Functions 11

Course Notes

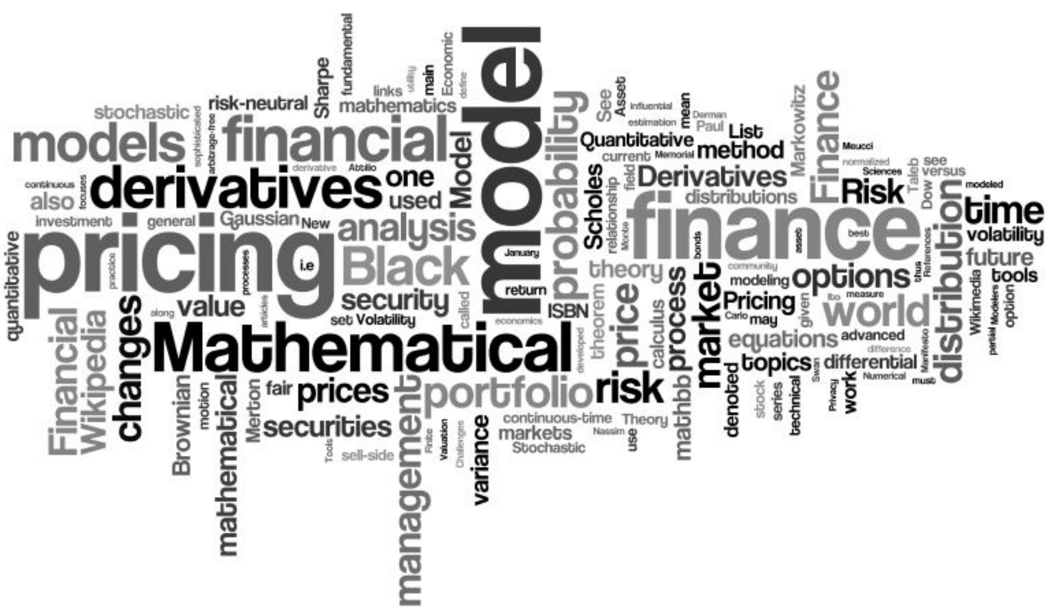
Unit 8 – Financial Math

Compound interest is the greatest Mathematical discovery of all time

Albert Einstein (maybe)

We will learn

- *understand the connections between simple interest and Arithmetic Sequences and linear growth*
- *understand the connections between compound interest and Geometric Sequences and exponential growth*
- *solve problems using the formulae, or using spreadsheets, for simple and compound interest, and for annuities/mortgages*



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Chapter 8 – Financial Math

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

8.1: Pg. 481 – 482 #5bce, 6, 7, 11, 8 (yes, try #8 last!)

8.2: Pg. 490 – 491 #1ab, 3cd (no timeline), 6 (guess and check!), 7 (option two is tricky), 9, 10

8.3: Pg. 498 – 499 #1ab, 3cd, 4, 6, 8 (ask for help if you need it), 10

8.4: Pg. 511 – 512 #3, 5bd, 6, 8 (guess and check!), 9, 10

8.5: Pg. 520 – 521 #1ac (*no timelines, no series*), 3bc, 4, 6 (*yes – CD/DVD players did cost that much in the past*!), 7, 10

Chapter 8 – Financial Applications (Discrete Functions)

8.1 Simple Interest

Learning Goal: We are learning to calculate simple interest.

This *should* be review, but it never hurts to review previously learned material.

Most people are *interested* in their personal financial situations. Obviously that's why we call the money earned on investments **interest**.

Simple **I**nterest is calculated using an interest **r**ate, ***r*** (% /a), over a period of time, ***t*** (in years).

per annum

year

We call the amount invested (borrowed) the **P**rincipal.

The Interest Formula

$$I = Prt$$

this tells you how much you earned

The **A**mount (of money) Formula

$$A = P + I$$

Example 8.1.1

From your text: Pg. 481 #5f

For each investment, calculate the interest earned and the total amount.

	Principal P	Rate of Simple Interest per Year r	Time t
a)	$500	4.8% $= 0.048$	8 years
b)	$3 200	9.8%	12 years
c)	$5 000	3.9% $= 0.039$	16 months $\rightarrow \frac{16}{12} = \frac{4}{3}$
d)	$128	18%	5 months
e)	$50 000	24%	17 weeks $\rightarrow \frac{17}{52}$
f)	$4 500	12%	100 days $\rightarrow \frac{100}{365}$

a) $I = Prt = (500)(0.048)(8) = \$192$

$A = 500 + 192 = \$692$

c) $I = (5000)(0.039)(\frac{4}{3}) = \$260$

$A = \$5260$

Example 8.1.2

Jasmine invests $\$4850$ at $\boxed{7.6\%/a}$ simple interest. If she wants her money to increase to $\$8000$, for how long will she need to keep her money invested?

