Chapter 5

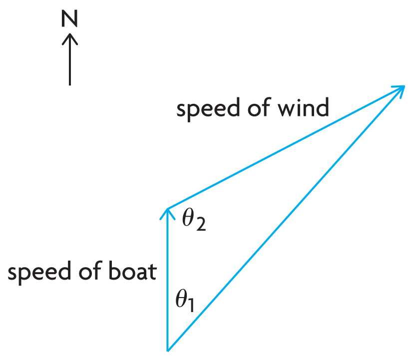
Trigonometric Ratios

GOALS

You will be able to

- Relate the six trigonometric ratios to the unit circle
- Solve real-life problems by using trigonometric ratios, properties of triangles, and the sine and cosine laws
- Prove simple trigonometric identities

? How would changes in the boat's speed and the wind's speed affect the angles in the vector diagram and the speed and direction of the boat?



5 Getting Started

SKILLS AND CONCEPTS You Need

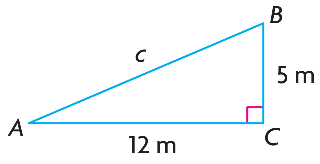
Study ***Aid***

- For help, see Essential Skills Appendix.

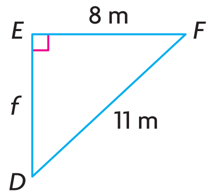
Question	Appendix
1	A-4
2–7	A-16
8	A-17

1. Use the Pythagorean theorem to determine each unknown side length.

a)



b)

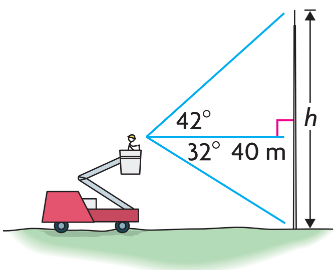


2. Using the triangles in question 1, determine the sine, cosine, and tangent ratios for each given angle.
   a) $\angle A$ **b)** $\angle D$

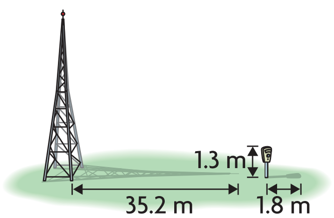
3. Using the triangles in question 1, determine each given angle to the nearest degree.
   a) $\angle B$ **b)** $\angle F$

4. Use a calculator to evaluate to the nearest thousandth.
   a) $\sin 31°$ **b)** $\cos 70°$

5. Use a calculator to determine θ to the nearest degree.
   a) $\cos \theta = 0.3312$ **b)** $\sin \theta = 0.7113$ **c)** $\tan \theta = 1.1145$

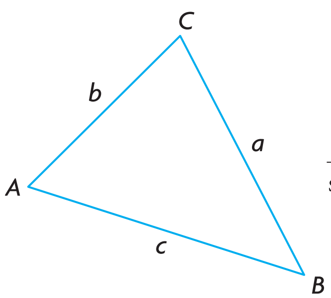


6. Mario is repairing the wires on a radio broadcast tower. He is in the basket of a repair truck 40 m from the tower. When he looks up, he estimates the **angle of elevation** to the top of the tower as 42°. When he looks down, he estimates the **angle of depression** to the bottom of the tower as 32°. How high is the tower to the nearest metre?



7. On a sunny day, a tower casts a shadow 35.2 m long. At the same time, a 1.3 m parking meter that is nearby casts a shadow 1.8 m long. How high is the tower to the nearest tenth of a metre?

8. The **sine law** states that in any triangle, the side lengths are proportional to the sines of the opposite angles.



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use a graphic organizer to show how to use the **sine law** to calculate an unknown angle.

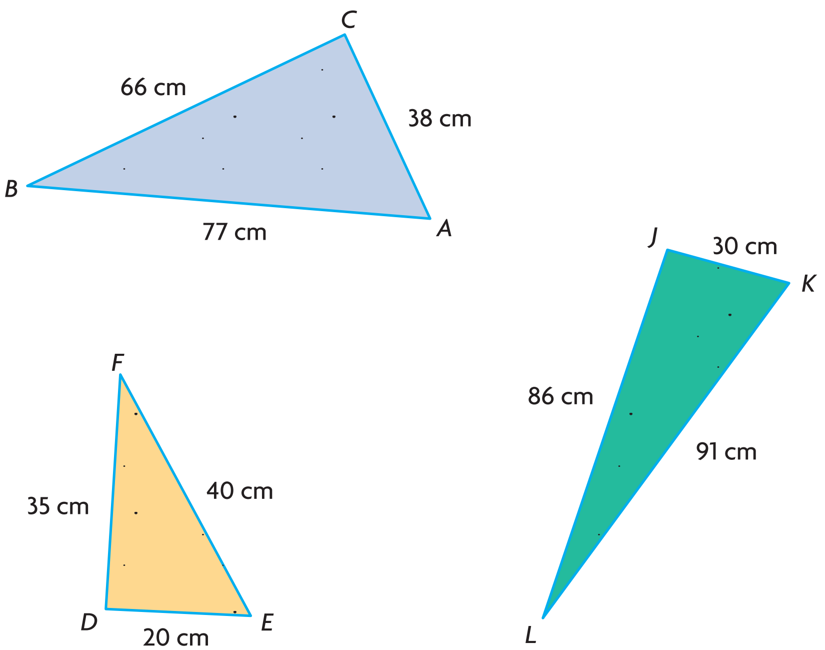
Tech ***Support***

For help using the inverse trigonometric keys on a graphing calculator, see Technical Appendix, B-13.

APPLYING What You Know

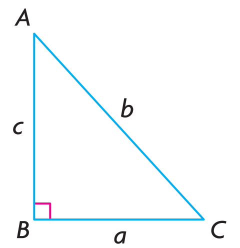
Finding a Right-Angled Triangle

Raymond and Alyssa are covering a patio with triangular pieces of stone tile. They need one tile that has a right angle for the corner of the patio. They don't have a protractor, so they use a tape measure to measure the side lengths of each triangle. The measurements are shown.



Communication *Tip*

It is common practice to label the vertices of a triangle with upper case letters. The side opposite each angle is labelled with the lower case letter corresponding to that angle.



? Which of these triangles can be used for the corner of the patio?

A. In $\triangle ABC$, which angle is most likely a right angle? Justify your decision.

B. Assuming that $\triangle ABC$ is a right triangle, write down the mathematical relationship that relates the three sides.

