Functions 11

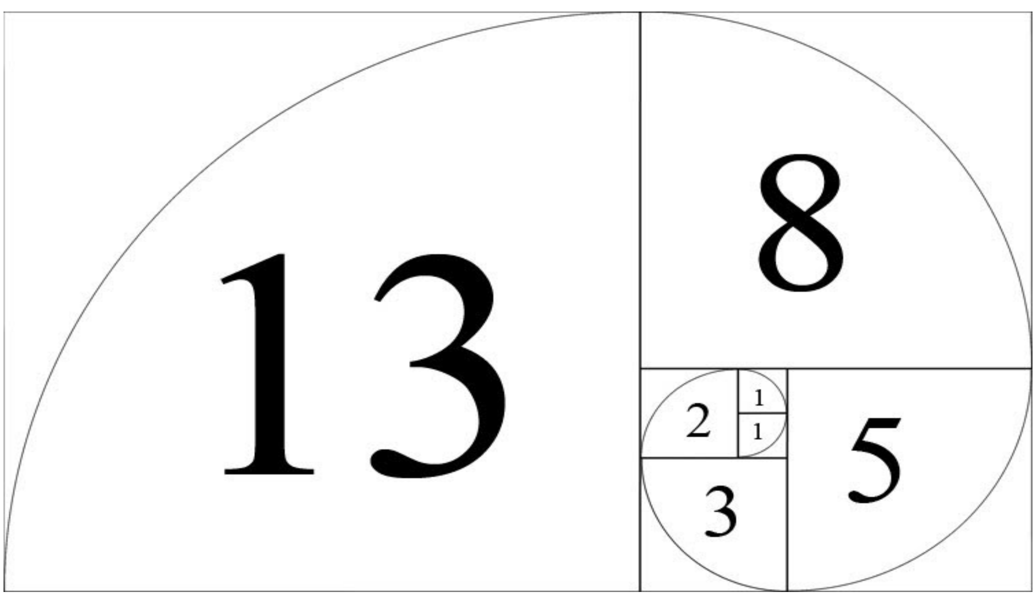
Course Notes

Unit 7 – Sequences and Series

AS EASY AS 1,1,2,3…

We will learn

- *about the nature of a sequence and how to represent a sequence using recursive and general formulas*
- *about the Fibonacci Sequence and/or Pascal's Triangle*
- *about Arithmetic and Geometric Sequences and Series and how to use them in problem solving*



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Chapter 7 – Sequences & Series

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

7.1: Pg. 424 – 425 #2 – 5 (general terms only), 8i) iii), 9bcd, 10, 11, 13, 15

7.2: Pg. 430 – 432 #2bc, 4 , 5, 6i) iii), 7, 9bcd, 10, 11, 13,
15 (interesting question…make sure you understand the text answer…ask for help if you need it), 16

7.3: Pg. 452 – 453 #1bc, 2 , 4ace, 5abe, 6, 7abcf, 11, 13, 15

7.6: Pg. 459 – 461 #1abc, 3abde , 5 – 7 (what would the common ratio be?), 11

Chapter 7 – Sequences and Series (Discrete Functions)

↳ finite

7.1 – Arithmetic Sequences

Learning Goal: We are learning to recognize the characteristics of arithmetic sequences, and express the general terms in a variety of ways.

Definition 7.1.1

A **mathematical sequence** is a list of numbers, usually with some kind of order. Each number is called a "term" of the sequence.

e.g. $3 \xrightarrow{+4} 7 \xrightarrow{+4} 11, 15, \ldots t_n$

term 1   term 2

General Term, or, term in the n^{th} position.

