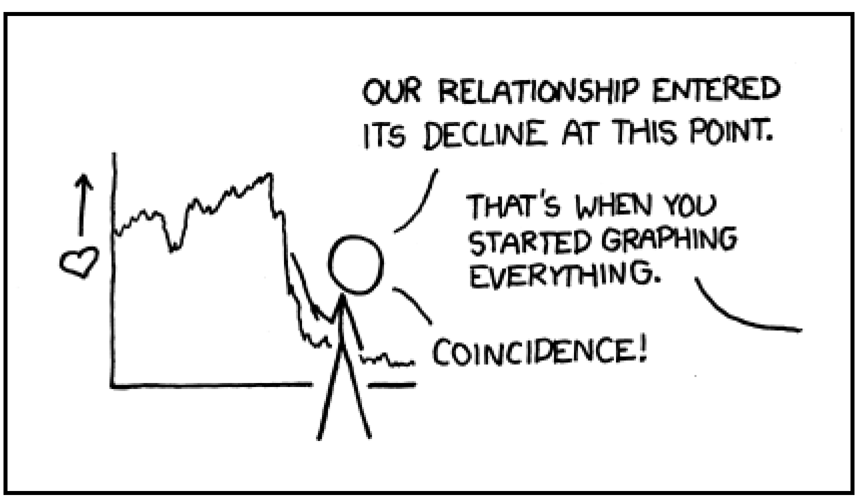
Functions 11

Course Notes 2021

Chapter 1 – Introduction to Functions

We will learn

- *the meaning of the term Function and how to use function notation to calculate and represent functions*
- *the meanings of the terms domain and range, and how a function's structure affects domain and range*
- *how to use transformations to represent and sketch graphs*
- *how to determine the inverse of a function*



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MATH@TD

Chapter 1 – Introduction to Functions

Contents with suggested problems from the Nelson Textbook. You are welcome to ask for help with any of the following problems from your peers or myself. They will be handed in on the day of the Unit Test as a homework check.

Section 1.1

Pg. 10 – 12 #1, 2 (*no ruler needed…*), 6, 7, (*no need for the VLT, but do sketch graphs even if you use Desmos to do the sketching*!), 9, 11, 12 (*think carefully about the idea that the domain and range are "limited"*)

Section 1.2

Page 23 #1-2, 5, 8b, 10, 11cd, 15, 16, challenge #17

Section 1.3/1.4

Domain and Range Handout

Section 1.5

Pg. 47 – 49 #1, 8, 10, 16, 17

Also, determine the inverse (your method of choice) of:

a) $f(x) = 2\sqrt{x-3} + 5$ b) $g(x) = \frac{1}{x+3}$ c) $h(x) = \frac{1}{2}(x+3)^2 - 1$

Section 1.6-1.8

Big Handout

Chapter 1 – Introduction to Functions

1.1 Relations and Functions (*This is a KEY lesson*!)

Learning Goal: We are learning to recognize functions in various representations.

This course is called **FUNCTIONS**, so it seems rather important that you know what a function actually is. Thus you need to know, very well, the following (algebraic) definition:

Definition 1.1.1

A **FUNCTION** is rule which connects two sets of numbers in a special way. A function assigns/connects exactly one number in the set called $\underbrace{\text{range}}_{y\text{-values}}$ to each number in a set called $\underbrace{\text{domain}}_{x\text{-values}}$.

→ in an equation, each "x" produces only one "y"!

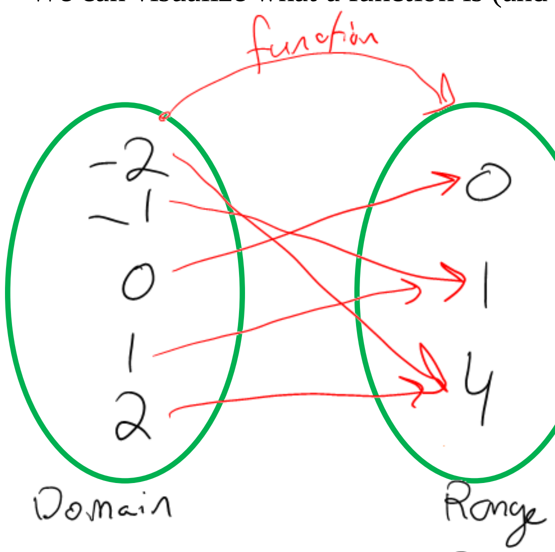
$y^2 = x$

plug in 4 → x

$\sqrt{y^2} = \sqrt{4}$

$y = \pm 2$

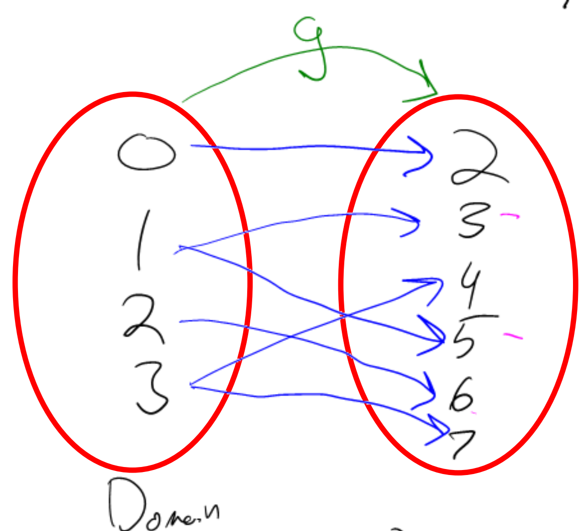
We can visualize what a function is (and **isn't**) by using so-called "**arrow diagrams**":



D_f : $\{-2, -1, 0, 1, 2\}$

R_f : $\{0, 1, 4\}$

This is a function because every "x" is connected to only one "y"!



D_g: $\{0, 1, 2, 3\}$ Input

R_g: $\{2, 3, 4, 5, 6, 7\}$ Output

This is not a function because 1 produces 3 AND 5.

We need a few more definitions before moving on, so that we can "speak the language" of functions (and that language is mathematics!)