C. Check to see if $\triangle ABC$ is a right triangle by evaluating each side of the relationship you wrote in part B. Compare both sides.

D. Is $\triangle ABC$ a right triangle? Justify your decision.

E. Repeat parts A to D for the remaining triangles.

F. Which triangular stone would you use for the corner of the patio? Justify your decision.

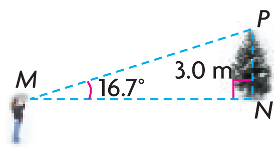
5.1 Trigonometric Ratios of Acute Angles

GOAL

Evaluate reciprocal trigonometric ratios.

LEARN ABOUT the Math

From a position some distance away from the base of a tree, Monique uses a clinometer to determine the angle of elevation to a treetop. Monique estimates that the height of the tree is about 3.0 m.



? How far, to the nearest tenth of a metre, is Monique from the base of the tree?

Communication *Tip*

A clinometer is a device used to measure the angle of elevation (above the horizontal) or the angle of depression (below the horizontal).

EXAMPLE 1 Selecting a strategy to determine a side length in a right triangle

In $\triangle MNP$, determine the length of MN.

Clive's Solution: Using Primary Trigonometric Ratios

$$\tan 16.7° = \frac{3.0}{MN}$$

I knew the opposite side but I needed to calculate the adjacent side *MN*. So I used tangent.

$$MN(\tan 16.7°) = 3.0$$

$$MN = \frac{3.0}{\tan 16.7°}$$

I multiplied both sides of the equation by *MN*, then divided by tan 16.7°.

I used my calculator to evaluate.

$$MN \doteq 10.0 \text{ m}$$

Monique is about 10.0 m away from the base of the tree.

Communication *Tip*

The symbol ≐ means "approximately equal to" and indicates that a result has been rounded.

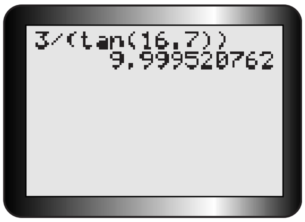
Tony's Solution: Using Reciprocal Trigonometric Ratios

$$\cot 16.7° = \frac{MN}{3.0}$$

NP is opposite the 16.7° angle, and *MN* is adjacent. I used the **reciprocal trigonometric ratio** cot 16.7°. This gave me an equation with the unknown in the numerator, making the equation easier to solve.

$$(3.0)\cot 16.7° = MN$$

To solve for *MN*, I multiplied both sides by 3.0.



I evaluated $\frac{1}{\tan 16.7°}$ to get cot 16.7°.

$$10.0 \text{ m} \doteq MN$$

Monique is about 10.0 m away from the base of the tree.

reciprocal trigonometric ratios

the reciprocal ratios are defined as 1 divided by each of the primary trigonometric ratios

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent}}{\text{opposite}}$$

Cot θ is the short form for the cotangent of angle θ, sec θ is the short form for the secant of angle θ, and csc θ is the short form for the cosecant of angle θ.

Tech *Support*

Most calculators do not have buttons for evaluating the reciprocal ratios. For example, to evaluate

- csc 20°, use $\frac{1}{\sin 20°}$
- sec 20°, use $\frac{1}{\cos 20°}$
- cot 20°, use $\frac{1}{\tan 20°}$

Reflecting

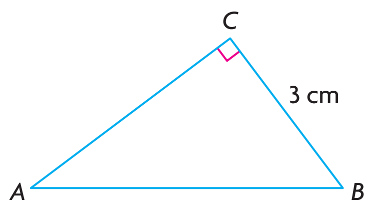
A. What was the advantage of using a reciprocal trigonometric ratio in Tony's solution?

B. Suppose Monique wants to calculate the length of *MP* in $\triangle MNP$. State the two trigonometric ratios that she could use based on the given information. Which one would be better? Explain.

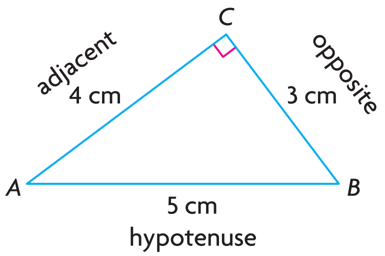
APPLY the Math

EXAMPLE 2 Evaluating the six trigonometric ratios of an angle

$\triangle ABC$ is a right triangle with side lengths of 3 cm, 4 cm, and 5 cm. If $CB = 3$ cm and $\angle C = 90°$, which trigonometric ratio of $\angle A$ is the greatest?



Sam's Solution



I labelled the sides of the triangle relative to ∠*A*, first in words and then with the side lengths. The hypotenuse is the longest side, so its length must be 5 cm. If the side opposite ∠*A* is 3 cm, then the side adjacent to ∠*A* is 4 cm.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.60 \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.80 \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

First, I used the definitions of the primary trigonometric ratios to determine the sine, cosine, and tangent of ∠*A*.

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \doteq 1.67 \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} = 1.25 \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3} \doteq 1.33$$

Then I evaluated the reciprocal trigonometric ratios for ∠*A*. I wrote the reciprocal of each primary ratio to get the appropriate reciprocal ratio.

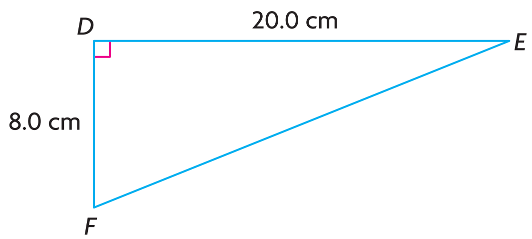
I expressed these ratios as decimals to compare them more easily.

The greatest trigonometric ratio of ∠*A* is csc *A*.

EXAMPLE 3 Solving a right triangle by calculating the unknown side and the unknown angles

a) Determine *EF* in △*DEF* to the nearest tenth of a centimetre.

b) Express one unknown angle in terms of a primary trigonometric ratio and the other angle in terms of a reciprocal ratio. Then calculate the unknown angles to the nearest degree.



Lina's Solution

a) $EF^2 = (8.0)^2 + (20.0)^2$

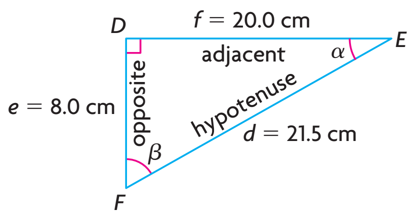
$EF^2 = 464.0 \text{ cm}^2$

$EF = \sqrt{464.0}$

$EF \doteq 21.5 \text{ cm}$

Since $\triangle DEF$ is a right triangle, I used the Pythagorean theorem to calculate the length of *EF*.

b)



I labelled $\angle E$ as α. Side *e* is opposite α and *f* is adjacent to α. So I expressed α in terms of the primary trigonometric ratio $\tan \alpha$.