Definition 7.1.2

An **Arithmetic Sequence** is a sequence where each term differs from the previous term by a common difference d.

add or subtract.

e.g. $3,\ 3+4,\ 3+2(4), \ldots$

(7) 11

$d = 4$

Definition 7.1.3

The general **term** of an arithmetic sequence, usually labelled t_n, is given by a formula. The subscript n gives the position of the term in the sequence, with one exception: the first term of a sequence, t_1, is called a

e.g. In the sequence $4, -3, -10, -17, -24, -31, \ldots$

-7 -7

$a = t_1 = 4$

$t_2 = -3$

$t_5 = -24$

$d = -7$

$t_n = ?$

$-3 - 4 = -7$

$-10 - (-3) = -7$

$-31 - (-24) = -7$

The General Term of an Arithmetic Sequence

Formula

The formula can be arrived at using some simple logic. Consider some arithmetic sequence with first term a and common difference d. Then the sequence can be written:

$t_1 \quad t_{\boxed{2}}^{n} \quad t_{\boxed{3}}^{n} \quad t_{\boxed{4}}^{n} \quad \ldots \quad t_n$

$a, \quad a+d, \quad a+2d, \quad a+3d$

$n-1$

$$\boxed{t_n = a + (n-1)d}$$

general term formula

Example 7.1.1

From your text: Pg. 424 # 1

Determine which sequences are arithmetic. For those that are, state the common difference.

a) 1, 5, 9, 13, 17, … ✓ $d=4$ ($+4$, $+4$, $+4$)

b) 3, 7, 13, 17, 23, 27, … ✗ ($+4$, $+6$, $+4$, $+6$)

c) 3, 6, 12, 24, … ✗ ($+3$, $+6$, $+12$)

d) 59, 48, 37, 26, 15, … ✓ $d=-11$ (-11, -11, -11, -11)

Example 7.1.2

From your text: Pg 424 #6

Determine ~~the recursive formula and~~ the general term for the arithmetic sequence in which

a) the first term is 19 and consecutive terms increase by 8

b) $t_1 = 4$ and consecutive terms decrease by 5

c) the first term is 21 and the second term is 26

d) $t_4 = 35$ and consecutive terms decrease by 12

$t_n = a + (n-1)d$

a) $t_1 = 19 = a$

$d = 8$

$t_n = 19 + (n-1)(8)$

$t_n = 19 + 8n - 8$

$\underbrace{t_n = 8n + 11}_{\text{linear equation}}$

d) $\underbrace{71}_{t_1}$ $\underbrace{59}_{t_2}$ $\underbrace{47}_{t_3}$ $\underbrace{35}_{t_4}$ $d = -12$

a

$+12$ $+12$ -12

$t_n = 71 + (n-1)(-12)$

$t_n = 71 - 12n + 12$

$t_n = -12n + 83$

Example 7.1.3

From your text: Pg. 425 #9a

i) Determine whether each general term defines an arithmetic sequence.

ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_n = 8 - 2n$

ii) $t_1 = 8 - 2(1)$ $t_2 = 8 - 2(2)$

$t_1 = 6$ $t_2 = 4$

$6, 4, 2, 0, -2$

$d = -2$

Example 7.1.4

From your text: Pg 424 #13b

Determine the number of terms in the arithmetic sequence

$-20, -25, -30, \ldots, -205$

a d=-5 the end, the LAST term

which term # is -205.

$t_n = a + (n-1)d$

$t_n = -205$

$-205 = -20 + (n-1)(-5)$ (+20, +20)

$\frac{-185}{-5} = \frac{(n-1)(-5)}{-5} \Rightarrow 37 = n - 1$

$38 = n$

∴ there are 38 terms in this sequence.

Example 7.1.5

Given an arithmetic sequence with $t_7 = 25$ and $t_{20} = 77$ determine the general term for the sequence, and also determine t_{150}.

$t_7 = 25 = a + 6d$

$t_{20} = 77 = a + 19d$

Subtract!

$-52 = -13d \Rightarrow d = 4$

$25 = a + 6(4)$

$25 = a + 24$

$1 = a$

$\therefore t_n = 1 + (n-1)(4)$

$t_n = 4n - 3$

$t_{150} = 4(150) - 3$

$= 597$

Success Criteria:

- I can identify when a sequence is arithmetic, by seeing if it has a common difference
- I can use the General Term Formula to develop an equation for an arithmetic sequence
- I can use the General Term to find any term in a sequence OR to find out how many terms are in a sequence
- I can recognize that an arithmetic sequence is always a linear function

Chapter 7 – Sequences and Series (Discrete Functions)

7.2 – Geometric Sequences

Learning Goal: We are learning the characteristics of geometric sequences and how to express the general terms in a variety of ways.