Definition 1.1.2

A **SET** is collection (real numbers) of objects. We use numbers.

ex: $\{x \in \mathbb{R} \mid x > 4\}$

$\in$: belongs to; $\mid$: there is a condition, "so that"

Definition 1.1.3

A **RELATION** is any relationship between domain and range values

Definition 1.1.4

The **DOMAIN** of a function (or a relation) is the set of numbers which are allowed to be plugged into the relation.

ex: $y = \frac{1}{x}$; $y = \frac{1}{0} =$ undefined

Definition 1.1.5

The **RANGE** of a function (or a relation) is the set of numbers which are calculated from the domain.

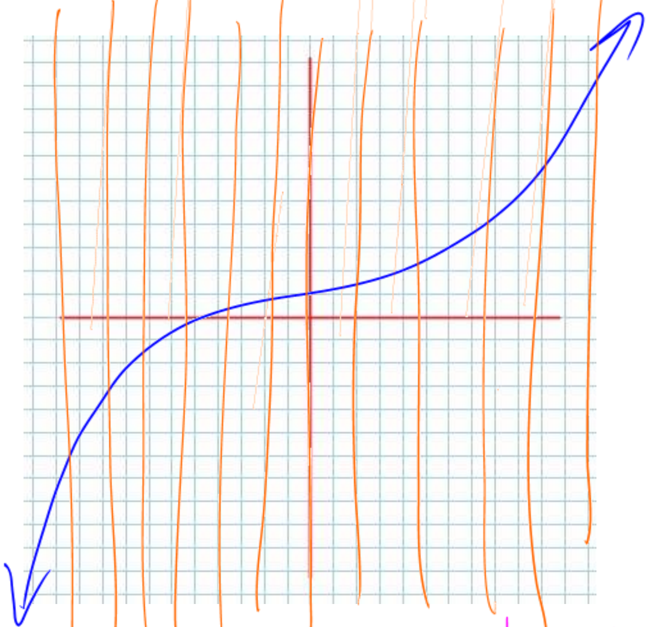
Two other important terms to know are:

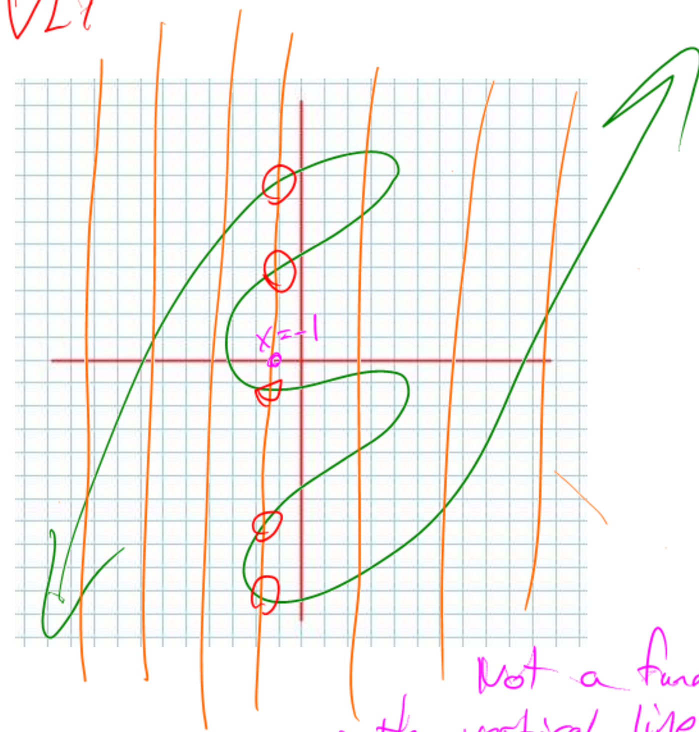
1) The **INDEPENDENT VARIABLE** is the "x" variable or the domain.

2) The **DEPENDENT VARIABLE** is the "y" variable or the range.

Knowing when a Relation is, and isn't, a Function

Graphically: *The Vertical Line Test* VLT





This is a function because each vertical line passes the graph only once

Not a function as the vertical lines cross more than once

Algebraically: **(*NOTE: this is a "rough" way of thinking about the problem*)**

If the Dependent Variable is raised to an even exponent, the relation is not a function.

y

e.g. $y^2 = x+3$

$y = \pm\sqrt{x+3}$

→ 2 possibilities

$y^3 = x+3$

$y = \sqrt[3]{x+3}$

cubic root

this is a function

$2^3 = 2\times2\times2$

$= 8$

$(-2)^3 = (-2)\times(-2)\times(-2)$

$= -8$

Success Criteria:

- I can determine the domain of a relation or function as the set of all values of the independent variable
- I can determine the range of a relation or function as the set of all values of the dependent variable
- I can apply the vertical line test to determine if a graph is a function
- I can recognize whether a relation is a function from its equation

Chapter 1 – Introduction to Functions

1.2 Function Notation

Learning Goal: We are learning to use function notation to represent linear and quadratic functions

Here we learn a **NEW AND IMPROVED WAY** for describing a function, algebraically. You have been using the following form for functions (in this example, for a quadratic):

$$y = 3(x-2)^2 + 1$$

A much more useful way of writing function is to use **FUNCTION NOTATION**. The above quadratic (***which we call a "function of x" because the domain is given as x-values***) can be written as:

dependent variable

function $\boxed{f(x)} = 3(x-2)^2 + 1$

the independent variable

$f(x)$ replaces y

$f(x) = y$

"f of x", "f at x"

This new notation is so useful because the "symbol"

shows **BOTH** the **DOMAIN** and the **RANGE** values. Because of that, the function notation shows us **points** on the graph of the function.

Let's do some examples (from your text on pages 23 – 24)

Example 1.2.1

i)

4. Evaluate $f(-1)$, $f(3)$, and $f(1.5)$ for

a) $f(x) = (x-2)^2 - 1$

b) $f(x) = 2 + 3x - 4x^2$

i) $f(\boxed{x}) = (\boxed{x} - 2)^2 - 1$

$f(-1) = (-1-2)^2 - 1$

$f(-1) = 9 - 1$

$\boxed{f(-1) = 8}$

- leave left side alone.
- calculate the right side

The function at $x=-1$ is 8.