I labelled $\angle F$ as β. Side *d* is the hypotenuse and *e* is adjacent to β.

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} \qquad \sec \beta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$= \frac{e}{f} \qquad = \frac{d}{e}$$

$$= \frac{8.0}{20.0} \qquad = \frac{21.5}{8.0}$$

$$= 0.40 \qquad \doteq 2.69$$

I expressed β in terms of the reciprocal trigonometric ratio $\sec \beta$.

$\alpha = \tan^{-1}(0.40)$

$\alpha \doteq 22°$

To determine angle α, I used my calculator to evaluate $\tan^{-1}(0.40)$ directly.

$\sec \beta \doteq 2.69$

$\cos \beta \doteq \frac{1}{2.69}$

$\beta \doteq \cos^{-1}\left(\frac{1}{2.69}\right)$

Since my calculator doesn't have a $\sec^{-1}$ key, I wrote $\sec \beta$ in terms of the primary trigonometric ratio $\cos \beta$ before determining β.

$\beta \doteq 68°$

I determined angle β directly by evaluating $\cos^{-1}\left(\frac{1}{2.69}\right)$ with my calculator.

EF is about 21.5 cm long, and $\angle E$ and $\angle F$ are about 22° and 68°, respectively.

Communication *Tip*

Unknown angles are often labelled with the Greek letters θ (theta), α (alpha), and β (beta).

Communication *Tip*

Arcsine ($\sin^{-1}$), arccosine ($\cos^{-1}$), and arctangent ($\tan^{-1}$) are the names given to the inverse trigonometric functions. These are used to determine the angle associated with a given primary ratio.

In Summary

Key Idea

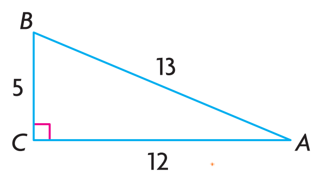
- The reciprocal trigonometric ratios are reciprocals of the primary trigonometric ratios, and are defined as 1 divided by each of the primary trigonometric ratios:
  - $\csc \theta = \dfrac{1}{\sin \theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
  - $\sec \theta = \dfrac{1}{\cos \theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
  - $\cot \theta = \dfrac{1}{\tan \theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

Need to Know

- In solving problems, reciprocal trigonometric ratios are sometimes helpful because the unknown variable can be expressed in the numerator, making calculations easier.
- Calculators don't have buttons for cosecant, secant, or cotangent ratios.
- The sine and cosine ratios for an acute angle in a right triangle are less than or equal to 1 so their reciprocal ratios, cosecant and secant, are always greater than or equal to 1.
- The tangent ratio for an acute angle in a right triangle can be less than 1, equal to 1, or greater than 1, so the reciprocal ratio, cotangent, can take on this same range of values.

CHECK Your Understanding

1. Given $\triangle ABC$, state the six trigonometric ratios for $\angle A$.



2. State the reciprocal trigonometric ratios that correspond to $\sin \theta = \frac{8}{17}$, $\cos \theta = \frac{15}{17}$, and $\tan \theta = \frac{8}{15}$.

3. For each primary trigonometric ratio, determine the corresponding reciprocal ratio.

a) $\sin \theta = \dfrac{1}{2}$

b) $\cos \theta = \dfrac{3}{4}$

c) $\tan \theta = \dfrac{3}{2}$

d) $\tan \theta = \dfrac{1}{4}$

4. Evaluate to the nearest hundredth.

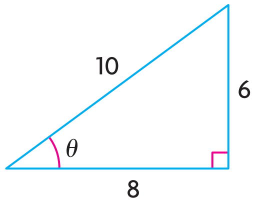
a) $\cos 34°$ **b)** $\sec 10°$ **c)** $\cot 75°$ **d)** $\csc 45°$

PRACTISING

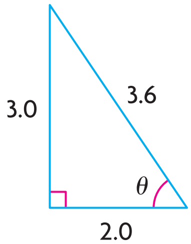
5. **a)** For each triangle, calculate csc θ, sec θ, and cot θ.

K **b)** For each triangle, use one of the reciprocal ratios from part (a) to determine θ to the nearest degree.

i)



iii)



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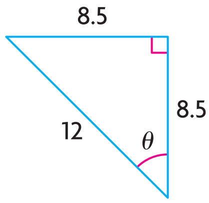

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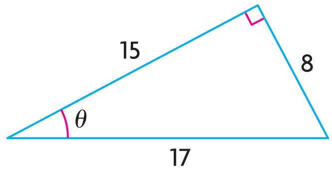


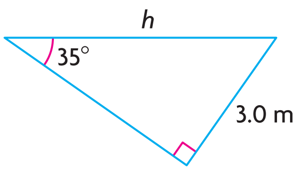
6. Determine the value of θ to the nearest degree.

a) $\cot \theta = 3.2404$ **c)** $\sec \theta = 1.4526$

b) $\csc \theta = 1.2711$ **d)** $\cot \theta = 0.5814$

7. For each triangle, determine the length of the hypotenuse to the nearest tenth of a metre.

a)



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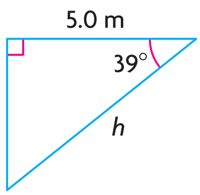
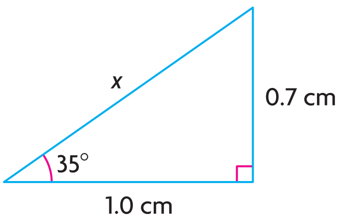


8. For each triangle, use two different methods to determine x to the nearest tenth of a unit.

a)



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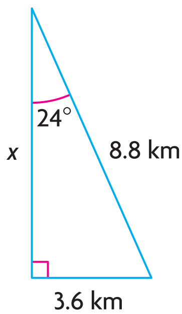
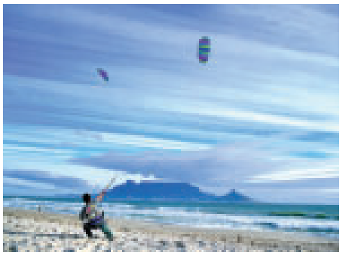


9. Given any right triangle with an acute angle θ,

a) explain why csc θ is always greater than or equal to 1

b) explain why cos θ is always less than or equal to 1

10. Given a right triangle with an acute angle θ, if $\tan\ \theta = \cot\ \theta$, describe what this triangle would look like.