In the last lesson we considered sequences of the form:

$$3, 7, 11, 15, 19, 23, \ldots$$

This sequence is arithmetic because there is a common **difference** between successive terms. We can write the general term of the above sequence because we know the first term $(a = 3)$, and the common difference $(d = 4)$.

$$t_n = 3 + (n-1)(4)$$

Note that if we "simplify" the general term, we can actually consider that simplification as a **function** of *n*!!

i.e. $f(n) = 4n - 1$

Consider the following sequence:

$$3, 6, \underset{\times 2}{\overset{\times 2}{12}}, 24, 48, \ldots \qquad \frac{48}{24} = 2 \qquad \frac{12}{6} = 2$$

This sequence is not arithmetic, but there is a discernible pattern as we look at moving from one term to the **next** term. In this case, we see that each new term is generated by multiplying the previous term by 2.

Such a sequence is called a **Geometric Sequence**. There isn't a common difference between two successive terms, but there is a **common ratio** (r) between two successive terms.

Comparing Two Successive Terms

Arithmetic

$$t_n - t_{n-1} = d$$

n previous

Geometric

$$\frac{t_n}{t_{n-1}} = r$$

previous

The General Term of a Geometric Sequence

Again, using simple logic will allow us to arrive at a formula (or even a function depending on how you interpret things). Consider some geometric sequence with first term a and common ratio r. The sequence can be written:

$t_1 \quad t_2 \quad t_3 \quad t_4 \; \ldots$

$a \quad a(r) \quad a(r)(r) \quad a(r)(r)(r) \quad n-1$

$ar^2 \quad ar^3$

$$t_n = ar^{n-1}$$

Example 7.2.1

From your text: Pg. 430 #1

Determine which sequences are geometric. For those that are, state the common ratio.

a) 15, 26, 37, 48, ... No!

b) 5, 15, 45, 135, ...

c) 3, 9, 81, 6561, ... NO

d) 6000, 3000, 1500, 750, 375, ...

$\frac{15}{5} = \boxed{3} = r$

$\frac{45}{15} = 3$

$\div 2 \quad \div 2 \quad \div 2 \quad \div 2$

$r = \frac{3000}{6000} = \frac{1}{2}$

Example 7.2.2

Determine the general term and t_{10} of the geometric sequence

$81, 27, 9, 3, \ldots$

$a = 81$

$r = \frac{3}{9} = \frac{1}{3}$

$t_n = 81\left(\frac{1}{3}\right)^{n-1}$

$t_{10} = 81\left(\frac{1}{3}\right)^{\cancel{10-1}\;9}$

$t_{10} = 81\left(\frac{1}{3^9}\right) = 81\left(\frac{1}{19683}\right) = \frac{1}{243}$

$= 0.0041$

Example 7.2.3

From your text: Pg. 430 #8

Determine the ~~recursive formula~~ and the general term for the geometric sequence in which

a) the first term is 19 (a) and the common ratio is 5 (r)

b) $t_1 = -9 = a$ and $r = -4$

c) the first term is 144 (a) and the second term is 36

d) $t_1 = 900$ and $r = \dfrac{1}{6}$

$$t_n = ar^{n-1}$$

a) $t_n = 19(5)^{n-1}$

b) $t_n = -9(-4)^{n-1}$

c) $r = \dfrac{t_2}{t_1} = \dfrac{36}{144} = \dfrac{1}{4}$

$t_n = 144\left(\dfrac{1}{4}\right)^{n-1}$

Example 7.2.4

Given a geometric sequence with $t_6 = -486$ and $t_9 = 13122$, determine the general term and the first 4 terms of the sequence.

