= x^2 - 2xy + 3xz - 2xy + 4y^2 - 6yz + 3xz - 6yz + 9z^2$

$= x^2 - 4xy + 6xz + 4y^2 - 12yz + 9z^2$

2.3: *Factoring* UNDOES *The Distributive Property*

On the class video page (linked in Edsby) you will find videos demonstrating the skills necessary for factoring. Watch those videos. Next day we will have a full period to practice the skills seen in the videos.

Success Criteria:

- I can expand polynomials using the distributive property (FOIL)
- I can factor polynomials using a variety of methods (GCF, a=1, decomposition, difference of squares, perfect squares)

Chapter 2 – Polynomial and Rational Expressions

2.4: Simplifying Rational Expressions

Learning Goal: We are learning to define rational functions, and explore methods of simplifying the related rational expression.

A **RATIONAL EXPRESSION** is constructed by "**DIVIDING**" one **POLYNOMIAL EXPRESSION** by another.

e.g. $\frac{3x^2-4x+1}{x^2-1}$ ✓ is a rational expression, but $\cancel{\frac{3\sqrt{x}-4}{x^2}}$ is not.

SIMPLIFYING RATIONAL EXPRESSIONS REQUIRES 4 THINGS:

algorithm

1) Factoring any polynomials which can be factored
2) **Stating any restrictions on the variables** we cannot divide by zero!!
3) Cancelling any common factors, top to bottom
4) Writing the rational expression in simplified form

Note that stating restrictions **MUST BE DONE BEFORE CANCELLING**!!!! If you cancel before stating the restrictions, **YOU RUN THE RISK OF EXPLODING THE UNIVERSE**. Don't do it. **FOR THE SAKE OF ALL HUMANITY, PLEASE DON'T DO CANCEL BEFORE STATING RESTRICTIONS!**

RESTRICTIONS ON A RATIONAL EXPRESSION

Consider the rational expression $\frac{2x-5}{x+2}$. Because x is a variable, we can substitute different (varying) values for it and calculate different values for the rational expression.

$x=1 \rightarrow \frac{2(1)-5}{(1)+2} = \frac{-3}{3} = -1$ | $x=5 = \frac{2(5)-5}{5+2} = \frac{5}{7}$

However, there is one value which we cannot substitute:

$x=-2 \rightarrow \frac{2(-2)-5}{-2+2} = \frac{-9}{0}$ but we cannot divide by zero!! Therefore, $x=-2$ is called a restriction because it does work in the expression

$$\frac{8}{12} = \frac{2(\cancel{4})}{3(\cancel{4})} = \frac{2}{3}$$

Let a be "the" variable

Example 2.4.1

Simplify, stating any restrictions on the variable: $\dfrac{3a(\cancel{a-2b})}{(a+b)(\cancel{a-2b})}$ $= \dfrac{3a}{a+b}$

$a \neq -b, 2b$

Look in the denominator, each factor can be a restriction.

① $a + b \neq 0$ → $a \neq -b$

② $a - 2b \neq 0$ → $a \neq 2b$

Example 2.4.2

Difference of Squares

Simplify $\dfrac{\boxed{x^2-9}}{(2+x)(3-x)}$

$x = 5$: $\frac{2}{-2} = -1$

$$= \frac{(x-3)(x+3)}{(2+x)(3-x)}$$

$x \neq -2, 3$

$\frac{-2}{3} = \frac{2}{-3} = -\frac{2}{3}$

$$= \frac{\cancel{(x-3)}(x+3)}{-\cancel{(x-3)}(2+x)}$$

$$= \frac{-(x+3)}{2+x}$$

Trick: opposite factors

$3 - x$

$= -x + 3$

$= -(x-3)$

Example 2.4.3

Simplify, stating any restrictions on the variable: $\dfrac{3t^2+t-2}{9t^3-6t^2}$.

M: -6
A: $+1$
$3, -2$

$= \underbrace{3t^2+3t}_{3t}\underbrace{-2t-2}_{-2}$

$= (3t-2)(t+1)$

① Factor
② $t \neq$
③ Simplify
④ Right it out

$$= \frac{\cancel{(3t-2)}(t+1)}{3t^2\cancel{(3t-2)}}$$

$t \neq 0, \frac{2}{3}$

$$= \frac{t+1}{3t^2}$$

$3t^2 \neq 0$ → $t^2 \neq 0$ → $t \neq 0$

$3t - 2 \neq 0$ → $3t \neq 2$ → $t \neq \frac{2}{3}$

Success Criteria:

- I can simplify rational expressions by dividing out the GCF
- I can determine the restrictions from the factored form of the rational expression

Chapter 2 – Polynomial and Rational Expressions

2.6: Multiplying and Dividing Rational Expressions

Learning Goal: We are learning to multiply and divide rational expressions.

