Functions 11

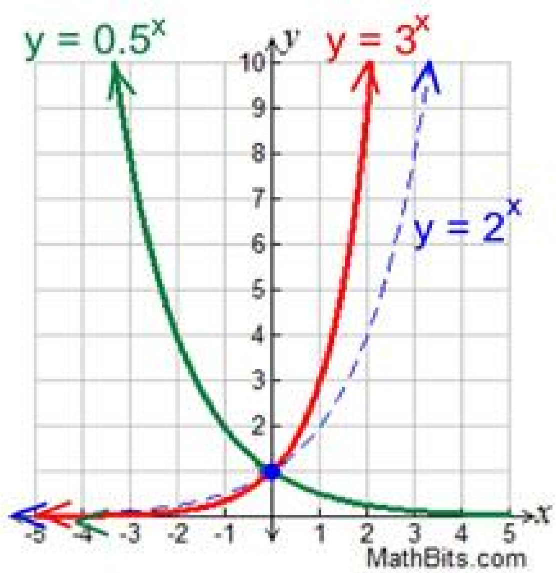
Course Notes

Unit 4 – Exponential Functions

LOCAL TITLE

We are learning to

- *understand the meaning of a zero, and learn how to find them algebraically*
- *determine the max or min value of a quadratic algebraically and graphically*
- *sketch parabolas (using transformations, zeroes, the vertex and y-intercept)*
- *solve real-world problems, including linear-quadratic systems*



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Chapter 4 – Exponential Functions

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

Section 4.2

Pg. 222 – 223 #5 – 8, 13

Section 4.3

Pg. 229: #2de, 3cdef, 4cd, 5, 6, 8, 10 (*a question of awesomeness*), 12 (*we may take up next day*)

Section 4.4

Pg. 236 – 237 #2acef, 4acdf, 5, 6, 7ac (simplify BEFORE substituting!), 9ad

Section 4.7

READ Example 2 on pages 256 – 257 (which method do you prefer: Guess and Check, or Graphing Calculator?)

READ Example 4 on pages 259 – 260.

Pg. 261 – 263 #1, 3 – 9, 11 – 16

Chapter 4 – Exponential Functions

4.2 – Integer Exponents

Learning Goal: We are learning to work with integer exponents.

Before beginning, we should quickly review (***ominous music plays***):

THE POWER LAWS

Consider a typical "power" a^n. We call "***a***" the base. We call "***n***" the exponent and the entire expression a^n is called a power

The Laws: Given the powers a^m and a^n, with exponents *m* and *n*, and the number $\frac{a}{b}$, then

1) $1^m = 1$

2) $a^1 = a$

3) $a^0 = 1$ Zero law. $(a^2 + 2z - 18\pi)^0 = 1$

4) $a^m \times a^n = a^{m+n}$ Product Law

5) $(a \cdot b)^m = a^m \cdot b^m$

6) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ ex: $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

7) $\frac{a^m}{a^n} = a^{m-n}$ Quotient Law

8) $(a^m)^n = a^{mn}$ Power Law

Until now, for the most part, the exponents you've been working with have always been **NATURAL NUMBERS**. But, we now will examine **INTEGER EXPONENTS**!!

↳ positive integers

↳ $-\infty$ to ∞

ADDITIONAL POWER LAWS:

9) $a^{-n} = \frac{1}{a^n}$ reciprocal | $\frac{1}{a^{-n}} = a^n$

ex: $2^{-2} = 0.25 = \frac{1}{4}$

↳ $= \frac{1}{2^2} =$

10) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

11) $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$

Fun: $0^m = 0$

- Zero to the power of anything is zero
- anything to the power of zero is one

$0^0 = ?$ undefined.

Example 4.2.1

Write each expression as a single power with a positive exponent:

a) $(4)^{-5}$

$= \frac{1}{4^5}$

b) $\left(\frac{3}{2}\right)^{-4}$

$= \left(\frac{2}{3}\right)^4$

c) $\frac{7^3}{7^9}$

$= 7^{3-9}$

$= 7^{-6} = \frac{1}{7^6}$

Example 4.2.2 write as a single power

Simplify, then evaluate each expression and state your answers in rational form:

↳ calculate.