$\underset{t_6}{-486} \xrightarrow{\times r} \underset{t_7}{\_\_\_} \xrightarrow{\times r} \underset{t_8}{\_\_\_} \xrightarrow{\times r} \underset{t_9}{13122}$

$\dfrac{-486r^3}{-486} = \dfrac{13122}{-486}$

$r^3 = -27$

$r = -3$

$\sqrt[3]{-27}$ or $(-27)^{1/3}$

$\boxed{\begin{aligned} t_9 &= 13122 = ar^8 \\ t_6 &\div\; -486 = ar^5 \end{aligned}}$

$-27 = r^3$

$-3 = r$ ✓

$-486 = a(-3)^5$

$-486 = a(-243)$

$2 = a$

$\therefore t_n = 2(-3)^{n-1}$

2, -6, 18, -54

Success Criteria:

- I can identify when a sequence is geometric, by seeing if it has a common ratio
- I can use the General Term Formula to develop an equation for an geometric sequence
- I can use the General Term to find any term in a sequence OR to find out how many terms are in a sequence
- I can recognize that an geometric sequence is always an exponential function

Chapter 7 – Sequences and Series (Discrete Functions)

7.5 – Arithmetic Series

Learning Goal: We are learning to calculate the sums of the terms of an arithmetic sequence.

We have been studying sequences (which are ordered lists of numbers). We examined two types of sequences: Arithmetic and Geometric. We now turn our attention to a concept very closely related to sequences – Series.

Definition 7.5.1

A **Series** is constructed by **adding together the terms of a sequence**.

So an Arithmetic Series arises when we add together the terms of an Arithmetic Sequence.

Example 7.5.1

Given the 8 term arithmetic sequence $3, 7, 11, 15, 19, 23, 27, 31$ determine the associated series. Determine the **PARTIAL SUM** S_4.

A partial sum occurs when you add up ***PART* of a series**

Series

$S_8 = 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31$
$= 136$

$S_4 = 3 + 7 + 11 + 15$
$= 36$

Obtaining a Partial Sum Formula

Karl Friedrich Gauss was really smart. He found a cool and quick way to add up the numbers from 1 to 100.

$$S_{100} = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$+ \quad S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$

$$2(S_{100}) = 101 + 101 + 101 \cdots + 101 + 101 + 101$$

$$2(S_{100}) = 100(101)$$

$$S_{100} = \frac{100(101)}{2} = 5050$$

Consider now another generic arithmetic sequence. You tell me the first number, and the common difference. Let's generate 7 terms and add them up using Gauss's method.

$$S_7 = 12 + 32 + 52 + 72 + 92 + 112 + 132$$

$$+ \quad S_7 = 132 + 112 + 92 + 72 + 52 + 32 + 12$$

$$2(S_7) = 144 + 144 + 144 + 144 + 144 + 144 + 144$$

$$2(S_7) = 7(144)$$

$$S_7 = \frac{7(144)}{2} = \frac{\boxed{7}(\overset{t_1 = a}{12} + \overset{t_7}{132})}{2}$$

n

$$S_7 = 504$$

First equation for an arithmetic series:

$$S_n = \frac{n(t_1 + t_n)}{2}$$

t_1 = first term

t_n = last term

n = # of terms being added

if you know t_n

What happens if we don't know t_n? Is there a formula for t_n?

$$t_n = a + (n-1)d$$

Second equation for an arithmetic series:

$$S_n = \frac{n(a + a + (n-1)d)}{2}$$

$$S_n = \frac{n(2a + (n-1)d)}{2}$$

use if you don't know t_n

Example 7.5.2

Determine the sum of the first 20 terms of the arithmetic sequence:

$5, 2, -1, -4, \ldots$

$a = 5$

$d = -3$

$n = 20$

$t_{20} = ?$

$$S_n = \frac{n\left(2a + (n-1)d\right)}{2}$$

$$S_{20} = \frac{\cancel{20}^{10}\left(2(5) + (20-1)(-3)\right)}{\cancel{2}}$$

$$S_{20} = 10(10 - 57)$$

$$S_{20} = 10(-47)$$

$$S_{20} = -470$$

Example 7.5.3

From your text: Pg. 452 #5d

For the given arithmetic series determine t_{12} and S_{12}: $\frac{1}{5} + \frac{7}{10} + \frac{6}{5} + \frac{17}{10} + \ldots$

