Functions 11

Course Notes

Unit 6 – Sinusoidal Functions

We will learn how to

- *identify situations that can be modelled by sinusoidal or periodic functions*
- *interpret the graphs of sinusoidal or periodic functions*
- *graph sinusoidal functions with transformations*
- *determine the equations of sinusoidal functions from real-world situations*

Chapter 6 – Sinusoidal Functions

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

Section 6.1

Pg. 352 – 355 #4, 5, 7 – 10

Section 6.5

Pg. 383 – 3385 #1, 2, 4 – 7, 9

Section 6.6

Pg. 391 – 393 #4b, 5bcd, 6acd, 7, 11

Section 6.7

Pg. 398 – 401 #4 – 6, 8, 10 (*a question of **beauty***)

Chapter 6 – Sinusoidal Functions

6.1 – Properties of Periodic Functions

Learning Goal: We are learning to interpret and describe graphs that repeat at regular intervals.

Definition 6.1.1

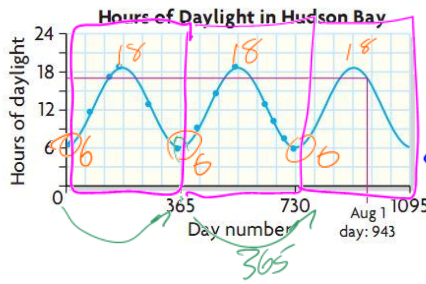
y or f(x)

A **PERIODIC FUNCTION** is one in which ***the functional values repeat***. in the same amount of "time"/x-values

e.g. Consider the following pictures: Determine which are periodic.

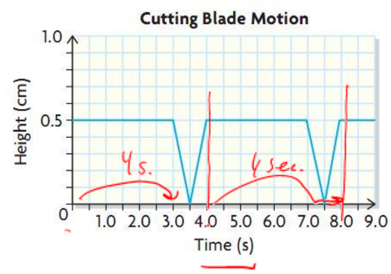
a)

This is periodic



also sinusoidal

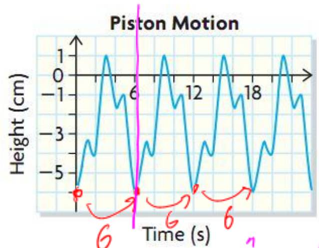
b)



Periodic

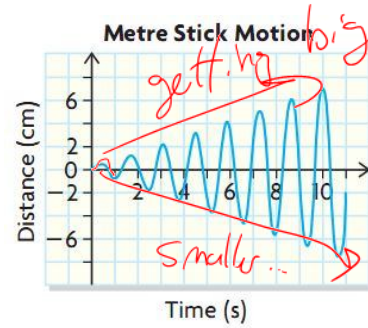
c)

Periodic



each "part" needs to be identical

d)



not repeating.

not periodic

Definition 6.1.2

The **Period** of a periodic function is the amount of the **domain values** where **one cycle** takes place.

- one pattern.
- the repeating shape

Example 6.1.1

Determine the periods of the above periodic functions:

a) 365 days b) 4 seconds c) 6 seconds

OR: amp = middle − min.

Definition 6.1.3

a) The **Amplitude** of a periodic function is half of the distance between a maximum value and a minimum value.

$$\text{Amplitude} = \frac{\max - \min}{2}$$

Distance from the middle to the top

amp = max − middle

b) The **Central Axis** is half way between the maximum value and the minimum value.

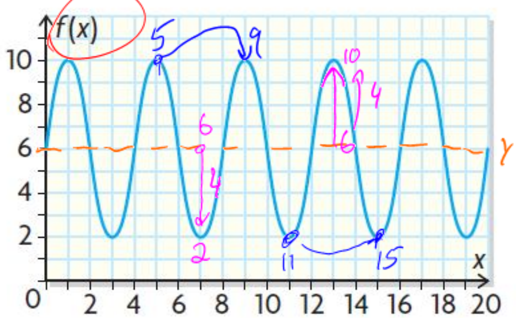
The equation of The Central Axis is given by $y = \dfrac{\max + \min}{2}$.

horizontal line

Average of max and min

Example 6.1.1

Determine the range, period, equation of the central axis, and amplitude of the function shown.