a) $3^5(3^{-2})$

$= 3^{5+(-2)}$

$= 3^3$

$= 27$

b) $\left(2^{-3}(2^4)\right)^{-5}$

$= (2^1)^{-5}$ Power Law → multiply

$= 2^{-5}$

$= \frac{1}{2^5} = \frac{1}{32}$

c) $\frac{5^{-3}}{(5^2)^{-2}}$

$= \frac{5^{-3}}{5^{-4}}$

$= 5^{-3-(-4)}$

$= 5^1 = 5$

Example 4.2.3

Evaluate and express in rational form:

fraction

a) $3^2(6^{-3})$

$= \frac{3^2}{6^3}$

$= \frac{9}{216} \begin{matrix}\div 9\\ \div 9\end{matrix}$

$= \frac{1}{24}$

b) $2^{-3}+10^{-3}-3(5^{-3})$

$= \frac{1}{2^3}+\frac{1}{10^3}-\frac{3}{5^3}$

$= \frac{1\times125}{8\times125}+\frac{1}{1000}-\frac{3\times8}{125\times8}$

$= \frac{125+1-24}{1000}$

$= \frac{102}{1000} = \frac{51}{500}$

c) $13^{-5}\times\left(\frac{13^2}{13^8}\right)^{-1}$

multiply

$= 13^{-5}\times(13^{-6})^{-1}$

add

$= 13^{-5}\times13^{6}$

$= 13$

Example 4.2.4

Evaluate using the laws of exponents (the power rules):

a) $3^2\times9^{-3}\div3^{-7}$

$= 3^2\times(3^2)^{-3}\div3^{-7}$

$= 3^2\times3^{-6}\div3^{-7}$

$= 3^{2+(-6)-(-7)}$

$= 3^3$

$= 27$

b) $\frac{4^{-2}+3^{-1}}{5^{-1}+2^{-2}}$

$= \frac{\frac{1}{4^2}+\frac{1}{3^1}}{\frac{1}{5^1}+\frac{1}{2^2}}$

$= \frac{\frac{1\times3}{16\times3}+\frac{1\times16}{3\times16}}{\frac{1\times4}{5\times4}+\frac{1\times5}{4\times5}}$

$= \frac{\frac{3+16}{48}}{\frac{4+5}{20}}$

$= \frac{\frac{19}{48}}{\frac{9}{20}}$ ÷

$= \frac{19}{\cancel{48}_{12}}\times\frac{\cancel{20}^{5}}{9}$

$= \frac{95}{108}$

Success Criteria:

- I can apply the exponent laws
- I can recognize that a negative exponent represents a reciprocal expression

$(ab)^n = a^n b^m$

$a^2 b^2 = (ab)^2$

$3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^1$

$(3 \times 3)^{\frac{1}{2}} = 3^1$

$(9)^{\frac{1}{2}} = 3$

$\sqrt{9} = 3$

an exponent of $\frac{1}{2}$ is a square root.

Chapter 4 – Exponential Functions

4.3 – Rational Exponents

↳ fractions/decimals.

Learning Goal: We are learning to work with powers involving rational (fractional) exponents and to evaluate expressions containing them.

A **RATIONAL EXPONENT** can be a **FRACTION**. For example, we can consider the number $(16)^{\frac{3}{4}}$. Of course, the question we need to ask is:

What the rip is that thing??

As you know, a fraction has two parts: a numerator, and a denominator. When a fraction is used as an exponent, the two parts of the fraction carry two related (but different) meanings in terms of "powers".

Recall that 4^3 means $4 \times 4 \times 4$. Now $4^{\frac{1}{2}}$ does not mean $4 \div 4$! Your text has a nice explanation of the meaning of numbers like $4^{\frac{1}{2}}$. See (i.e. **READ** examples 1 and 2 on pages 224 and 225. For now, we will simply take the meaning of $\frac{1}{2}$ to be a square root.

Definition 4.3.1

Given a power with a "rational" (fractional) exponent $a^{\frac{m}{n}}$, the **numerator** of the exponent is a "**power**" in the usual sense, and the **denominator** represents a "**root**" or "**radical**".

e.g. For the number $16^{\frac{3}{4}}$ → 3: the power, 4: 4th root

$\left(16^{\frac{1}{4}}\right)^3$?

$2 \times 2 \times 2 \times 2 = 16$

$= (2)^3$

$= 8$

$\left(16^3\right)^{\frac{1}{4}}$

$= (4096)^{\frac{1}{4}}$

Harder....

$= 8$

$\sqrt[2]{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{\frac{1}{3}}$ cubic root

$\sqrt[4]{x} = x^{\frac{1}{4}}$ 4th root

$\sqrt[n]{x} = x^{\frac{1}{n}}$ nth root.