11. **A** A kite is flying 8.6 m above the ground at an angle of elevation of 41°. Calculate the length of string, to the nearest tenth of a metre, needed to fly the kite using

a) a primary trigonometric ratio

b) a reciprocal trigonometric ratio

12. A wheelchair ramp near the door of a building has an incline of 15° and a run of 7.11 m from the door. Calculate the length of the ramp to the nearest hundredth of a metre.

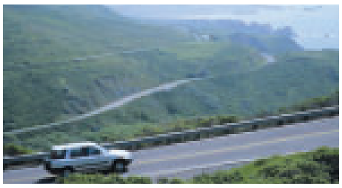
13. **T** The hypotenuse, c, of right $\triangle ABC$ is 7.0 cm long. A trigonometric ratio for angle A is given for four different triangles. Which of these triangles has the greatest area? Justify your decision.

a) $\sec A = 1.7105$ **c)** $\csc A = 2.2703$

b) $\cos A = 0.7512$ **d)** $\sin A = 0.1515$

14. The two guy wires supporting an 8.5 m TV antenna each form an angle of 55° with the ground. The wires are attached to the antenna 3.71 m above ground. Using a reciprocal trigonometric ratio, calculate the length of each wire to the nearest tenth of a metre. What assumption did you make?

15. From a position some distance away from the base of a flagpole, Julie estimates that the pole is 5.35 m tall at an angle of elevation of 25°. If Julie is 1.55 m tall, use a reciprocal trigonometric ratio to calculate how far she is from the base of the flagpole, to the nearest hundredth of a metre.



16. The maximum grade (slope) allowed for highways in Ontario is 12%.

a) Predict the angle θ, to the nearest degree, associated with this slope.

b) Calculate the value of θ to the nearest degree.

c) Determine the six trigonometric ratios for angle θ.

17. **C** Organize these terms in a word web, including explanations where appropriate.

sine	cosine	tangent	opposite
cotangent	hypotenuse	cosecant	adjacent
secant	angle of depression	angle	angle of elevation

Extending

18. In right $\triangle PQR$, the hypotenuse, r, is 117 cm and $\tan P = 0.51$. Calculate side lengths p and q to the nearest centimetre and all three interior angles to the nearest degree.

19. Describe the appearance of a triangle that has a secant ratio that is greater than any other trigonometric ratio.

20. The tangent ratio is undefined for angles whose adjacent side is equal to zero. List all the angles between 0° and 90° (if any) for which cosecant, secant, and cotangent are undefined.

5.2 Evaluating Trigonometric Ratios for Special Angles

GOAL

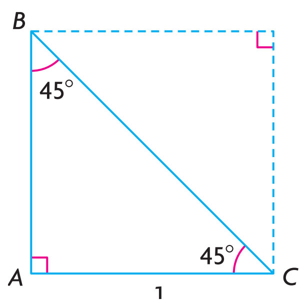
Evaluate exact values of sine, cosine, and tangent for specific angles.

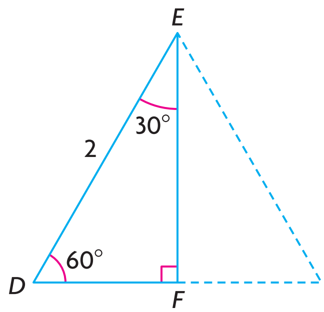
YOU WILL NEED

- ruler
- protractor

LEARN ABOUT the Math

The diagonal of a square of side length 1 unit creates two congruent right isosceles triangles. The height of an equilateral triangle of side length 2 units creates two congruent right scalene triangles.



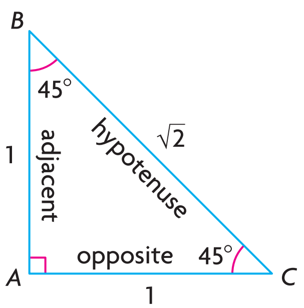


? How can isosceles $\triangle ABC$ and scalene $\triangle DEF$ be used to determine the exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles?

EXAMPLE 1 Evaluating exact values of the trigonometric ratios for a 45° angle

Use $\triangle ABC$ to calculate exact values of sine, cosine, and tangent for 45°.

Carol's Solution



$$BC^2 = AB^2 + AC^2$$
$$BC^2 = 1^2 + 1^2$$
$$BC^2 = 2$$
$$BC = \sqrt{2}$$

I labelled the sides of the triangle relative to $\angle B$. The triangle is isosceles with equal sides of length 1. I used the Pythagorean theorem to calculate the length of the hypotenuse.

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for ∠*B*.

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{1}{1}$$

$$= \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = 1$$

If I multiplied both the numerator and denominator by $\sqrt{2}$, I would get an equivalent number with a whole-number denominator.

$$= \frac{\sqrt{2}}{2} \qquad = \frac{\sqrt{2}}{2}$$

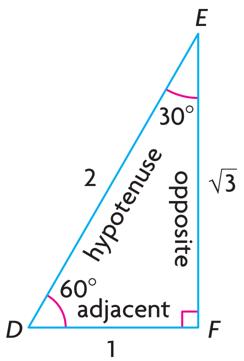
This would be an easier number to use to estimate the size, since I knew that $\sqrt{2}$ is about 1.4, so half of it is about 0.7.

The exact values of sine, cosine, and tangent for 45° are $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, and 1, respectively.

EXAMPLE 2 Evaluating exact values of the trigonometric ratios for 30° and 60° angles

Use △*DEF* to calculate exact values of sine, cosine, and tangent for 30° and 60°.

Trevor's Solution



$$DE^2 = DF^2 + EF^2$$
$$2^2 = 1^2 + EF^2$$
$$4 = 1 + EF^2$$
$$3 = EF^2$$
$$\sqrt{3} = EF$$

I labelled the sides of the triangle relative to ∠*D*. Since the height of an equilateral triangle divides the triangle into two smaller identical triangles, *DF* is equal to $\frac{1}{2}$ *DE*. So *DF* must be 1. I used the Pythagorean theorem to calculate the length of *EF*.

$$\sin D = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos D = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan D = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for ∠*D*.

$$\sin D = \frac{EF}{DE} \qquad \cos D = \frac{DF}{DE} \qquad \tan D = \frac{EF}{DF}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1}$$

$$= \sqrt{3}$$

$$\sin E = \frac{DF}{DE} \qquad \cos E = \frac{EF}{DE} \qquad \tan E = \frac{DF}{EF}$$

I wrote the primary trigonometric ratios for ∠*E* in terms of the sides of the triangle.