Max: 10

Min: 2

Range: $\{ f(x) \in \mathbb{R} \mid 2 \le f(x) \le 10 \}$

f(x) belongs to the Real #'s, such that f(x) is inbetween 2 and 10.

Central Axis

$y = \dfrac{10+2}{2} = \dfrac{12}{2} = 6$

Amp:

amp $= \dfrac{10-2}{2} = 4$

amp: $10 - 6 = 4$

Period = 4

Example 6.1.2

3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.

a) What is the period of one complete cycle?

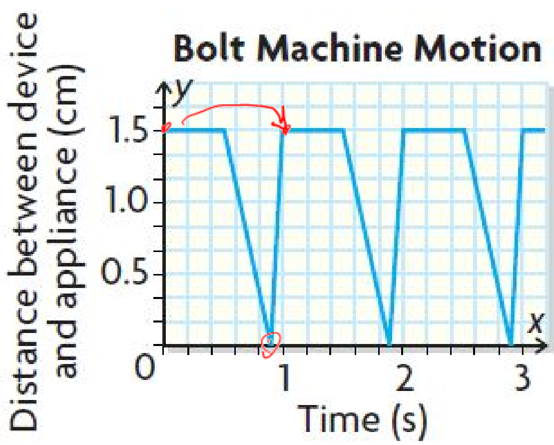
b) What is the maximum distance between the device and the appliance?

c) What is the range of this function?

d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.

e) Determine the equation of the axis.

f) Determine the amplitude.



a) Period is one second/cycle.

b) 1.5 cm = max

c) $\{y \in \mathbb{R} \mid 0 \le y \le 1.5\}$

d) $\{x \in \mathbb{R} \mid 0 \le x \le 5\}$

one period is 1 sec.

five periods is 5 seconds

e) $y = \frac{1.5+0}{2} = 0.75$

f) $amp = \frac{1.5-0}{2} = 0.75$ cm

Success Criteria:

- I can find the range, period, central axis, and amplitude of a periodic function
- I can determine IF a function is periodic

Chapter 6 – Sinusoidal Functions

6.5 – Sketching Sinusoidal Functions

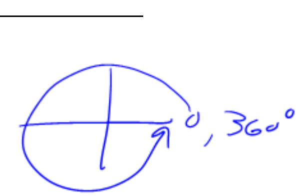
Learning Goal: We are learning to sketch the graphs of sinusoidal functions using transformations.

By this point in your illustrious High School careers, you have a solid understanding of Transformations of Functions in general. In terms of the sinusoidal functions Sine and Cosine in particular, the concepts are as you expect, but the transformations have specific meanings relating to nature of the sinusoidal "wave".

General Form of the Sine and Cosine Functions

$$f(\theta) = a\sin\left(k(\theta - d)\right) + c \qquad g(\theta) = a\cos\left(k(\theta - d)\right) + c$$

Transformation	Properties
$a =$ Vertical stretch if $a < 0$, flips affects y's	$a = \frac{\max - \min}{2}$ $a =$ amplitude
$k =$ horizontal stretch factor $\frac{1}{k} =$ horizontal stretch affects x's	Period $= \frac{360^\circ}{K}$ $K = \frac{360}{\text{period}}$
$d =$ horizontal shift or the phase shift x's	Note: To determine d you MUST Factor "K" away. this just means how far from the y-axis does the function/graph "start"
$c =$ vertical shift y's	$c = \frac{\max + \min}{2}$ $c =$ central axis.



Example 6.5.1

Determine the amplitude, period, phase shift and the equation of the central axis for:

a) $f(\theta) = \overset{a}{2}\sin\left(\theta + 60^\circ\right) + 1$ $k=1$

amp = 2

period = $\frac{360}{1} = 360°$

phase shift = $-60°$

central axis: $y = 1$

b) $g(\theta) = \overset{a}{3}\cos\left(2\theta - 90^\circ\right)$

$2(\theta - 45)$

amp = 3

period = $\frac{360}{2} = 180°$

phase shift = 45°

central axis: $y = 0$

The Graphs of Sin and Cos

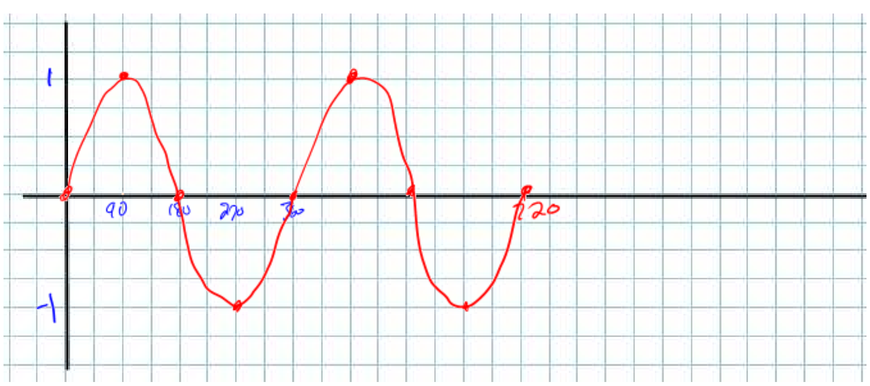
Using the special angles of 0º, 90º, 180º, 270º, and 360º, graph the following functions:

$f(\theta) = \sin(\theta) \quad 0^\circ \le \theta \le 720^\circ$

2 periods / 2 cycles

x	y
0	0
90	1
180	0
270	−1
360	0

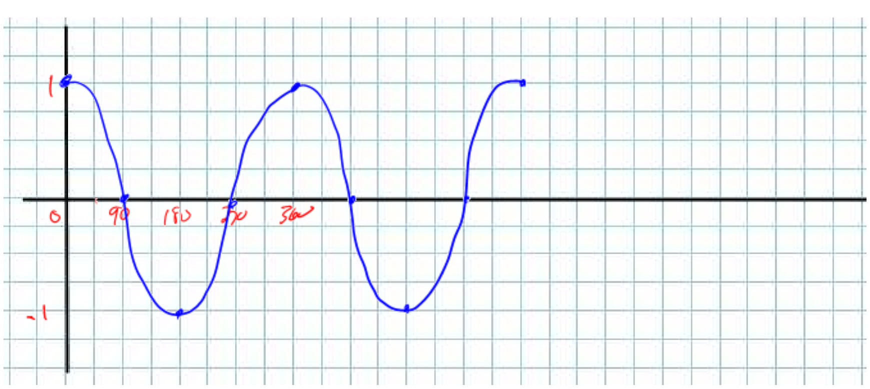
starts at the central axis



$g(\theta) = \cos(\theta) \quad 0^\circ \le \theta \le 720^\circ$

x	y
0°	1
90	0
180	−1
270	0
360	1

starts at the max



Example 6.5.2

Sketch $f(x)=\sin(x)$ and $g(x)=\sin(2x)$ for $0^\circ \le x \le 360^\circ$ on the same set of axes.

k

for $g(x)$, period $= \frac{360}{k} = \frac{360}{2} = 180$

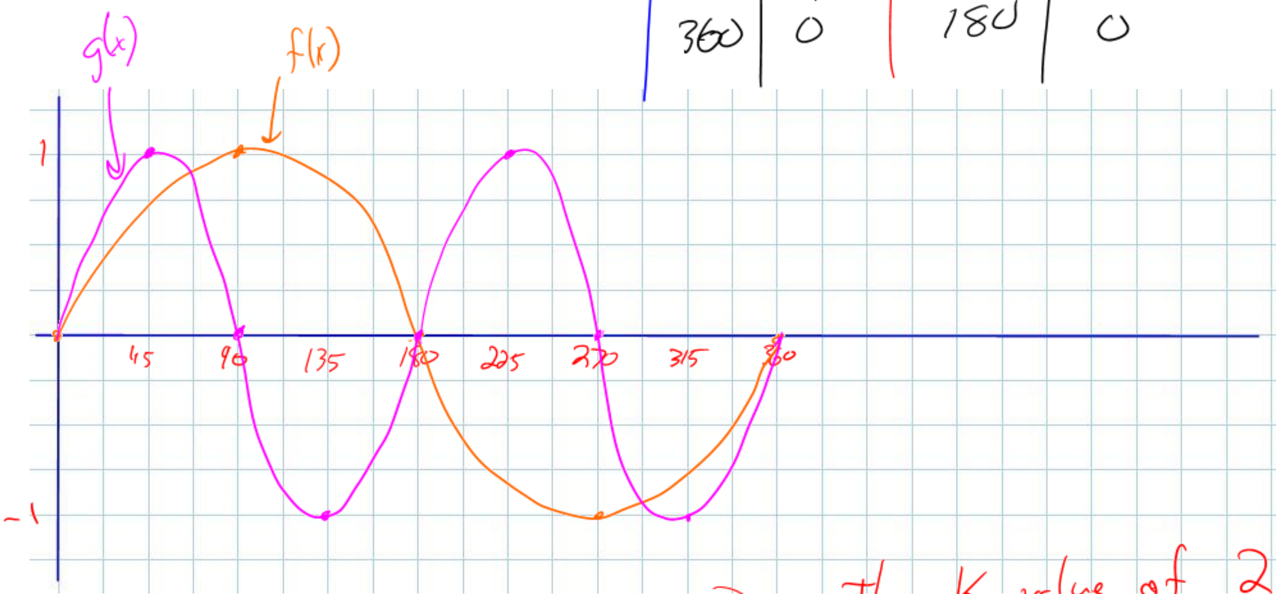
Also, Horizontal Stretch of $\frac{1}{2}$