$= P \quad = r = 0.076$

$A \quad t$

① $A - P = I$

$8000 - 4850 = I$

$3150 = I$

② $I = Prt$

$$3150 = (4850)(0.076)t$$

$$\frac{3150}{(4850)(0.076)}$$

$$8.54 = t$$

$\therefore$ approximately 8.5 years to reach \$8000.

Example 8.1.3

Philip (the unwise) borrows $\$1540$ for $\boxed{90 \text{ days}}$ by taking a cash advance from the company YourCashIsOurCash. The interest rate Philip (unwisely) agrees to is $\boxed{26\%/a}$ (simple interest). How much money will Philip have to pay back at the end of 90 days, and how much interest does he pay?

$= P \quad t \quad = r = 0.26$

$P = 1540$

$r = 0.26$

$t = \frac{90}{365}$

$I = ?$

$$I = (1540)(0.26)\left(\frac{90}{365}\right)$$

$$I = \$98.73$$

$$A = \$1,638.73$$

$\therefore$ Philip has to pay back \$1,638.73

Success Criteria:

- I can recognize that simple interest is calculated only on the principal
- Simple interest is an example of a linear function

Chapter 8 – Financial Mathematics

8.2 – Compound Interest and Future Value

Learning Goal: We are learning to determine the future value of a principal amount, using compound interest. This applies to both savings and loans.

Last day we look at the idea of Simple Interest (with formula $A = P(1+rt)$). Now we consider the notion of Compound Interest. Compound Interest is such that your savings grows much more quickly than it could if you were just earning simple interest.

$1000 → $100
$1100 → $110
$1210

Compound Interest Formulae:

Future Value

$$A = P(1+i)^n$$

where: $A =$ Amount (in Future) → FV

$P =$ Principal (Present Value)

$i =$ interest rate per compounding period

$n =$ total number of compounding periods

Now, there is one major challenge to any calculations. Interest is normally stated on a yearly basis, but interest is actually charged more frequently (the bank earns more money when you pay interest on interest ☹). So if you pay 12% interest per year, but pay interest monthly, you actually pay 1% each month. You must incorporate this into your calculations.

12%/a

5 years

Here is a helpful chart:

Periods	How often you pay (Times per year)	i value in your equation	n value in your equation
Annually	1	i | 0.12	n years | 5
Semi-annually	2	$\frac{i}{2}$ | 0.06	$2n$ | 10
Quarterly	4	$\frac{i}{4}$ | 0.03	$4n$ | 20
Monthly	12	$\frac{i}{12}$ | 0.01	$12n$ | 60

$$A = P(1+i)^n$$

Example 8.2.1

a) You deposit $10 000 into an account which pays 2.4% per year, compounded annually. What is the amount of money in the account in 10 years?

$P = 10{,}000$

$i = \frac{0.024}{1}$

$n = 10(1)$

$$A = 10000(1+0.024)^{10}$$

1.024

$$A = \$12{,}676.51$$

b) You deposit $10 000 into an account which pays 2.4% per year, compounded monthly. What is the amount of money in the account in 10 years?

$P = 10000$

$i = \frac{0.024}{12} = 0.002$

$n = 10(12) = 120$

$$A = 10000(1+0.002)^{120}$$

$$A = \$12{,}709.44$$

Example 8.2.2

Beth deposits $500 into an account which pays 6% compounded monthly. Fred deposits $500 into an account which pays 6% **simple interest**. What is the difference in the value of their accounts after 5 years?

Beth

$P = \$500$

$i = \frac{0.06}{12} = 0.005$

$n = 5(12) = 60$

$$A = P(1+i)^n$$

$$A = 500(1+0.005)^{60}$$

$$A = \$674.43$$

Fred

$P = \$500$

$r = 0.06$

$t = 5$

$$I = Prt$$

$$I = (500)(0.06)(5)$$

$$I = 150$$

$$A = 500 + 150$$

$$A = \$650$$

$\therefore$ Beth is the wiser of the two as she earned $24.43

$A = P(1+i)^n$

Example 8.2.3

On her 15th birthday, being very wise, Susan invests $10,000 in an account which pays 2.4% compounded monthly. The not-so-wise John waits until his 45th birthday to invest $10,000 in an account which pays 2.4% compounded monthly. How much is each account worth when they reach 65 years old?