$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ or $\left(a^m\right)^{\frac{1}{n}}$

$= \left(\sqrt[n]{a}\right)^m$

$\sqrt[n]{a^m}$

$\sqrt[x]{\ }$ $9 \sqrt[x]{\ } 512$

Example 4.3.1

From your text: Pg. 229 #2.

Write in exponent form, and then evaluate:

a) $\sqrt[9]{512}$

$= 512^{\frac{1}{9}}$

$= 2$

c) $\sqrt[3]{27^2}$

$= 27^{\frac{2}{3}}$

$= \left(27^{\frac{1}{3}}\right)^2$

$= (3)^2$

$= 9$

b) $\sqrt[3]{-27}$

$= (-27)^{\frac{1}{3}}$

$= -3$

Note: We ***CANNOT*** take an **Even Root** of a **negative radicand**.
We ***CAN*** take an **Odd Root** of a negative radicand, however.

f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$ n $= m$

flip the base

$= \left(\frac{16}{81}\right)^{\frac{-1}{4}}$

$= \left(\frac{81}{16}\right)^{\frac{1}{4}} = \frac{81^{\frac{1}{4}}}{16^{\frac{1}{4}}} = \frac{3}{2}$

Example 4.3.2

From your text: Pg. 229 #3

Write as a single power:

a) $\left(8^{\frac{2}{3}}\right)\left(8^{\frac{1}{3}}\right)$

$= 8^{\frac{2}{3}+\frac{1}{3}}$

$= 8^{1} = 8$

b) $8^{\frac{2}{3}} \div 8^{\frac{1}{3}}$

$= 8^{\frac{2}{3}-\frac{1}{3}}$

$= 8^{\frac{1}{3}}$

$= 2$

Example 4.3.3

From your text: Pg. 229 #4

Write as a single power, then **evaluate**. Express answers in **rational form**.

a) $\sqrt{5}\cdot\sqrt{5}$

$= (5)^{\frac{1}{2}}\cdot(5)^{\frac{1}{2}}$ $\frac{1}{2}+\frac{1}{2}$

$= 5$

b) $\dfrac{\sqrt[3]{-16}}{\sqrt[3]{2}}$

$= \dfrac{(-16)^{\frac{1}{3}}}{(2)^{\frac{1}{3}}}$

$= \left(\dfrac{-16}{2}\right)^{\frac{1}{3}}$

$= (-8)^{\frac{1}{3}}$

$= -2.$

Success Criteria:

- I can understand that the numerator of a fractional exponent is the power, while the denominator is the root.

Chapter 4 – Exponential Functions

4.4 – Simplifying Expressions Involving Exponents

Learning Goal: We are learning to simplify algebraic expressions involving powers and radicals.

Keep the **EXPONENT RULES** in your mind at all times.

One of the Keys of the exponent rules is "**SAMENESS**".

- When you have the **SAME BASE**, (but possibly different exponents) you can combine powers.

e.g. $\frac{x^3 \times x^4}{x^7} = \frac{x^7}{x^7} = x^0 = 1$

- When you have the **SAME EXPONENT** (but possibly different bases) you can "combine the bases under the same exponent".

e.g. $\frac{\sqrt[3]{12} \times \sqrt[3]{36}}{\sqrt[3]{16}} = \frac{(12)^{1/3}(36)^{1/3}}{(16)^{1/3}}$

$= \left(\frac{\cancel{12}^{3} \times \cancel{36}^{9}}{\cancel{16}_{4}}\right)^{1/3}$

$= 27^{1/3}$

$= 3$

Now we turn to problems involving both numbers and variables being exponentized (not a word, but it should be because of how awesome it sounds).

Example 4.4.1

Simplify, leaving you answer with only positive exponents:

a) $\left(x^{3}\right)^{2}\left(x^{-8}\right)$

First

$= \left(x^{6}\right)\left(x^{-8}\right)$

$= x^{-2}$

$= \dfrac{1}{x^{2}}$ or $\left(\dfrac{1}{x}\right)^{2}$

b) $\dfrac{z^{4}\left(z^{-2}\right)}{z^{-3}}$

$= \dfrac{z^{2}}{z^{-3}}$

$= z^{2-(-3)}$

$= z^{5}$

c) $\sqrt{36x^{-6}}$

$(ab)^{m}$

$a^{m}b^{m}$

$= \left(36x^{-6}\right)^{\frac{1}{2}}$

Evaluate

$= \left(36\right)^{\frac{1}{2}}\left(x^{-6}\right)^{\frac{1}{2}}$

multiplying

$= 6x^{-3}$

$= \dfrac{6}{x^{3}}$

d) $\left(\dfrac{\left(3x^{2}y\right)^{-1}\left(x^{3}y^{-4}\right)}{\left(x^{3}y^{2}\right)^{-2}}\right)^{-2}$ Leave it for last.