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

$$= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

If I multiplied both the numerator and denominator by $\sqrt{3}$, I would get an equivalent number with a whole-number denominator. This is an easier number to estimate, since $\sqrt{3}$ is about 1.7, so a third of it is about 0.57.

$\sin E = \cos D$ $\cos E = \sin D$ $\tan E = \cot D$

$\sin 30° = \cos 60°$ $\cos 30° = \sin 60°$ $\tan 30° = \cot 60°$

I noticed that sin *E* and cos *E* are equal to cos *D* and sin *D*, respectively. I also noticed that tan *E* is equal to the reciprocal of tan *D*.

The exact values of sine, cosine, and tangent for 30° are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{3}}{3}$, respectively and for 60° are $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, and $\sqrt{3}$, respectively.

Reflecting

A. In Example 1, would you get the same results if you used $\angle C$ for the 45° angle instead of $\angle B$? Explain.

B. Explain how sin 30° and cos 60° are related.

C. In Example 2, explain why the reciprocal ratios of tan 30° and cot 60° are equal.

D. How can remembering that a 30°–60°–90° triangle is half of an equilateral triangle and that a 45°–45°–90° triangle is isosceles help you recall the exact values of the primary trigonometric ratios for the angles in those triangles?

APPLY the Math

EXAMPLE 3 Determining the exact value of a trigonometric expression

Determine the exact value of $(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$.

Tina's Solution

$(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

I substituted the exact values of each trigonometric ratio.

$$= \frac{2}{4} + \frac{\sqrt{3}}{4}$$

I evaluated the expression by multiplying, then adding the numerators.

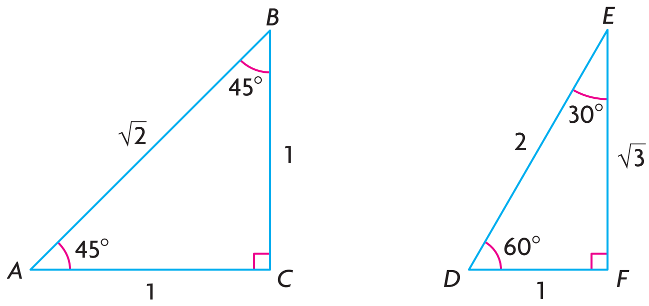
$$= \frac{2 + \sqrt{3}}{4}$$

The exact value is $\frac{2 + \sqrt{3}}{4}$.

In Summary

Key Idea

- The exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles can be found by using the appropriate ratios of sides in isosceles right triangles and half-equilateral triangles with right angles. These are often referred to as "special triangles."



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{\sqrt{3}}{3} \doteq 0.5774$
45°	$\frac{\sqrt{2}}{2} \doteq 0.7071$	$\frac{\sqrt{2}}{2} \doteq 0.7071$	1
60°	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{1}{2} = 0.5$	$\sqrt{3} \doteq 1.7321$

Need to Know

- Since tan 45° = 1, angles between 0° and 45° have tangent ratios that are less than 1, and angles between 45° and 90° have tangent ratios greater than 1.
- If a right triangle has one side that is half the length of the hypotenuse, the angle opposite that one side is always 30°.
- If a right triangle has two equal sides, then the angles opposite those sides are always 45°.

CHECK Your Understanding

1. a) Draw a right triangle that has one angle measuring 30°. Label the sides using the lengths $\sqrt{3}$, 2, and 1. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to the 30° angle.
c) Identify the adjacent and opposite sides relative to the 60° angle.

2. a) Draw a right triangle that has one angle measuring 45°. Label the sides using the lengths 1, 1, and $\sqrt{2}$. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to one of the 45° angles.

3. State the exact values.
a) sin 60° **b)** cos 30° **c)** tan 45° **d)** cos 45°

PRACTISING

4. Determine the exact value of each trigonometric expression.

K

a) $\sin 30° \times \tan 60° - \cos 30°$

b) $2 \cos 45° \times \sin 45°$

c) $\tan^2 30° - \cos^2 45°$

d) $1 - \dfrac{\sin 45°}{\cos 45°}$

5. Using exact values, show that $\sin^2 \theta + \cos^2 \theta = 1$ for each angle.

a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

6. Using exact values, show that $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$ for each angle.

a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

7. Using the appropriate special triangle, determine θ if $0° \le \theta \le 90°$.

a) $\sin \theta = \dfrac{\sqrt{3}}{2}$

b) $\sqrt{3} \tan \theta = 1$

c) $2\sqrt{2} \cos \theta = 2$

d) $2 \cos \theta = \sqrt{3}$

8. A 5 m stepladder propped against a classroom wall forms an angle of 30° with the wall. Exactly how far is the top of the ladder from the floor? Express your answer in radical form. What assumption did you make?

A

9. Show that $\tan 30° + \dfrac{1}{\tan 30°} = \dfrac{1}{\sin 30° \cos 30°}$.

10. A baseball diamond forms a square of side length 27.4 m. Sarah says that she used a special triangle to calculate the distance between home plate and second base.

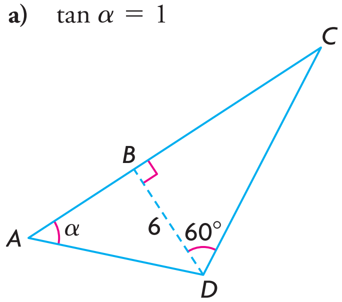
a) Describe how Sarah might calculate this distance.

b) Use Sarah's method to calculate this distance to the nearest tenth of a metre.

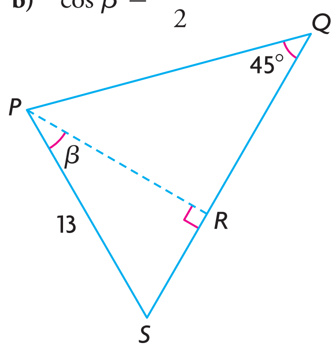
11. Determine the exact area of each large triangle.

T

a) $\tan \alpha = 1$



b) $\cos \beta = \dfrac{\sqrt{3}}{2}$



12. To claim a prize in a contest, the following skill-testing question was asked: Calculate $\sin 45°(1 - \cos 30°) + 5 \tan 60°(\sin 60° - \tan 30°)$.