The point is $(-1, 8)$

$f(3) = 2 + 3(3) - 4(\cancel{3^2})$ 9

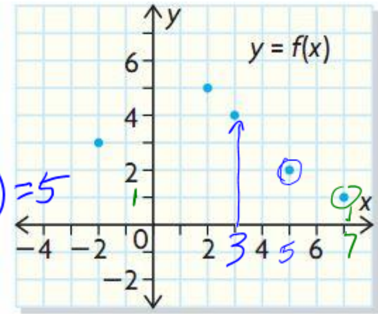
$f(3) = 2 + 9 - 36$

$f(3) = -25$

The point is $(3, -25)$

Example 1.2.2

6. The graph of $y = f(x)$ is shown at the right.



a) State the domain and range of f.

b) Evaluate.

i) $f(3) = 4$ (x, y)

ii) $f(5) = 2$

iii) $f(5 - 3) = f(2) = 5$

iv) $f(5) - f(3)$

$= 2 - 4$

$= -2$

v) $f(x) = 1$ $= y$

$(7, 1)$

$x = 7$

a) Domain: $\{-2, 2, 3, 5, 7\}$ x-values

Range: $\{1, 2, 3, 4, 5\}$ y-values

Example 1.2.3

11. For $g(x) = 4 - 5x$, determine the input for x when the output of $g(x)$ is =

a) -6 $\quad$ b) 2

$\boxed{g(x) = -6}$

$\boxed{g(x)} = 4 - 5x$

$-6 = 4 - 5x$

$-4 \qquad -4$

$\frac{-10}{-5} = \frac{-5x}{-5}$

$2 = x$

$\therefore g(2) = -6$

$g(x) = 2$

$g(x) = 4 - 5x$

$2 = 4 - 5x$

$\frac{-2}{-5} = \frac{\cancel{-5}x}{\cancel{-5}}$

$\frac{2}{5} = x$

$\therefore g\left(\frac{2}{5}\right) = 2$

$y = 4 - 5x$

$y = -6$

$-6 = 4 - 5x$

Example 1.2.4

7. For $h(x) = 2x - 5$, determine

a) $h(a)$ **c)** $h(3c - 1)$

b) $h(b + 1)$ **d)** $h(2 - 5x)$

a) $h(a) = 2a - 5$

b) $h(b+1) = 2(b+1) - 5$

$h(b+1) = 2b + 2 - 5$

$h(b+1) = 2b - 3$

Example 1.2.5

12. A company rents cars for \$50 per day plus \$0.15/km. "x"

a) Express the daily rental cost as a function of the number of kilometres travelled.

R d

b) Determine the rental cost if you drive 472 km in one day. =d

c) Determine how far you can drive in a day for \$80.

rental cost

a) $R(d) = 50 + 0.15d$ $f(x) = 50 + 0.15x$

b) $R(472) = 50 + 0.15(472)$

$R(472) = \$120.80$

c) $\$80 = \$50 + 0.15d$

$\frac{30}{0.15} = \frac{0.15d}{0.15}$

$200\text{ km} = d$

Success Criteria:

- I can evaluate functions using function notation, by substituting a given value for x in the equation for f(x)
- I can recognize that f(x) = y corresponds to the coordinate (x, y)
- I can, given y = f(x), determine the value of x

Chapter 1 – Introduction to Functions

1.3 and 1.4 Parent Functions and Domain and Range

Learning Goal: We are learning the graphs and equations of five basic functions; and using their tables, graphs, or equations to find their domains and ranges.

Two **INCREDIBLY IMPORTANT** aspects of functions are their

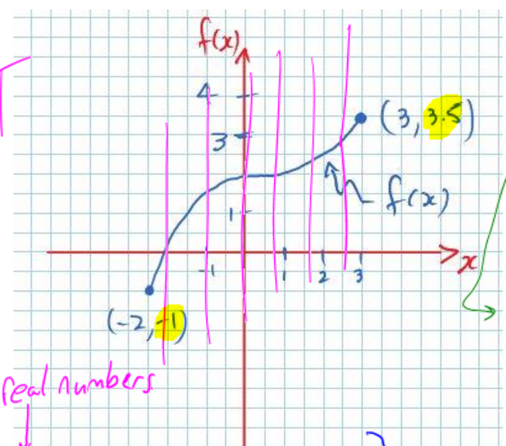
Again, the Domain is the set of all x-values that work (input)

And, the Range is the set of all functional values that come out (output)
y-values

Example 1.4.1

Given the **SKETCH OF THE GRAPH** of the **RELATION** determine: the domain, the range of the relation, and whether the relation is, or is not, a function.

a)



Passes VLT so it is a function.

$x \geq -2$

$x \leq 3$

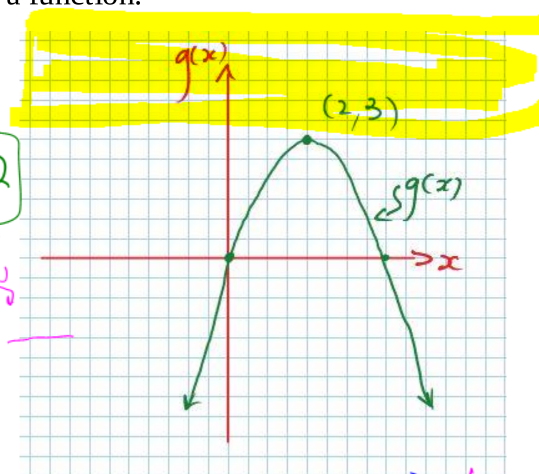
$-2 \leq x$

real numbers

$D_f = \{x \in \mathbb{R} \mid -2 \leq x \leq 3\}$ $\quad R_f = \{f(x) \in \mathbb{R} \mid -1 \leq f(x) \leq 3.5\}$

condition

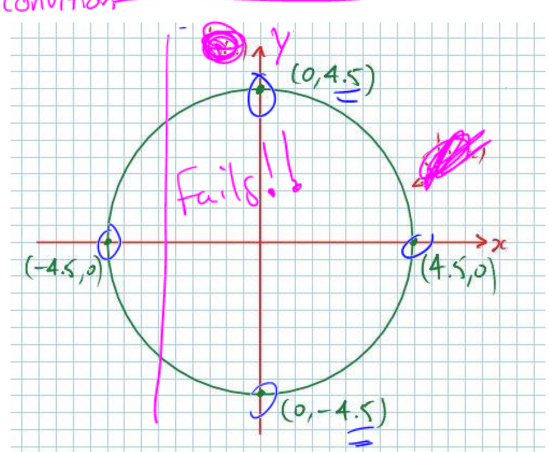
b)



b) $D_g = \{x \in \mathbb{R}\}$

$R_g = \{g(x) \in \mathbb{R} \mid g(x) \leq 3\}$

c)