reciprocal

$= \left(\dfrac{\left(3^{-1}x^{-2}y^{-1}\right)\left(x^{3}y^{-4}\right)}{x^{-6}y^{-4}}\right)^{-2}$

$= \left(\dfrac{x^{1}y^{-5}}{3^{1}x^{-6}y^{-4}}\right)^{-2}$

$1-(-6)=7$

$-5-(-4)=-1$

$= \left(\dfrac{x^{7}y^{-1}}{3}\right)^{-2}$

$= \dfrac{x^{-14}y^{2}}{3^{-2}} = \dfrac{9y^{2}}{x^{14}}$

e) $\left(\dfrac{\sqrt{16a^6}}{(a^3)^{-1}}\right)^{\frac{3}{2}}$

$= \left(\dfrac{(16a^6)^{\frac{1}{2}}}{a^{-3}}\right)^{3/2}$

$= \left(\dfrac{4a^3}{a^{-3}}\right)^{3/2}$ $3-(-3)$

$= (4a^6)^{3/2}$

power $\sqrt{\ }$

$= (4)^{3/2}(a^6)^{3/2}$ $4^{\wedge}1.5$ $4^{\wedge}(3/2)$

$= 8a^9$

f) $\left(\dfrac{(6x^3)^2(6y^3)}{(9xy)^6}\right)^{-\frac{1}{3}}$ negative

multiply

$= \left(\dfrac{(36x^6)(6y^3)}{531441x^6y^6}\right)^{-\frac{1}{3}}$

9^6

$\dfrac{216}{531441} \div 9$

$\dfrac{24}{59049} \div 3$

$\dfrac{8}{19683}$

$= \left(\dfrac{216x^6y^3}{531441x^6y^6}\right)^{-\frac{1}{3}}$

$= \left(\dfrac{8y^{-3}}{19683}\right)^{-1/3}$ flip the fraction

$= \left(\dfrac{19683}{8y^{-3}}\right)^{1/3}$

$= \dfrac{27}{2y^{-1}}$

$= \dfrac{27y}{2}$

Success Criteria:

- I can simplify algebraic expressions containing powers by using the exponent laws
- I can simplify algebraic expressions involving radicals

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Chapter 4 – Exponential Functions

4.7 – Applications of Exponential Functions

Learning Goal: We are learning to use exponential functions to solve problems involving exponential growth and decay.

Anything in the real world which grows, or decays can be "**MODELED**" (or in some sense "***DESCRIBED***") with words, or pictures or mathematics. **Mathematical models** are useful for getting solutions to problems, and making predictions.

So far in Mathematics 11U we have studied the basics of functions in general (chapter 1), we've done some algebra (chapter 2), and we've examined Quadratic functions (chapter 3). Part of our study of Quadratics was learning how to use the vertex of a parabola to answer questions about maxima and minima for some real word problems. For example we saw a question where we tried to maximize revenue for a school store. Quadratic **MODELS** are very useful for solving max/min problems.

In this lesson we want to work on **LEARNING HOW TO SOLVE PROBLEMS DEALING WITH GROWTH AND DECAY.** We have to decide what type of function will best model (or describe) the type of growth/decay seen in the problem (hint: for this lesson we'll be examining Exponential Growth and Decay, and therefore we expect that exponential functions will be used…shocking, I know)

Q. What is Exponential Growth or Decay?

Consider the following:

A single cell divides into two "daughter" cells. Both daughter cells divide resulting in four cells. Those four cells each divide and we now have a population of 8 cells

Describe, using mathematics, how the cell population changes from generation to generation.

The cells are doubling each generation

so $g(x) = 2^x$

Doubling model.

Example 4.7.1

Being a financial wizard, you deposit \$1,000 into an account which pays 3.5% interest, annually. ↳ 0.035

a) Determine who much money is in your account after t =1,2,3, and 4 years.

b) Determine a mathematical model which can describe how the value of the account is changing from year to year.