$f(x)$

x	y
0	0
90	1
180	0
270	-1
360	0

$g(x)$

$\frac{1}{2}x$	y
0	0
45	1
90	0
135	-1
180	0



The K value of 2 compresses the graph in half.

Now…what would $\sin\left(\frac{1}{2}x\right)$ look like?

period $= \frac{360}{\frac{1}{2}} = 720$. The period doubles

Notes about Domain and Range: Consider the function $f(x) = \overset{a}{-2}\cos(\overset{k}{3}x+90^\circ)\overset{c}{+3}$.

Determine all the transformations for this function. Without graphing, determine the range of the function. Determine the domain of the function for: 1 cycle; 2 cycles; 3 cycles.

need: amp, central axis, max, and min.

amp: $|a| = |-2| = 2$

central axis: $y = 3$

$\therefore$ Range $= \{f(x) \in \mathbb{R} \mid 1 \le f(x) \le 5\}$

5
3
1
$\uparrow 2$
$\downarrow 2$

max $= 5$, min $= 1$

Domain of sinusoidal functions is $\{x \in \mathbb{R}\}$

A cycle = one period

Period $= \frac{360}{3} = 120^\circ$

one cycle: $\{x \in \mathbb{R} \mid 0^\circ \le x \le 120^\circ\}$

two cycles: $\{x \in \mathbb{R} \mid 0^\circ \le x \le 240^\circ\}$

three cycles: $\{x \in \mathbb{R} \mid 0^\circ \le x \le 360^\circ\}$

Example 6.5.3

Sketch $f(\theta) = -2\cos(\theta - 60^\circ) + 1$ on $0^\circ \le \theta \le 360^\circ$. State transformations, create tables, and state domain and range of the function.

a k d c

Vertical Stretch: −2

Horizontal Stretch: 1

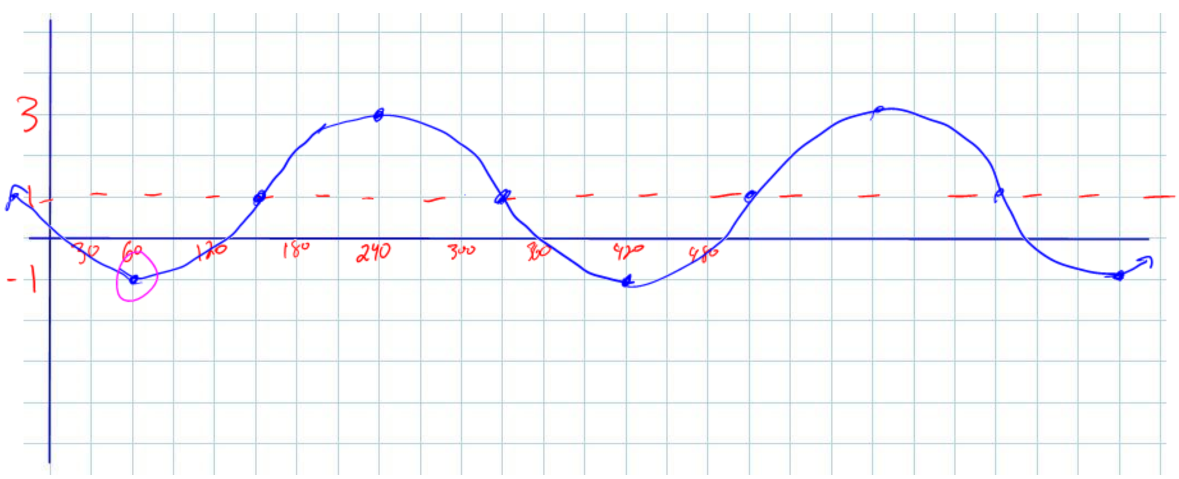
Vertical Shift: +1 (up 1)