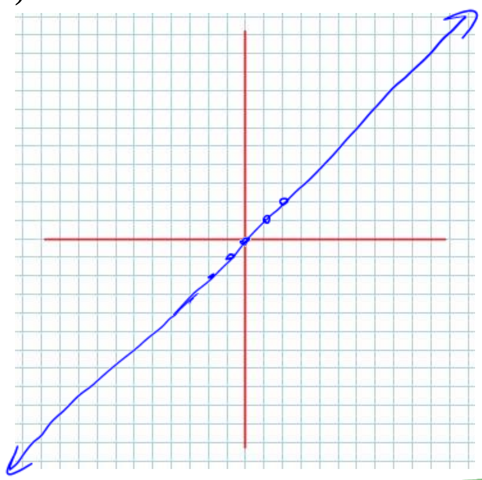


c) $D = \{x \in \mathbb{R} \mid -4.5 \leq x \leq 4.5\}$

$R = \{y \in \mathbb{R} \mid -4.5 \leq y \leq 4.5\}$

Together we will explore (graphically) basic properties of the five *parent* functions:

a) Linear



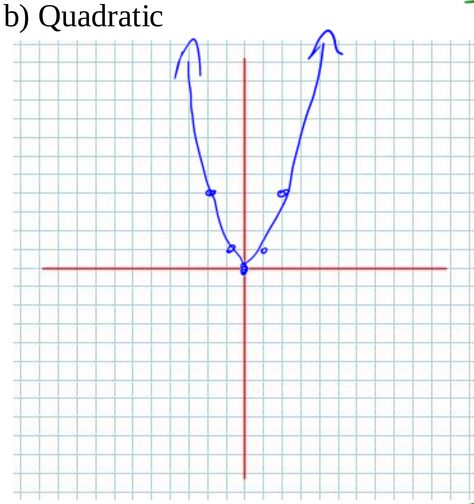
$f(x) = x$

x	$f(x)$
-2	-2
-1	-1
0	0
1	1
2	2

$D = \{x \in \mathbb{R}\}$

$R = \{f(x) \in \mathbb{R}\}$

b) Quadratic



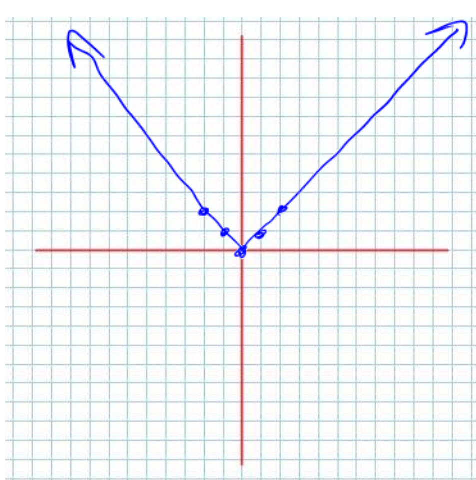
$g(x) = x^2$

x	$g(x)$
-2	4
-1	1
0	0
1	1
2	4

$D = \{x \in \mathbb{R}\}$

$R = \{g(x) \in \mathbb{R} \mid g(x) \geq 0\}$

c) Absolute Value



$h(x) = |x|$

Absolute Value is the distance from $\underline{x}$ to zero

↳ is always positive

ex: $|3| = 3$

$|-10| = 10$

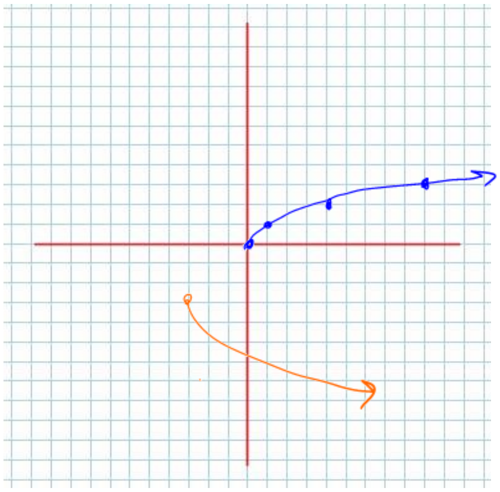
x	$h(x)$
-2	2
-1	1
0	0
1	1
2	2

$D = \{x \in \mathbb{R}\}$

$R = \{h(x) \in \mathbb{R} \mid h(x) \geq 0\}$

only positive square roots.

d) Square Root $f(x)=\sqrt{x}$

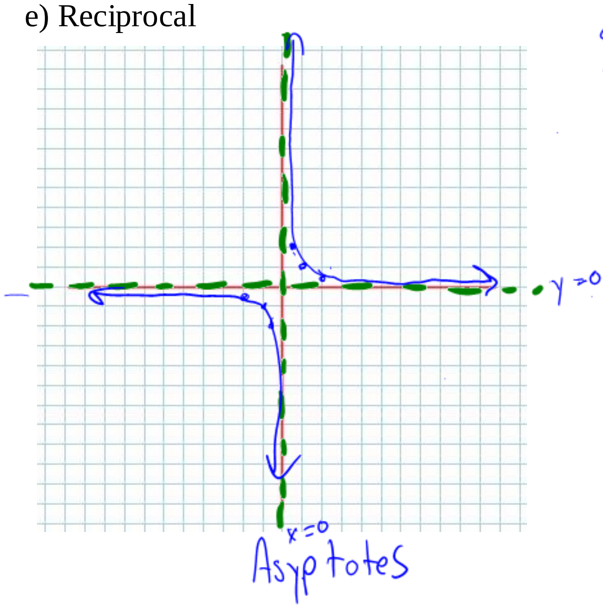


x	$f(x)$
0	0
1	1
4	2
9	3
16	4

$D_f=\{x\in\mathbb{R} \mid x\geq 0\}$

$R_f=\{f(x)\in\mathbb{R} \mid f(x)\geq 0\}$

e) Reciprocal $g(x)=\frac{1}{x}$



x	$g(x)$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	0
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

$\frac{1}{0}$ = undefined

$1\div\frac{1}{2}=1\times\frac{2}{1}=2$

$\frac{1}{100}=0.01$

$\frac{1}{100000}=0.000001$

$\frac{1}{1 \text{ trillion}}=$

$\frac{1}{0.01}=100$

$\frac{1}{1/\text{trillion}}=$ trillion

$D_g=\{x\in\mathbb{R} \mid x\neq 0\}$

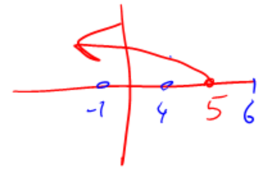
$R_g=\{g(x)\in\mathbb{R} \mid g(x)\neq 0\}$

Example 1.4.2 (*From Pg. 36 in your text*)

8. Write a function to describe coffee dripping into a 10-cup carafe at a rate of 1 mL/s. State the domain and range of the function (1 cup = 250 mL).