Year 0

\$1000

Year 1

\$1000 + interest

$\$1000 + (0.035)(1000)$

$1000(1 + 0.035)$

$1000(1.035)^1$

$= \$1035$

Year 2

\$1035 + interest

$= 1035 + (0.035)(1035)$

$= 1035(1 + 0.035)$

$= \boxed{1035}(1.035)$

$= \underline{1000(1.035)(1.035)}$

$= 1000(1.035)^2$

$= \$1,071.23$

Year 3

\$1071.23 + interest

$= 1071.23 + (0.035)(1071.23)$

$= \underbrace{1071.23}(1.035)$

$= \underline{1000(1.035)^2(1.035)}$

$= \underbrace{1000}_{\text{What you started with}}(\underbrace{1.035}_{\%\text{ growth}})^{3}$

3 → Year

$$A(t) = 1000(1.035)^t$$

Definition 4.7.1

A function describing Exponential Growth is of the form:

$$\underbrace{A(x)}_{\text{Final \#}} = \underbrace{A_0}_{\text{A naught (initial value)}}(1 + r)^{\boxed{x}}$$

x → time/periods

r → growth rate, $r > 0$

A function describing Exponential Decay is of the form:

$$A(x) = A_0(1 - r)^x$$

r → $0 < r < 1$

between 0 and 1.

Example 4.7.2

From your text, Pg. 263

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.

a) the percent of colour left if blue jeans lose $\boxed{1\%}$ $= 0.01 = r$ of their colour every time they are washed → decay

b) the population if a town had $\boxed{2500}$ residents in 1990 and grew at a rate of 0.5% each year after that for *t* years → $r = 0.005$

a) Start @ 100%.

$C(w) = 100(1-0.01)^w$

$C(w) = 100(0.99)^w$

b) $P(t) = 2500(1+0.005)^t$

$P(t) = 2500(1.005)^t$

Example 4.7.3

A new car depreciates (loses value) at a rate of 20% (0.2) per year. Steve bought a new car for \$26,000 (initial).

a) Write the equation that models this scenario.

$A(x) = 26000(1-0.2)^x = 26000(0.8)^x$

b) How much will Steve's car be worth in 3 years (x)?

$A(3) = 26000(0.8)^3$

$= \$13,312$

c) When will Steve's car be worth $\boxed{\$4000}$? $= A(x)$, $x = ?$

$4000 = 26000(0.8)^x$

$x = 6,\quad A(6) = 6,815$

$x = 8,\quad A(8) = 4,362$

$x = 9,\quad A(9) = 3,489$

∴ Between years 8 and 9, Steve's car will be worth \$4000.

Additional Applications – *DOUBLING AND HALF-LIFE*

Thus far, we have only seen examples with single period rates: "yearly" "monthly" "daily"

Unfortunately, it's not always that simple…Our rates could be…

Every 3 years $\frac{x}{3}$ Every 6 hours $\frac{x}{6}$ Every 4 days $\frac{x}{4}$

How do we deal with the exponent in these cases?

The exponent changes. The special rate becomes the denominator of the exponent.

Example (Doubling)

base is 2 100% + 100%

A species of bacteria has a population of $\boxed{300}$ (Po) at 9 am. It doubles every 3 hours.

a) Write the function that models the growth of the population, P, at any hour, t

$$P(x) = 300(2)^{x/3}$$

b) How many will there be at 6 pm?

9am to 6pm = 9 hours

$$P(9) = 300(2)^{9/3}$$

$$P(9) = 2400$$

c) How many will there be at 11 pm?

9am to 11pm = 14 hours

$$P(14) = 300(2)^{14/3}$$

$$= 7,619$$

d) Determine the time at which the population first exceeds 3000.

$9 \Rightarrow 2400$

$10 \Rightarrow 3023$

It took 10 hours, or 7pm.

Example (Half-Life) $\to$ base $= \frac{1}{2}$

A 200g sample of radioactive material has a half-life of 138 days. How much will be left in 5 years?

$$A(x) = 200\left(\frac{1}{2}\right)^{\frac{x}{138}}$$

Units must match!!

$$5 \text{ years} = 5(365) = 1825 \text{ days}$$

$$A(1825) = 200\left(\frac{1}{2}\right)^{\boxed{\frac{1825}{138}}} \qquad \frac{1825}{138} = 13.22$$

$$A(1825) = 0.02 \text{ grams}$$

Success Criteria:

- I can differentiate between exponential growth and exponential decay
- I can use the exponential function $f(x) = ab^x$ to model and solve problems involving exponential growth and decay
  - Growth rate is $b = 1 + r$. Decay rate is $b = 1 - r$.
  - r is a DECIMAL, not a percent!!!!!