C

a) Louise used a calculator to evaluate the expression. Determine her answer to three decimal places.

b) Megan used exact values. Determine her answer in radical form.

c) Only Megan received the prize. Explain why this might have occurred.

Communication Tip

$\tan^2 30° = (\tan 30°)(\tan 30°)$. The expression is squared, not the angle.

Extending

13. If $\cot \alpha = \sqrt{3}$, calculate $(\sin \alpha)(\cot \alpha) - \cos^2 \alpha$ exactly.

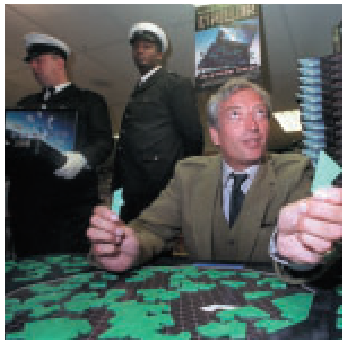
14. If $\csc \beta = 2$, calculate $\dfrac{\tan \beta}{\sec \beta} - \sin^2 \beta$ exactly.

15. Using exact values, show that $1 + \cot^2 \theta = \csc^2 \theta$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

Curious | Math

The Eternity Puzzle

Eternity, a puzzle created by Christopher Monckton, consists of 209 different pieces. Each piece is made up of twelve $30°-60°-90°$ triangles. The puzzle was introduced in Britain in June 1999, and the goal was to arrange the pieces into the shape of a dodecagon (12-sided polygon). Monckton provided six clues to solve his puzzle, and a £1 000 000 award (about $2 260 000 Canadian dollars) was offered for the first solution. It turned out that the puzzle didn't take an eternity to solve after all! Alex Selby and Oliver Riordan presented their solution on May 15, 2000, and collected the prize.

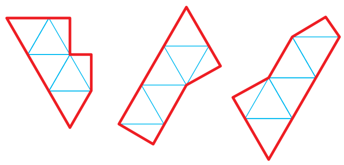


A second solution was found by Guenter Stertenbrink shortly afterwards. Interestingly, all three mathematicians ignored Monckton's clues and found their own answers. Monckton's solution remains unknown.

1. Consider the first three pieces of the Eternity puzzle. Each contains twelve $30°-60°-90°$ triangles. Suppose one such triangle has side lengths of 1, $\sqrt{3}$, and 2, respectively.

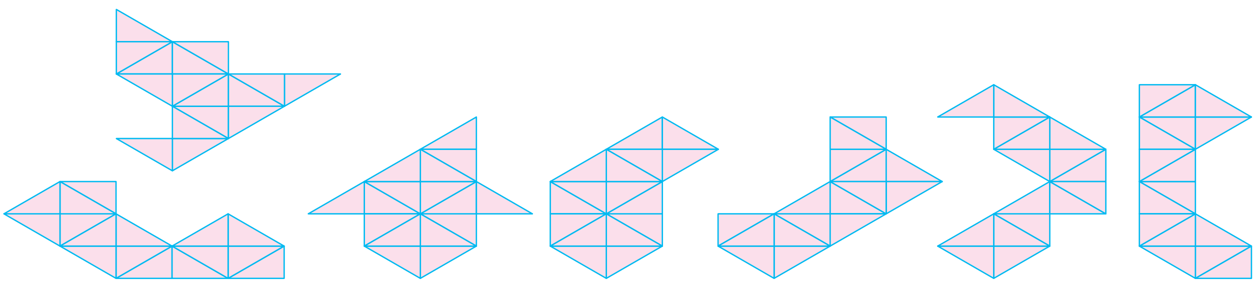
a) For each puzzle piece, determine the perimeter. Write your answer in radical form.

b) Calculate the area of each puzzle piece. Round your answer to the nearest tenth of a square unit.



piece 1 piece 2 piece 3

2. The seven puzzle pieces shown can be fit together to form a convex shape. Copy these pieces and see if you can find a solution.



5.3 Exploring Trigonometric Ratios for Angles Greater than 90°

GOAL

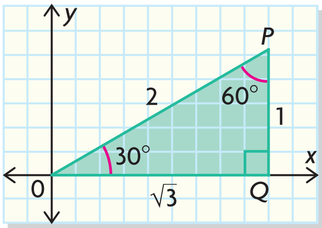
Explore relationships among angles that share related trigonometric ratios.

YOU WILL NEED

- graph paper
- dynamic geometry software (optional)

EXPLORE the Math

Raj is investigating trigonometric ratios of angles greater than 90°. He drew one of the special triangles on a Cartesian grid as shown.

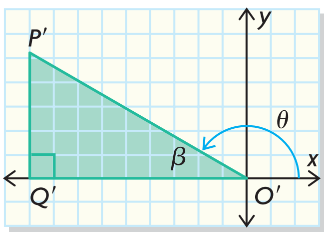


Next he performed a series of reflections in the y- and x-axes.

? Which angles in the Cartesian plane, if any, have primary trigonometric ratios related to those of a 30° angle?

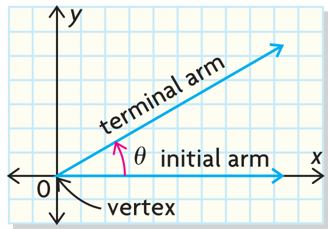
A. Use Raj's sketch of a 30° angle in standard position in the Cartesian plane to record the lengths of all sides and the primary trigonometric ratios for 30° to four decimal places.

B. Reflect the triangle from part A in the y-axis. $\angle P'O'Q'$ is now called the related acute angle β. What is its angle measure? What is the size of the principal angle θ and in which quadrant does the terminal arm lie?



standard position

an angle in the Cartesian plane whose vertex lies at the origin and whose initial arm (the arm that is fixed) lies on the positive x-axis. Angle θ is measured from the initial arm to the terminal arm (the arm that rotates).

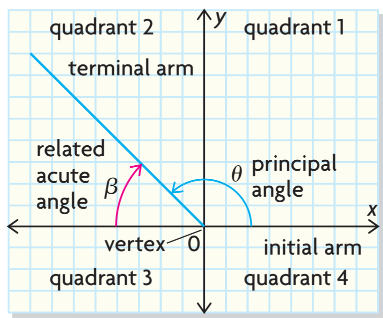


related acute angle

the acute angle between the terminal arm of an angle in standard position and the x-axis when the terminal arm lies in quadrants 2, 3, or 4

principal angle

the counterclockwise angle between the initial arm and the terminal arm of an angle in standard position. Its value is between 0° and 360°.