Example 1.4.3 (*From Pg. 37 in your text…use Desmos*)

9. Determine the domain and range of each function. positive ∴ opens up



5 − (−1)
= √6

a) $f(x) = -3x + 8$

Linear Function

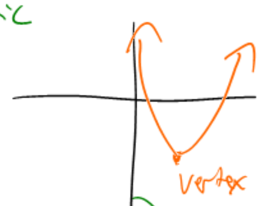
$D_f = \{x \in \mathbb{R}\}$

$R_f = \{f(x) \in \mathbb{R}\}$

d) $p(x) = \frac{2}{3}(x - \boxed{2})^{2} \boxed{-\ 5}$

h k

Quadratic



$D_p = \{x \in \mathbb{R}\}$

from a

k

$R_p = \{p(x) \in \mathbb{R} \mid p(x) \geq -5\}$

f) $r(x) = \sqrt{5 - x}$

5 − 5 = 0

$D_r = \{x \in \mathbb{R} \mid x \leq 5\}$

$R_r = \{r(x) \in \mathbb{R} \mid r(x) \geq 0\}$

Example 1.4.4

10. A ball is thrown upward from the roof of a 25 m building. The ball reaches a height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

a) Sketch a graph that shows the height of the ball as a function of time.

b) State the domain and range of the function.

c) Determine an equation for the function.

Success Criteria:

- I can identify the unique characteristics of five basic types of functions
- I can identify the domain and ranges of five basic types of functions
- I can identify when there are restrictions given real-world situations

Chapter 1 – Introduction to Functions

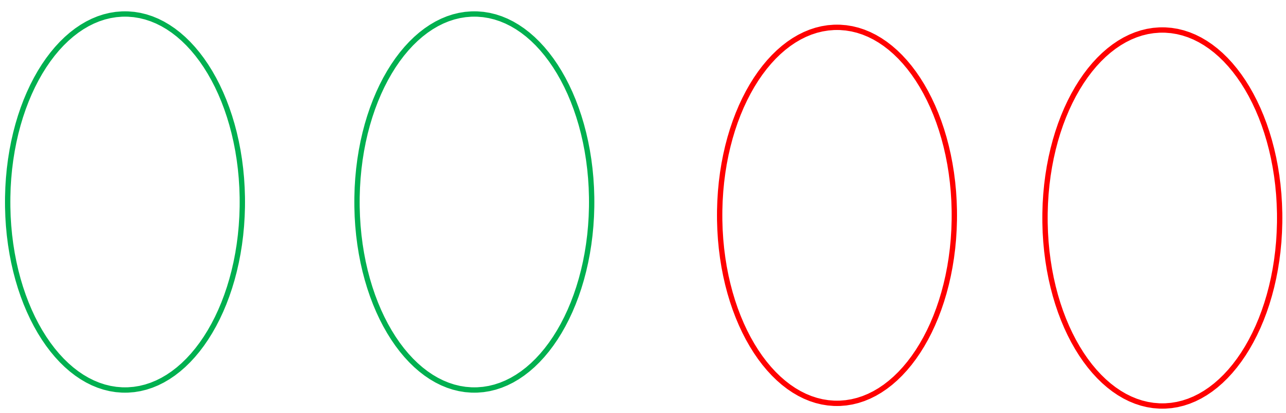
1.5: Inverses of Functions

Learning Goal: We are learning to determine inverses of functions and investigate their properties.

Definition 1.5.1 (*very rough definition!*)

Given a function $f(x)$, the inverse of the function (which we write as $f^{-1}(x)$) can be considered to "**undo**" what $f(x)$ originally did.

Consider the **Arrow Diagrams**:



Big Idea

Example 1.5.1

Given the graph of $f(x)$ determine: $D_f, R_f, f^{-1}(x), D_{f^{-1}}, R_{f^{-1}}$

$f(x) = \{(2,3),(4,2),(5,6),(6,2)\}$. Is $f^{-1}(x)$ a function?

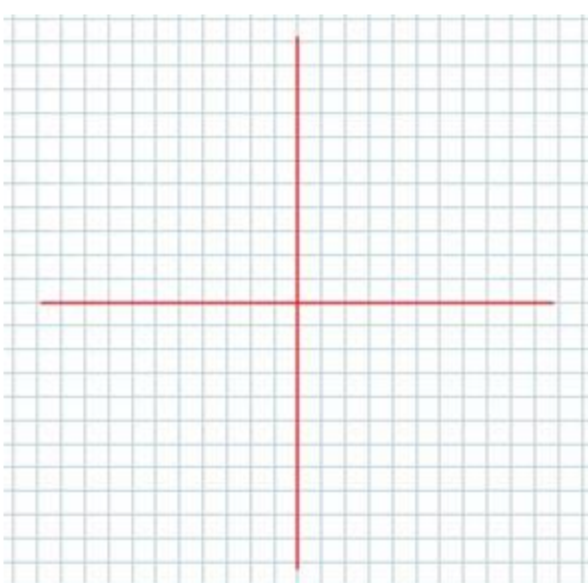
Determining the Inverse of a Function

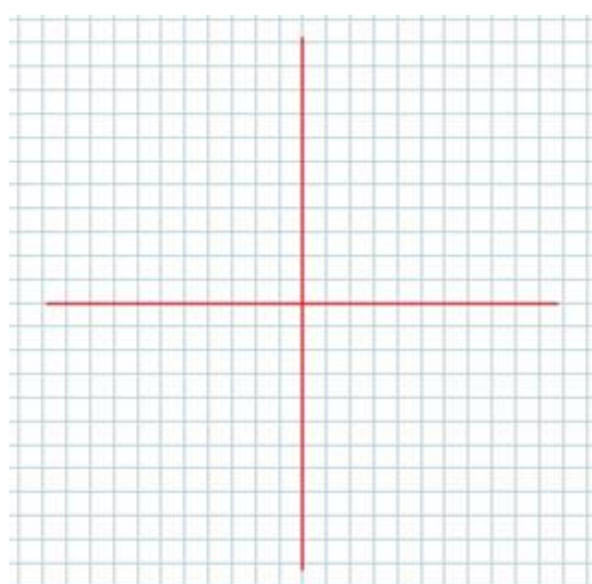
We can determine the inverse of some given function in either of two ways: Graphically and Algebraically.