Horizontal Shift: +60° (right 60°)

x	y
0	1
90	0
180	−1
270	0
360	1

1x + 60	−2y + 1
60	−1 min (−2(1) + 1)
150	1 middle
240	3 max
330	1
420	−1

graph this



$f(\theta) = \overset{a}{3}\sin\left(\overset{k}{2}\left(\theta - \overset{d}{45}\right)\right) \overset{c}{-1}$

Example 6.5.4

Sketch $f(\theta) = 3\sin(2\theta - 90) - 1$. State transformations, create tables, and state domain and range of the function.

V. Stretch is 3 → amp

H. Stretch is $\frac{1}{2}$ $\left(\frac{1}{k}\right)$

Period is $\frac{360}{2} = 180$

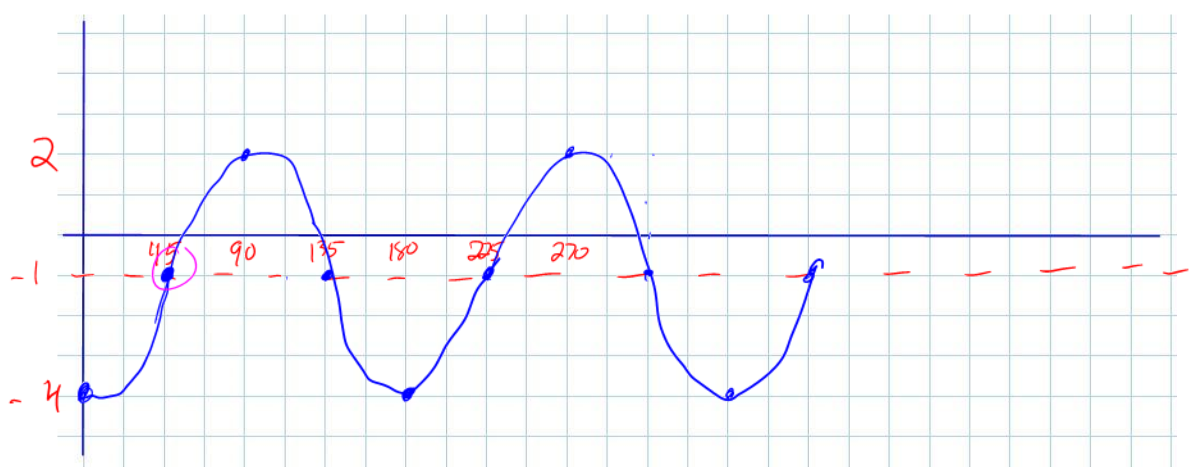
H. Shift is 45°

V. Shift of −1 → middle

$\sin\theta$

x	y
0	0
90	1
180	0
270	−1
360	0

$\frac{x}{2} + 45$	$3y - 1$	
45	−1	
90	2	max
135	−1	
180	−4	min
225	−1	



Success Criteria

- I can sketch the graph of a sinusoidal function by applying the transformations to the parent function.

Chapter 6 – Sinusoidal Functions

6.6 – Models of Sinusoidal Functions

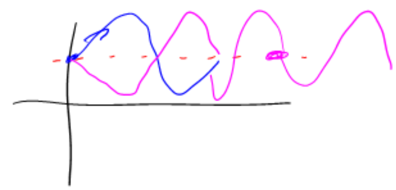
↳ equation/function

Learning Goal: We are learning to create a sinusoidal function from a graph or table of values.

In this section we will look at how to develop a sinusoidal function which can explain given information. In essence we will be writing sine or cosine functions based on given transformations.
are the same.