This concept extends what we learned in section **2.4: Simplifying Rational Expressions**., BUT we are adding a small (but **ridiculously fun**) twist.

Now we will consider more than one rational expression at a time, and MULTIPLY OR DIVIDE them. Since Rational Expressions are analogous to fractions, we need to remind ourselves of the rules for multiplying and dividing fractions.

MULTIPLYING FRACTIONS

The Rule is:

$$\frac{\text{Top}}{\text{Bottom}} \times \frac{\text{Top}}{\text{Bottom}} = \frac{\text{Top} \times \text{Top}}{\text{Bottom} \times \text{Bottom}}$$

ex: $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$

Remember, you need to simplify by cancelling common factor.

DIVIDING FRACTIONS

The Rule is: You can't divide fractions!

In order to divide, we need to change the question to a multiplication by multiplying by the $\underbrace{\text{reciprocal}}_{\text{flip the fraction}}$ of the second fraction.

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

When **MULTIPLYING RATIONAL EXPRESSIONS** we need to do the following (familiar) things:

1 5) Factoring any polynomials which can be factored

2 6) **Stating any restrictions on the variable(s)** → check both denominators!

3 7) Cancelling any common factors, top to bottom (even from one expression to the other)

4 8) Writing the rational expression in simplified form

When **DIVIDING RATIONAL EXPRESSIONS**, we do the same four things, **BUT** when stating the restrictions, there is one small twist **BECAUSE WE ARE NOT ALLOWED TO DIVIDE BY ZERO**. (Sorry, I shouldn't be shouting, but this is important. It would be terrible if you divided by zero, and then were taken away by the math-cops for endangering the universe).

Examples

a) $\dfrac{3(\cancel{x-2})}{\cancel{2}x} \times \dfrac{\cancel{6}\ 3}{\cancel{x-2}}$ $\qquad x \neq 0, 2$

$= \dfrac{9}{x}$

Why have I coloured the denominators?

b) $\dfrac{x^2-25}{3x^2+x-2} \times \dfrac{6x^2-13x+6}{2x^2+7x-15}$

$= \dfrac{(x-5)\cancel{(x+5)}}{\cancel{(3x-2)}(x+1)} \times \dfrac{\cancel{(2x-3)}\cancel{(3x-2)}}{\cancel{(x+5)}\cancel{(2x-3)}}$

$x \neq -5, -1, \frac{2}{3}, \frac{3}{2}$

$= \dfrac{x-5}{x+1}$

$3x^2+x-2$ $\quad$ M: -6, A: 1 $\quad 3, -2$

$3x^2+3x-2x-2$

$(3x-2)(x+1)$

$6x^2-13x+6$ $\quad$ M: 36, A: -13 $\quad -4, -9$

$6x^2-4x-9x+6$

$(2x-3)(3x-2)$

$2x^2+7x-15$ $\quad$ M: -30, A: 7 $\quad 10, -3$

$2x^2-3x+10x-15$

$(x+5)(2x-3)$

When dividing, flip and factor at the same time.

c) $\frac{(2x+1)^2}{25x^2-1} \div \frac{-4x-2}{20x+4}$

$= \frac{(2x+1)\cancel{(2x+1)}}{\cancel{(5x+1)}(5x-1)} \times \frac{\cancel{4}2\cancel{(5x+1)}}{-\cancel{2}1\cancel{(2x+1)}}$

$x \neq -\frac{1}{2}, -\frac{1}{5}, \frac{1}{5}$

$= \frac{2(2x+1)}{-(5x-1)}$

d) $\frac{9y^2-4}{4y-12} \div \frac{9y^2+12y+4}{18-6y}$

opposite factor trick

$9y^2+12y+4$

m: 36
a: 12
6, 6
Perfect Square.

$= \frac{(3y-2)\cancel{(3y+2)}}{\cancel{4}2\cancel{(y-3)}} \times \frac{-\cancel{6}3\cancel{(y-3)}}{(3y+2)\cancel{(3y+2)}}$

$6(3-y)$

$y \neq 3, -\frac{2}{3}$

$= \frac{-3(3y-2)}{2(3y+2)}$

Success Criteria:

- I can multiply rational expressions by following the appropriate steps
- I can divide rational expressions by multiplying by the reciprocal of the divisor, then following the multiplication steps
- I can determine the restrictions from the factored form of the rational expression

Chapter 2 – Polynomial and Rational Expressions

2.7: Adding and Subtracting Rational Expressions

Learning Goal: We are learning to add and subtract rational expressions.