Note: Finding a function inverse graphically is not a very useful method, but it can be instructive.

Function Inverses Graphically

Graph $g(x) = \sqrt{x} - 2$ then $g^{-1}(x)$.





Function Inverses Algebraically

Determining algebraic representations of inverse relations for given functions can be done in (at least) two ways:

1) Use algebra in a "brute force" manner (keeping in mind the Big Idea)
2) Use Inverse Operation Method.

Example 1.5.2
Use the "brute force" method to determine the inverses of:

a) $f(x) = 2x - 5$ b) $g(x) = \frac{1}{2}\sqrt{x-1} + 2$

Example 1.5.3

Use the Inverse Operation method to determine the inverses of:

a) $f(x) = 2\sqrt{\frac{1}{3}(x-1)} + 2$.

b) $g(x) = 4(x+8)^2 - 12$

Success Criteria:

- I can determine the inverse of a function using various techniques
- I can determine the inverse of a coordinate (a , b) by switching the variables: (b , a)
- I can recognize that the domain of an inverse is the range of the original function
- I can recognize that the range of an inverse is the domain of the original function
- I can understand that the inverse of a function is a reflection along the line y = x

Chapter 1 – Introduction to Functions

1.6 – 1.8: Transformations of Functions *(Part 1)*

Learning Goal: We are learning to apply combinations of transformations in a systematic order to sketch graphs of functions.

To **TRANSFORM** something is to change or move

TRANSFORMATIONS OF FUNCTIONS can be seen in two ways: algebraically, and graphically. We'll begin by examining transformations graphically.

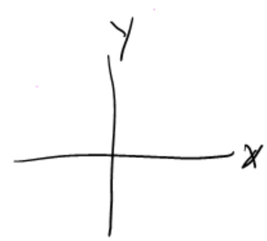
But before we do, we need to remember that the **GRAPH OF A FUNCTION**, $f(x)$, is given by:

$$f(x)=\left\{\underbrace{\left(x, f(x)\right)}_{\text{points}} \,\middle|\, x\in D_f\right\}$$

So, for functions we have two things (NUMBERS!) to "transform". We can apply transformations to → X

1) **Domain** values (which we call **HORIZONTAL TRANSFORMATIONS**)
2) **Range** values (which we call **VERTICAL TRANSFORMATIONS**)

↳ Y



There are **THREE BASIC FUNCTIONAL TRANSFORMATIONS**

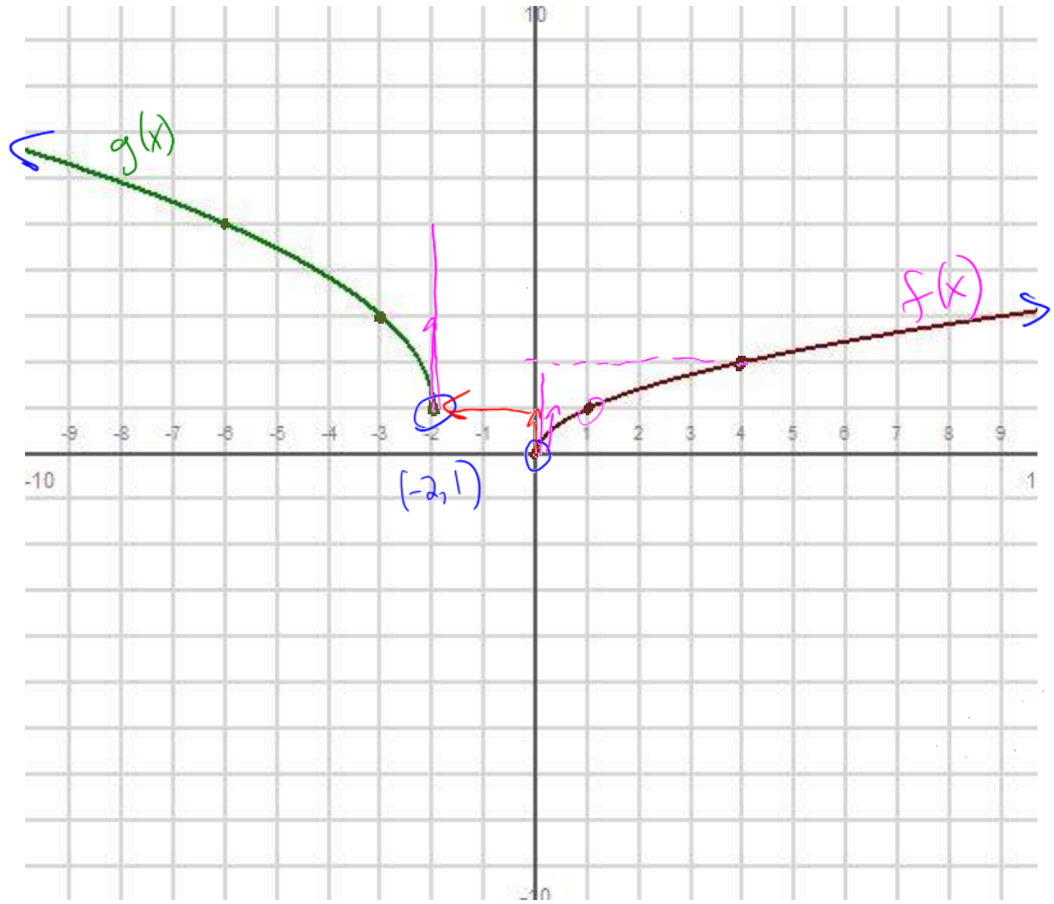
1) Flips (*Reflections "around" an axis*) → negatives on the stretches
2) Stretches (*Dilations*) multiplication
3) Shifts (*Translations*)
   addition/subtraction

So, we can have **Horizontal** flips, stretches and/or shifts, and **Vertical** flips, stretches and/or shifts. Now let's take a look at how transformations can be applied to functions.