Just as a reminder:

General Form of the Sine and Cosine Functions



$$f(\theta) = a\sin(k(\theta - d)) + c \qquad g(\theta) = a\cos(k(\theta - d)) + c$$

Another reminder (about the pattern of sinusoidal functions):

Sine functions "start" at the central axis and go up to a **max** if "*a*" is **positive**, or down to a **min** if "*a*" is **negative**.

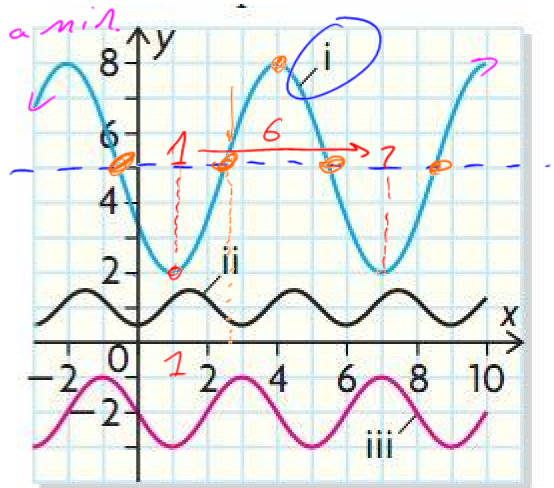
Cosine functions "start" at a **max** if "*a*" is **positive**, or at a **min** if "*a*" is **negative**.

↳ phase shift, any phase shift which represents either a max, middle, or a min.

Example 6.6.1

From your text: Pg. 391 #4a

Determine a sinusoidal equation for each function:



3 Universal Things

1. Amplitude $\left(\frac{max - min}{2}\right)$ $\quad \frac{8-2}{2} = 3 = \boxed{a}$

2. Central Axis $\left(\frac{max + min}{2}\right)$ $\quad \frac{8+2}{2} = 5 = \boxed{c}$

3. Period $= 6$ $\quad K = \frac{360}{period} = \frac{360}{6} = 60$

Now we need a "starting point": Phase Shift.

Max at $x = 4$ $\therefore d = 4$ and we use $+$ cosine $\quad f(x) = 3\cos(60(x-4)) + 5$

Middle at $x = 2.5$ ↑ $\therefore d = 2.5$, $+$sine $\quad f(x) = 3\sin(60(x-2.5)) + 5$

Min at $x = 1$ $\therefore d = 1$, $-$cosine $\quad f(x) = -3\cos(60(x-1)) + 5$

Example 6.6.2

From your text: Pg. 392 #5a)

5. For each table of data, determine the equation of the function that is the simplest model. = phase shift closest to $0°$

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1
	max	middle	min	mid	max	mid	min

d → − cosine

− sine

max = 3

min = 1

$\text{Amp} = \frac{3-1}{2} = \boxed{1 = a}$

$\text{Central Axis} = \frac{3+1}{2} = \boxed{2 = c}$

Period = 0 to 120

$= 120°$

$k = \frac{360}{120}$

$\boxed{k = 3}$

max at $0°$ ∴ $d = 0$, + cosine

$y = \overset{a}{1}\cos(\overset{k}{3}(x - \overset{d}{0})) + \overset{c}{2}$

$y = \cos(3x) + 2$

Example 6.6.3

From your text: Pg. 392 #6b)

6. Determine the equation of the cosine function whose graph has each of the following features.

	Amplitude	Period	Equation of the Axis	Horizontal ~~Translation~~ Shift
a)	3 = a	360°	$y = 11$ = c	0° = d
b)	4 = a	180°	$y = 15$ = c	30° = d

a) $K = \frac{360}{\text{period}}$

$K = \frac{360}{360} = 1$

∴ $f(x) = 3\cos(x) + 11$

b) $K = \frac{360}{180} = 2$

∴ $f(x) = 4\cos(2(x - 30)) + 15$

Example 6.6.4
A sinusoidal function has an amplitude of 4 units, a period of 120º, and a maximum at (0,9). Determine the equation of the function.

① amp = $\boxed{4 = a}$

② Period = 120

$k = \frac{360}{120}$

$\boxed{k = 3}$

③ Max of 9.