This is it. The pinnacle of Rational Expressions. **The most difficult thing you can do with Rational Expressions is to add or subtract them**. That's right! Adding and subtracting is the most difficult thing. Thankfully, you all can handle it!!

COMMON DENOMINATOR

Getting a Common Denominator is the key to the whole scene. A "**COMMON**" denominator must contain enough **FACTORS** (***or better***: **THE CORRECT FACTORS**) to "**KEEP EVERYONE HAPPY**", so to speak. Keep in mind that we **STILL MUST STATE RESTRICTIONS** when faced with factors containing variables!!!

In order to simply Rational Expressions being added or subtracted, you must:

1) Factoring any polynomials which can be factored
2) **Stating any restrictions on the variable(s)**
3) Determine the **COMMON DENOMINATOR** by identifying all **needed** factors
4) Writing the rational expressions with the **needed** factors so that each expression has the same denominator.
5) Add/Subtract the numerators of the expressions (***which may require some expanding so that you can collect like terms***!)
6) Check to see if the simplified numerator can be factored, and if it can, factor it.
7) Write the sum/difference in simplified (factored) form

$\frac{2(5)}{3(5)} + \frac{3(3)}{5(3)}$

Example 2.7.1

Simplify, stating any restrictions.

$$\frac{3}{t^4} + \frac{1}{2t^2} - \frac{3}{5t}$$

$t \neq 0$ $\quad$ CD: $10t^4$ $\quad$ $t^2 \times \boxed{t^2} = t^4$

$$= \frac{10(3)}{10(t^4)} + \frac{5t^2(1)}{5t^2(2t^2)} - \frac{2t^3(3)}{2t^3(5t)}$$

$$= \frac{30}{10t^4} + \frac{5t^2}{10t^4} - \frac{6t^3}{10t^4} \Rightarrow \frac{30 + 5t^2 - 6t^3}{10t^4} = \frac{-6t^3 + 5t^2 + 30}{10t^4}$$

Example 2.7.2

Simplify and state any restrictions.

$$\frac{4x}{x^2+6x+8}-\frac{3x}{x^2-3x-10}$$

$x \neq -2, -4, 5$

$$= \frac{4x(x-5)}{(x+2)(x+4)(x-5)} - \frac{3x(x+4)}{(x+2)(x-5)(x+4)}$$

$$= \frac{4x^2-20x-3x^2-12x}{(x+2)(x+4)(x-5)}$$ Do NOT EXPAND

$$= \frac{x^2-32x}{(x+2)(x+4)(x-5)} = \frac{x(x-32)}{(x+2)(x+4)(x-5)}$$

Example 2.7.3

Simplify. State restrictions. BEDMAS, folks, BEDMAS!!!!!!!!!!! (Seriously…bedmas)

$$\frac{p+1}{p^2+2p-35}+\frac{p^2+p-12}{p^2-2p-24}\times\frac{p^2-4p-12}{p^2+2p-15}$$

Always Factor First!

Multiply First!!

$p \neq -7, 5, 6, -4, -5, 3$

$$= \frac{p+1}{(p+7)(p-5)} + \frac{\cancel{(p+4)}\cancel{(p-3)}}{\cancel{(p-6)}\cancel{(p+4)}} \times \frac{\cancel{(p-6)}(p+2)}{(p+5)\cancel{(p-3)}}$$

$$= \frac{(p+1)(p+5)}{(p+7)(p-5)(p+5)} + \frac{(p+2)(p+7)(p-5)}{(p+5)(p+7)(p-5)}$$

$$= \frac{p^2+6p+5+(p+2)(p^2+2p-35)}{(p+7)(p-5)(p+5)}$$

$$= \frac{p^2+6p+5+p^3+2p^2-35p+2p^2+4p-70}{(p+7)(p-5)(p+5)}$$

$$= \frac{p^3+5p^2-25p-65}{(p+7)(p-5)(p+5)}$$

Success Criteria:

- I can add and subtract expressions through determining the LCD
- I can recognize that the LCD is not always the product of all the denominators
- I can identify restrictions from the factored form of the LCD

Chapter 2 – Polynomial and Rational Expressions

Chapter 2 Review

In this chapter we have not worked with functions. Instead we flexed our "algebraic muscles" through simplification of expressions.

Polynomial Expressions

The main thing here is that we add/subtract **LIKE TERMS**

Keep in mind the distinction between **Term** and **Factor**!

Rational Expressions

Restrictions…**Restrictions**…holy cow **RESTRICTIONS**

Cancelling…**Factors** are the only things which can be **cancelled**…FACTORS

Get your **Restrictions BEFORE YOU CANCEL**

Don't forget the small twist in finding restrictions for division problems!