***Note*:** We'll (mostly) be applying transformations to our so-called "parent functions" (although applying transformations to linear functions can seem pretty silly!)

Example 1.8.1

Consider, and make observations concerning the sketch of the graph of the parent function $f(x)=\sqrt{x}$ and the transformed function $g(x)=2\sqrt{-x-2}+1$.



Horizontal Transformations

Shift left two

flipped

Vertical Transformations

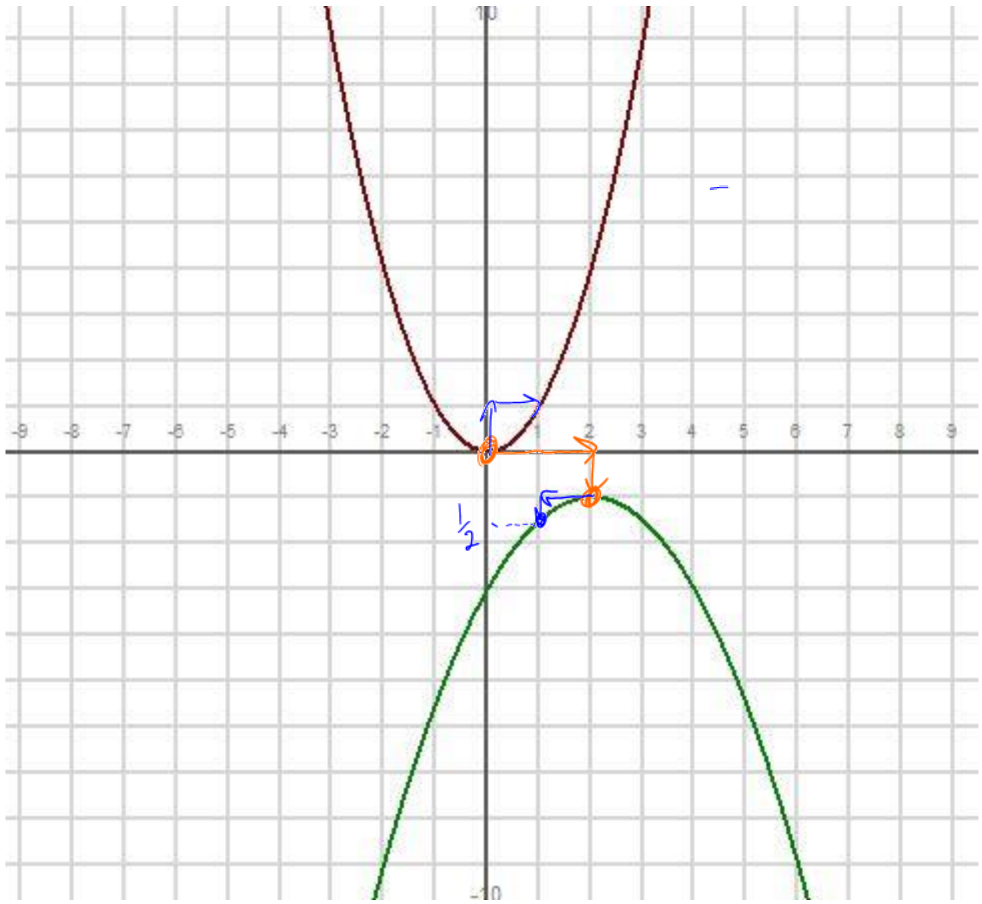
Shift up one

Stretch of two.

Note: In the above example we can **algebraically** describe $g(x)$ as a transformed $f(x)$ with the functional equation $g(x)=2f(-x-2)+1$

Example 1.8.2

Consider, and make observations concerning the sketch of the graph of the parent function $f(x)=x^2$ and the transformed function $g(x)=-\boxed{\frac{1}{2}}(x-2)^2-1$



Horizontal Transformations

right two

Vertical Transformations

down one

flipped

Note: In the above example we can **algebraically** describe $g(x)$ as a transformed $f(x)$ with the functional equation

$$g(x)=-\frac{1}{2}f(x-2)-1$$

Chapter 1 – Introduction to Functions

1.6 – 1.8: Transformations of Functions *(Part 2)*

We now turn to examining Transformations of Functions from an algebraic point of view (although a geometric perspective will still shine though!)

$(x-3)$ $d=3$ $(x+d)$

$(x+4)$ $d=-4$ $(x-(-d))$

Definition 1.8.1

Given a function $f(x)$ we can obtain a related function through functional transformations as

the function → must be factored away from the x and d.

$$g(x)=af\left(k(x-d)\right)+c$$, where

Horizontals are inside

$k \Rightarrow$ H. stretch of $\frac{1}{k}$ and the H. flip

→ multiply the x's

do opposites

$d =$ H. Shift (the d comes after the negative sign)

→ add/subtract the x's

Verticals happen outside the function

$a =$ V. Stretch and V. flip

→ multiply with the y's

$c =$ V. Shift

→ add or subtract to the y's

Example 1.8.3

Consider the given function. State its parent function, and all transformations.

$$f(x)=3\sqrt{-x+2}-1$$

$-|x+2$

The parent function is $f(x)=\sqrt{x}$

$f(x)=\underset{a}{3}\sqrt{\underset{k}{-1}(x-\boxed{2})}\underset{c}{-1}$ Proper Form

Horizontal Transformations

↳ inside square root

Horizontal Stretch of $\frac{1}{1}$ or 1

(no stretch)

Flip

Shift of 2

2 right

Vertical Transformations

Stretch of 3

No flip

Shift of -1

or 1 down

Example 1.8.4

The basic absolute value function $f(x)=|x|$ has the following transformations applied to it: **Vertical Stretch** $\overset{a}{-3}$, **Vertical Shift** $\boxed{\overset{c}{1\text{ up}}}$, **Horizontal Shift** $\boxed{5\text{ right}}$. -5
d

Determine the equation of the transformed function.

$$f(x)=a|k(x-d)|+c$$

$$f(x)=-3|x-5|+1$$

Back to a geometric point of view

Sketching the graph of a transformed function can be relatively easy if we know:

1) The shape of the parent function AND a few (3 or 4) points on the parent.
2) How transformations affect the points on the parent
   i) Horizontal transformations affect the domain values (***OPPOSITE!!!!!!***)
   ii) Vertical transformations affect the range values

Note: Given a point on some parent function which has transformations applied to it is called an ***IMAGE POINT*** on the transformed function.