$\downarrow$ 4/amplitude

middle = 5 = c

④ Max at x = 0

∴ d = 0, + cosine

$\therefore f(x) = 4\cos(3x) + 5$

Success Criteria:

- I can create an sinusoidal function based on information from a graph or table
- I can recognize when it is best to use a sine or cosine function

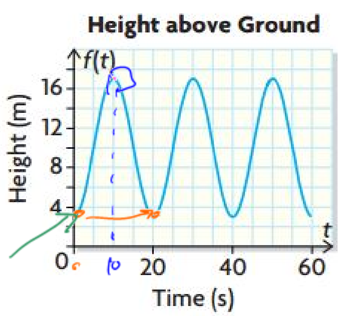
Chapter 6 – Sinusoidal Functions

6.7 – Problem Solving with Sinusoidal Functions

Learning Goal: We are learning to solve problems related to real-world applications of sinusoidal functions.

We can **use the sinusoidal properties** of **Period**, **Central Axis**, **Amplitude** and **Phase Shift** to describe and solve "real world" problems.

Example 6.7.1 (*From the text: Pg. 398 #2*)



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?

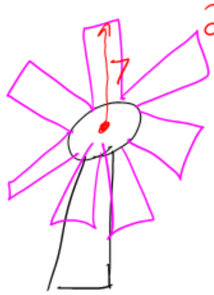
a) Max = 17, min = 3

Central Axis = $\frac{17+3}{2} = 10$

$y = 10 = c$

This represents the centre of the windmill

b) Amp: $\frac{17-3}{2} = 7 = a$



The length of the blade.

c) Period = 20 seconds $\Rightarrow k = \frac{360}{20} = 18$

The time it takes for Don to make one full rotation

d) D: $\{x \in \mathbb{R} \mid 0 \le x \le 140\}$

R: $\{h \in \mathbb{R} \mid 3 \le h \le 17\}$

e) $a = 7$, $k = 18$, $c = 10$,

Don is at the min at $x = 0$

$\therefore d = 0$, $-$ cosine

$f(t) = -7\cos(18t) + 10$

f) the period would change, and thus the K would also change.

Example 6.7.2

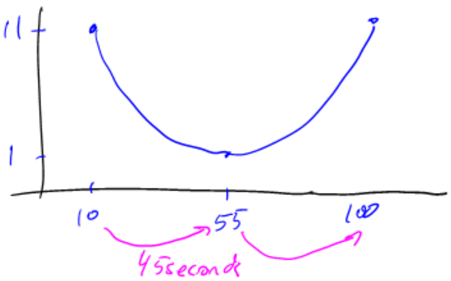
A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches a maximum height of 11m at 10s and then reaches a minimum height of 1m at 55s. How high is John after 2 minutes?

Max of 11m.

Min of 1m.

Central Axis: $\frac{11+1}{2} = 6 = c$

Amp: $\frac{11-1}{2} = 5 = a$



The period is 90 seconds

$k = \frac{360}{90} = 4$

You have a max at $x = 10$

$\therefore d = 10$, + cosine

$$h(t) = 5\cos(4(t-10)) + 6$$

time in seconds

2 minutes = 120 seconds

$$h(120) = 5\cos(4(120-10)) + 6$$
$$= 5\cos(440) + 6$$
$$= 6.9m$$

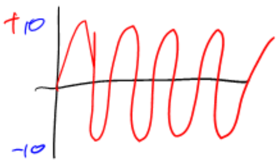
$\therefore$ John is 6.9m high after 2 minutes

Example 6.7.3 (Text pg. 396)

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right (+10 cm) and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At t = 0, the pole was momentarily at its resting position. Then it started moving to the right.

$d=0$ central axis / middle.

a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.



Max = 10

Min = -10

Central Axis = 0 = c

Amp = 10 = a

240 cycles or periods in one minute. ↙ 60 seconds

Need: $\frac{\text{time}}{\text{cycle}} = \frac{60 \text{ seconds} \div 240}{240 \text{ cycles} \div 240} = \frac{0.25 \text{ seconds}}{1 \text{ cycle}}$

$k = \frac{360}{0.25} = 1440$

t=0 is a middle, ∴ d=0, + sine going up.

$y = 10\sin(1440x)$

b) How does the situation affect the domain and range?

Domain is restricted to positive values since it is time.

Range is still the min → max

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Only the amplitude changes.

$10 \times 80\% = 8$

$\therefore y = 8\sin(1440x)$

Success Criteria:

- I can create a sinusoidal function that represents information from a real-life scenario
- I can use the function to solve further problems