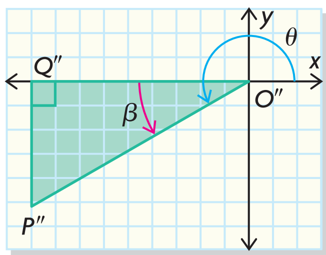


C. Use a calculator to determine the values of the primary trigonometric ratios for the principal angle and the related acute angle. Round your answers to four decimal places and record them in a table similar to the one shown.

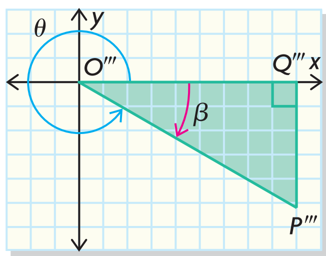
Angles	Quadrant	Sine Ratio	Cosine Ratio	Tangent Ratio
principal angle $\theta =$ _____				
related acute angle $\beta =$ _____				

How are the primary trigonometric ratios for the related acute angle related to the corresponding ratios for the principal angle?

D. Reflect the triangle from part B in the *x*-axis. What is the size of the related acute angle β? What is the size of the principal angle θ, and in which quadrant does the terminal arm lie? Use a calculator to complete your table for each of these angles. How are the primary trigonometric ratios for the related acute angle related to the corresponding ratios for the principal angle?

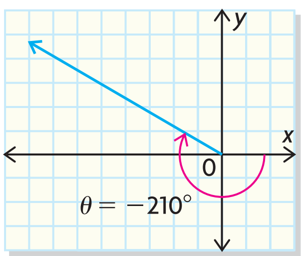


E. Repeat part D, but this time, reflect the triangle from part D in the *y*-axis.



negative angle

an angle measured *clockwise* from the positive *x*-axis



F. Repeat parts A to E, but this time start with a 45° and then a 60° angle in quadrant 1. Use negative angles for some of your trials.

G. Based on your observations, which principal angles and related acute angles in the Cartesian plane have the same primary trigonometric ratio?

Reflecting

H. **i)** When you reflect an acute principal angle θ in the y-axis, why is the resulting principal angle $180° - \theta$?

ii) When you reflect an acute principal angle θ in the y-axis and then in the x-axis, why is the resulting principal angle $180° + \theta$?

iii) When you reflect an acute principal angle θ in the x-axis, why is the resulting principal angle $360° - \theta$ (or $-\theta$)?

I. What does your table tell you about the relationships among the sine, cosine, and tangent of an acute principal angle and the resulting reflected principal angles?

J. How could you have predicted the relationships you described in part I?

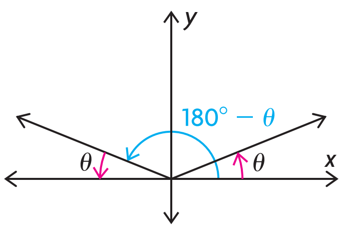
In Summary

Key Idea

- For any principal angle greater than 90°, the values of the primary trigonometric ratios are either the same as, or the negatives of, the ratios for the related acute angle. These relationships are based on angles in standard position in the Cartesian plane and depend on the quadrant in which the terminal arm of the angle lies.

Need to Know

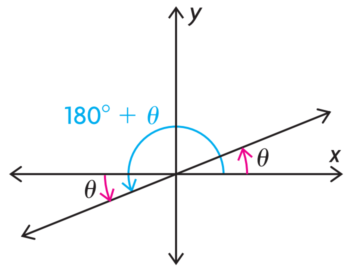
- An angle in the Cartesian plane is in standard position if its vertex lies at the origin and its initial arm lies on the positive x-axis.
- An angle in standard position is determined by a counterclockwise rotation and is always positive. An angle determined by a clockwise rotation is always negative.
- If the terminal arm of an angle in standard position lies in quadrants 2, 3, or 4, there exists a related acute angle and a principal angle.
- If θ is an acute angle in standard position, then
  - the terminal arm of the principal angle $(180° - \theta)$ lies in quadrant 2



$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

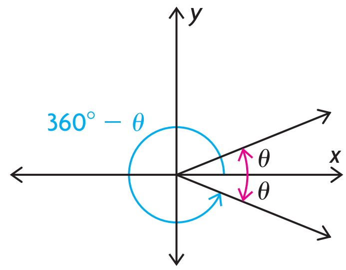
(*continued*)

- the terminal arm of the principal angle $(180° + \theta)$ lies in quadrant 3



$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

- the terminal arm of the principal angle $(360° - \theta)$ lies in quadrant 4



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = -\cos\theta$
$\tan(360° - \theta) = \tan\theta$

FURTHER Your Understanding

1. State all the angles between 0° and 360° that make each equation true.

a) $\sin 45° = \sin$ ______

b) $\cos$ ______ $= -\cos(-60°)$

c) $\tan 30° = \tan$ ______

d) $\tan 135° = -\tan$ ______

2. Using the special triangles from Lesson 5.2, sketch two angles in the Cartesian plane that have the same value for each given trigonometric ratio.

a) sine **b)** cosine **c)** tangent

3. Sylvie drew a special triangle in quadrant 3 and determined that $\tan(180° + \theta) = 1$.

a) What is the value of angle θ?

b) What would be the exact value of $\tan\theta$, $\cos\theta$, and $\sin\theta$?

4. Based on your observations, copy and complete the table below to summarize the signs of the trigonometric ratios for a principal angle that lies in each of the quadrants.

Trigonometric Ratio	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
sine	+			
cosine	+			
tangent	+			

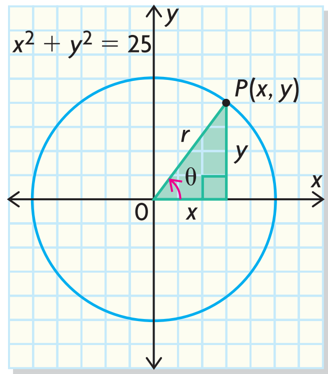
5.4 Evaluating Trigonometric Ratios for Any Angle Between 0° and 360°

GOAL

Use the Cartesian plane to evaluate the primary trigonometric ratios for angles between 0° and 360°.

YOU WILL NEED

- graph paper
- protractor
- dynamic geometry software (optional)



LEARN ABOUT the Math

Miriam knows that the equation of a circle of radius 5 centred at $(0, 0)$ is $x^2 + y^2 = 25$. She also knows that a point $P(x, y)$ on its circumference can rotate from 0° to 360°.

? For any point on the circumference of the circle, how can Miriam determine the size of the corresponding principal angle?