Example 1.8.5

Given the sketch of the function $f(x)$ determine the image points of the transformed function $\overset{a}{-2}f\left(\overset{k}{\frac{1}{3}}(x+\overset{d}{1})\right)+\overset{c}{3}$ and sketch the graph of the transformed function.

V. Stretch of 2 and a flip

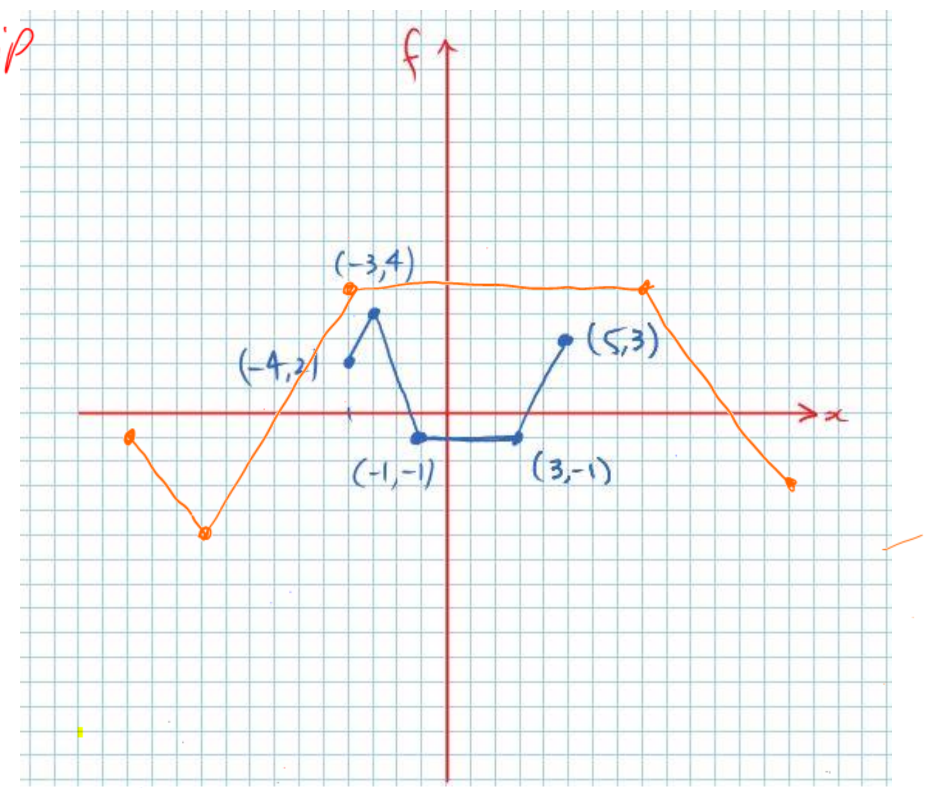
V. Shift of + 3

H. Stretch of 3

H. Shift of −1

Parent

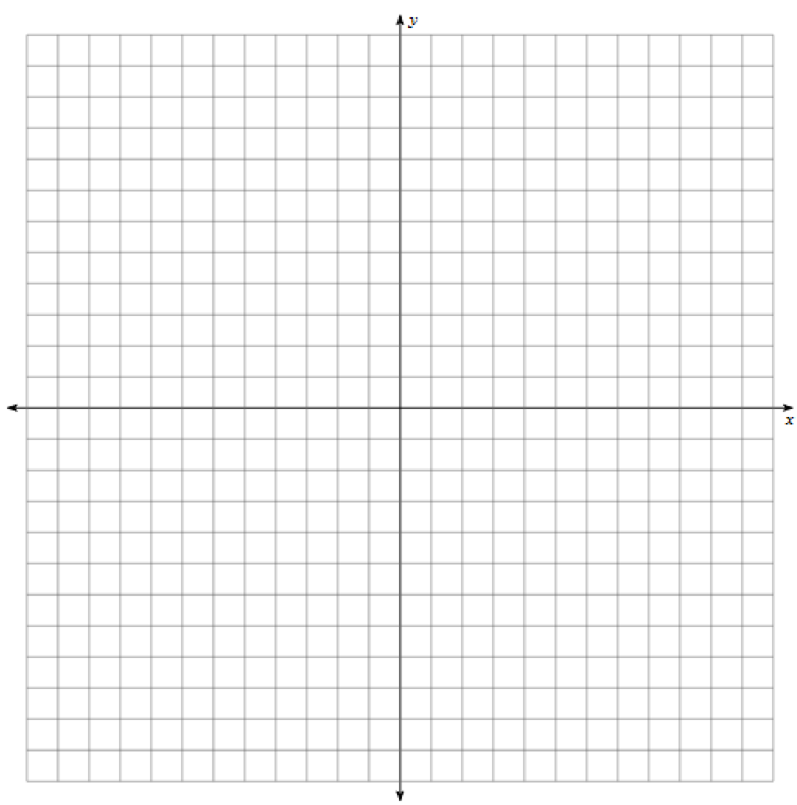
x	y	3x−1	−2y+3
−4	2	(−13	−1)
−3	4	(−10	−5)
−1	−1	(−4	5)
3	−1	(8	5)
5	3	(14	−3)



Function	Proper Function $f(x)=a\,f\left(k(x-d)\right)+c$	Vertical Stretch ***a***	Horizontal Stretch ***1/k***	Horizontal Shift ***d***	Vertical Shift ***c***
$e(x)=2\sqrt{-3x+12}-6$					

Domain		Range		y-int (x=0)	

Table Of Values	Parent Function:		Transformed Function	



Extra work space.

Success Criteria:

- I can use the value of a to determine if there is a vertical stretch/reflection in the x-axis
- I can use the value of k to determine if there is a horizontal stretch/reflection in the y-axis
- I can use the value of d to determine if there is a horizontal translation
- I can use the value of c to determine if there is a vertical translation
- I can transform x coordinates by using the expression $\frac{1}{k}x + d$
- I can transform y coordinates by using the expression $ay + c$