Susan:

$P = 10000$

$i = \frac{0.024}{12} = 0.002$

$n = 50(12) = 600$

$A =$

$A = 10000(1+0.002)^{600}$

$A = \$33,161.40$

John:

$P = 10000$

$i = \frac{0.024}{12} = 0.002$

$n = 20(12) = 240$

$A = 10000(1+0.002)^{240}$

$A = \$16,153$

Example 8.2.4

You find yourself in a furniture store. Looking around you become dazzled by the adverts promising a better and happier life if you only had one of their beautiful couches. The advertisement reads:

No Money Down!

No Payments for 3 years!!

Take this couch home today!!!!!

Only $1599*!!!!!!!!!!!!

You decide it's a good deal, but you neglect to read the asterisk until it's too late. You've already signed on the dotted line. After signing you decide to finally read the fine print which says:

**Financed at 18% compounded monthly*

How much do you have to pay after 3 years?

How much of that is **just** interest?

$P = \$1599$

$i = \frac{0.18}{12} = 0.015$

$n = 3(12) = 36$

$A = 1599(1 + 0.015)^{36}$

$A = \$2,732.91$

$I = 2732.91 - 1599$

$= \$1,133.91$

Success Criteria:

- I can determine the related interest (*i*) per compounding period (*n*)
- I can use the Future Value formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking A – P.

Chapter 8 – Financial Mathematics

8.3 – Compound Interest and Present Value

Learning Goal: We are learning to determine the present value remaining on a loan or investment.

Last day we looked at the idea of Compound Interest (Future Value). Today we continue looking at Compound Interest with a focus on Present Value.

Compound Interest Formulae:

Future Value

$$A = P(1+i)^n$$

Present Value

$$P = \frac{A}{(1+i)^n} = A(1+i)^{-n}$$

where: A = Future Amount

P = principal (present value)

i = interest rate per compounding period

n = compounding periods

Example 8.3.1

= Future = A

You want \$10 000 in your bank account 20 years from now. Your account pays 1.8% per year, compounded annually. What is the amount of money you have to deposit today?

$A = 10{,}000$

$i = 0.018$

$n = 20$

$$P = \frac{10000}{(1+0.18)^{20}}$$

$$= \$6{,}999.14$$

Example 8.3.2

You want to buy a house at 30 years old (14 years from now). You estimate that you will need a down payment of \$150 000. You find a bond which matures in 14 years paying 3.6% interest compounded monthly. How much do you need to invest in the bond today?

$A = 150{,}000$

$i = \frac{0.036}{12} = 0.003$

$n = 14(12) = 168$

$$P = \frac{150000}{(1+0.003)^{168}}$$

$$P = \$90{,}684.80$$

Example 8.3.3

Today Henry invests some money in an account which pays 2% compounded quarterly. 10 years from now he takes the money in his account and reinvests it in an account which pays 4.8% compounded monthly. After an additional 15 years [redacted] Henry has $125,345 in his account. How much did he originally invest?

Today

? —10 years→ $? —15 years→ $125,345

Stage 2

$A = 125,345$

$i = \frac{0.048}{12} = 0.004$

$n = (15)(12) = 180$

$$P = \frac{125345}{(1+0.004)^{180}}$$

$$= \$61,099.65$$

Stage 1

$A = 61,099.65$

$i = \frac{0.02}{4} = 0.005$

$n = (10)(4) = 40$

$$P = \frac{61099.65}{(1+.005)^{40}}$$

$$P = \$50,049.10$$

This is how much Henry began with

Success Criteria:

- I can use the Present Value formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking A – P.

Chapter 8 – Financial Mathematics

8.4 – Annuities: Future Value

Learning Goal: We are learning to determine the future value of an annuity earning compound interest.

The problem with the examples in 8.2 and 8.3 is that not many people have large amounts of money to be depositing into savings accounts. People usually make **regular deposits of smaller amounts of money**. **An account into which (or out of!) regular payments are made is called AN ANNUITY**. We will study two aspects of annuities: Future Value and Present Value.

Future Value of an Annuity

$$FV = \frac{R[(1+i)^n - 1]}{i}$$ ← be very mindful of brackets!

where: FV = Future Value (A) n = number of periods

R = Regular Payments / withdrawals
→ same every time.

i = interest rate per compounding period

Example 8.4.1

Dylan decides to deposit $200 monthly into an account which pays 3.6% per year, compounded monthly. What is the value of his annuity after 25 years? How much interest is earned? If Dylan leaves the money in his account for another 20 years, but makes no more regular payments, how much money will be in the account at the end of 45 years?