term

sum

$a = \frac{1}{5}$

$d = t_2 - t_1$

$= \frac{7}{10} - \frac{1}{5}$

$= \frac{7}{10} - \frac{2}{10}$

$= \frac{5}{10} = \frac{1}{2}$

$$t_n = a + (n-1)d$$

$$t_{12} = \frac{1}{5} + (11)\left(\frac{1}{2}\right)$$

$$t_{12} = \frac{1}{5} + \frac{11}{2}$$

$$t_{12} = \frac{2}{10} + \frac{55}{10}$$

$$t_{12} = \frac{57}{10}$$

$$S_{12} = \frac{n\left(t_1 + t_{12}\right)}{2}$$

$$S_{12} = \frac{\cancel{12}^{6}\left(\frac{1}{5} + \frac{57}{10}\right)}{\cancel{2}}$$

$$S_{12} = 6\left(\frac{2}{10} + \frac{57}{10}\right)$$

$$S_{12} = 6\left(\frac{59}{10}\right)$$

$$S_{12} = \frac{354}{10}$$

$$S_{12} = \frac{177}{5}$$

Example 7.5.4

From your text: Pg 452 #7e

Calculate the sum of the series: $\overset{t_1}{-31} - 38 - 45 - \ldots \overset{t_n}{-136}$

$a = -31$

$d = -7$

$n = ?$

$t_n = -136$

To find n, use the general term formula.

$t_n = a + (n-1)d$

$\overset{+31}{-136} = \overset{+31}{-31} + (n-1)(-7)$

$\dfrac{-105}{-7} = \dfrac{(n-1)(-7)}{-7}$

$15 = n - 1$

$\boxed{16 = n}$

$S_{16} = \dfrac{16(-31 + -136)}{2}$

$S_{16} = 8(-167)$

$S_{16} = -1336$

Success Criteria:

- I can calculate the sum of the first *n* terms of a arithmetic sequence by using one of the two formulas we learnt
  - o $S_n = \frac{n[2a+(n-1)d]}{2}$
  - o $S_n = \frac{n[t_1+t_n]}{2}$
- I can recognize when each formula is the most appropriate one to use

Chapter 7 – Sequences and Series (Discrete Functions)

7.6 – Geometric Series

Learning Goal: We are learning to calculate the sum of the terms of a geometric sequence.

Again, a series is associated with a sequence. A series arises by adding together the terms of a sequence, so a Geometric Series arises by adding together the terms of a geometric sequence.

need n.

do not need n.

The Partial Sum Formula for a Geometric Series

first term

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1, \qquad \text{OR} \qquad S_n = \frac{t_{n+1} - t_1}{r - 1}, \quad r \neq 1$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{\boxed{ar^n} - a}{r - 1} \quad a = t_1$$

$$t_{n+1} = ar^{n-\cancel{1+1}}$$

Remember that r is the common ratio between successive terms!

Example 7.6.1

Given the geometric series, determine t_7 and S_7

$3, 12, 48, \ldots$

$a = 3$
$r = 4$
$n = 7$

$t_7 = (3)(4)^{7-1}$

$t_7 = 12,288$

$S_7 = \frac{3(4^7 - 1)}{4 - 1}$

$S_7 = \frac{\cancel{3}(16383)}{\cancel{3}}$

$S_7 = 16,383$

Example 7.6.2

Determine S_{10} for the geometric series $1.3, 3.25, 8.125, 20.3125, \ldots$

$a = 1.3$

$r = \frac{t_2}{t_1} = \frac{3.25}{1.3} = 2.5$

$n = 10$

$S_{10} = \frac{1.3(2.5^{10} - 1)}{2.5 - 1}$

$S_{10} = 8,264.31$

Example 7.6.3

Calculate the sum of the geometric series $\underbrace{2}_{t_1} - 6 + 18 - 54 + ... + \underbrace{13122}_{t_n}$, $\underbrace{-39366}_{t_{n+1}}$

$a = 2$

$r = -3$

$n = ?$ and we don't care!

$$S_n = \frac{t_{n+1} - t_1}{r-1}$$

one after t_n

$$S_n = \frac{-39366 - 2}{-3-1}$$

$$S_n = \frac{-39368}{-4}$$

$$S_n = 9,842$$

Success Criteria:

- I can add the first n terms of a geometric sequence using:

$$S_n = \frac{a(r^n - 1)}{r-1}, \quad r \neq 1, \qquad \text{OR} \qquad S_n = \frac{t_{n+1} - t_1}{r-1}, \quad r \neq 1$$

- I can recognize when each formula is the most appropriate one to use