First 25 years:

$R = \$200$

$i = \frac{0.036}{12} = 0.003$

$n = (25)(12) = 300$

$FV = ??$

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

$$FV = \frac{200[(1+0.003)^{300} - 1]}{0.003}$$

$$FV = \frac{200(1.45629)}{0.003}$$

$$FV = 97{,}086.10$$

$I = FV - R(n)$ $\quad \$200 \times 300 = 60{,}000$

$I = \$37{,}086.10$

Next 20 years

→ $P = \$97{,}086.10$

$i = 0.003$

$n = 20(12) = 240$

$A = 97086.10(1+.003)^{240}$

$A = \$199{,}242.04$

Example 8.4.2

not realistic ...

Tingyi invests $300, every three months (i.e. quarterly) into an RRSP which pays (on average) 12% compounded quarterly. What is the value of her RRSP after 45 years?

$R = \$300$

$i = \frac{0.12}{4} = 0.03$

$n = 45(4) = 180$

$$FV = \frac{300[(1+.03)^{180} - 1]}{0.03}$$

$$FV = \$2,035,033.60$$

She invested $\$300 \times 180 = \$54,000$

Example 8.4.3

An absolutely wonderful student decides to give Mr. Templeton $100,000 when he retires in 15 years. The student finds an investment which pays 9% compounded monthly. What is the regular payment s/he would need to make to have $100,000 in 15 years?

$FV = \$100,000$

$i = \frac{0.09}{12} = 0.0075$

$n = (12)(15) = 180$

$R = ?$

$$100000 = \frac{R[(1+.0075)^{180} - 1]}{0.0075}$$

$$100000 = R(378.40572\ldots)$$

$$\$264.27 = R$$

Success Criteria:

- I can recognize that an annuity is a "regular payment" on a loan or investment
- I can use the Future Value Annuity formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking $I = FV - (n \times R)$.

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Chapter 8 – Financial Mathematics

8.5 – Annuities: Present Value

Learning Goal: We are learning to calculate the present value of an annuity earning compound interest.

Today we continue with the idea of an annuity, but we will be looking at **PRESENT VALUE (TODAY'S VALUE OF THE INVESTMENT/LOAN)**

Future Value of an Annuity

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

Present Value of an Annuity

$$PV = \frac{R[1-(1+i)^{-n}]}{i}$$

where: $FV =$ **Future Value**

$R =$ **The Regular Payment into or out of the annuity**

$i =$ **The interest rate per compounding period**

$n =$ **The total number of compounding periods over the life of the account**

$PV =$ **Present Value**

Example 8.5.1

While in college you want an annuity to pay you $250 a month, every month over the four years you are studying. How much money would you need to invest today in an account which pays 2.4% compounded monthly to guarantee the monthly payment? What is the total amount of money you receive from the annuity? How much interest do you earn?

PV.

$R = 250$

$i = \frac{0.024}{12} = 0.002$

$n = 4(12) = 48$

$PV = ?$

$$PV = \frac{250[1-(1+.002)^{-48}]}{0.002}$$

$PV = \$11,431.11$

You are actually receiving $\$250 \times 48 = \$12,000$

Interest is $\$12000 - 11431.11$

$= \$568.89$

Example 8.5.2

Henry borrows \$25,000 to buy a car. He pays \$3,000 down. He will take 6 years to pay off the loan. The bank charges 2.6% compounded biweekly. What are the regular payments Henry has to make to pay off the car? How much interest does he pay?

misleading

$\frac{52}{2} = 26$

$PV = \$25{,}000$

$n = (6)(26) = 156$

$i = \frac{0.026}{26} = 0.001$

$R = ?$

$$PV = \frac{R\left[1-(1+i)^{-n}\right]}{i}$$

$$25000 = R\,\boxed{\frac{\left[1-(1+0.001)^{-156}\right]}{0.001}}$$

$$25000 = R(144.37\ldots)$$

$$\$173.16 = R$$

Henry paid \$173.16 × 156

= \$27,012.96

−25,000

The interest is \$2,012.96.

Success Criteria:

- I can recognize the difference between present value and future value
- I can use the ~~Future~~ present Value Annuity formula to solve various financial quandaries
- I can calculate the total interest earned/paid by taking $I = (n \times R) - PV$.