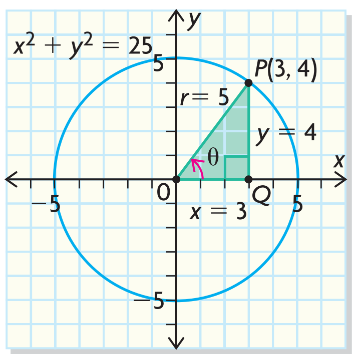
EXAMPLE 1 Relating trigonometric ratios to a point in quadrant 1 of the Cartesian plane

a) If Miriam chooses the point $P(3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle.

b) Determine the principal angle to the nearest degree.

Flavia's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. In $\triangle OPQ$, I noticed that the side opposite θ has length $y = 4$ units and the adjacent side has length $x = 3$ units. The hypotenuse is equal to the radius of the circle, so I labelled it r. In this case, $r = 5$ units. From the Pythagorean theorem, I also knew that $r^2 = x^2 + y^2$. Since r is the radius of the circle, it will always be positive.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{y}{r} \qquad = \frac{x}{r} \qquad = \frac{y}{x}$$

$$= \frac{4}{5} \qquad = \frac{3}{5} \qquad = \frac{4}{3}$$

I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

b) $\sin\theta = \frac{4}{5}$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 53°$$

The principal angle is about 53°.

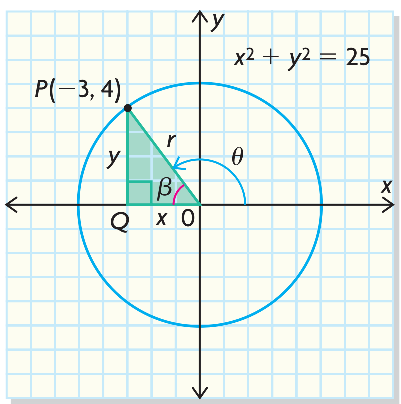
EXAMPLE 2 Relating trigonometric ratios to a point in quadrant 2 of the Cartesian plane

a) If Miriam chooses the point $P(-3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle to the nearest hundredth.

b) Determine the principal angle to the nearest degree.

Gabriel's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(-3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. Angle β is the related acute angle.

$$r^2 = x^2 + y^2$$
$$r^2 = 3^2 + 4^2$$
$$r^2 = 9 + 16$$
$$r^2 = 25$$
$$r = 5, \text{ since } r > 0$$

In $\triangle OPQ$, I knew that the lengths of the two perpendicular sides were $|x| = |-3| = 3$ and $y = 4$. The radius of the circle is still 5, so $r = 5$. I used the Pythagorean theorem to confirm this.

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{4}{5} \qquad \cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|x|}{r} = \frac{3}{5} \qquad \tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{|x|} = \frac{4}{3}$$

In $\triangle OPQ$, the side opposite β has length y and the adjacent side has length $|x|$. I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

$$\sin\theta = \sin\beta = \frac{4}{5} \qquad \cos\theta = -\cos\beta = -\frac{3}{5} \qquad \tan\theta = -\tan\beta = -\frac{4}{3}$$

Then I took into account the relationship among the trigonometric ratios of the related acute angle β and those of the principal angle θ. Since the terminal arm of angle θ lies in quadrant 2, the sine ratio is positive while the cosine and tangent ratios are negative.

b) $\sin\beta = \frac{4}{5}$

To determine angle β, I used a calculator to evaluate $\sin^{-1}\left(\frac{4}{5}\right)$ directly.

$$\beta = \sin^{-1}\left(\frac{4}{5}\right) \doteq 53°$$

$$\theta + \beta = 180°$$

I knew that θ and β add up to 180°. So I subtracted β from 180° to get θ.

$$\theta = 180° - \beta = 180° - 53° = 127°$$

The principal angle is about 127° because the related acute angle is about 53°.

Reflecting

A. In Example 2, explain why $\sin\theta = \sin\beta$, $\cos\theta \neq \cos\beta$, and $\tan\theta \neq \tan\beta$.

B. If Miriam chose the points $(-3, -4)$ and $(3, -4)$, what would each related acute angle be? How would the primary trigonometric ratios for the corresponding principal angles in these cases compare with those in Examples 1 and 2?

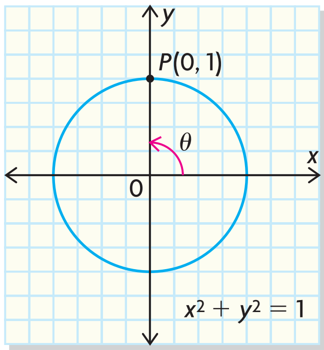
C. Given a point on the terminal arm of an angle in standard position, explain how the coordinates of that point vary from quadrants 1 to 4. How does this variation affect the size of the principal angle (and related acute angle, if it exists) and the values of the primary trigonometric ratios for that angle?

APPLY the Math

EXAMPLE 3 Determining the primary trigonometric ratios for a 90° angle

Use the point $P(0, 1)$ to determine the values of sine, cosine, and tangent for $90°$.

Charmaine's Solution



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(0, 1)$ on the circumference. Angle θ is the principal angle and is 90°.

In this case, I couldn't draw a right triangle by drawing a line perpendicular to the x-axis to P.

This meant that I couldn't use the trigonometric definitions in terms of opposite, adjacent, and hypotenuse.

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r} \qquad \tan \theta = \frac{y}{x}$$

$$= \frac{1}{1} \qquad = \frac{0}{1} \qquad = \frac{1}{0}$$

Since $P(0, 1)$, I knew that $x = 0$, $y = 1$, and $r = 1$.

I used the definitions of sine, cosine, and tangent in terms of x, y, and r to write each ratio.

$\sin 90° = 1$ $\quad \cos 90° = 0$ $\quad \tan 90°$ is undefined

Since $x = 0$ and it is in the denominator, $\tan 90°$ is undefined.

The point $P(0, 1)$ defines a principal angle of $90°$. The sine and cosine of $90°$ are 1 and 0, respectively. The tangent of $90°$ is undefined.

EXAMPLE 4 Determining all possible values of an angle with a specific trigonometric ratio

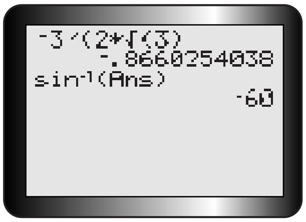
Determine the values of θ if $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$.

Jordan's Solution

$$\csc \theta = -\frac{2\sqrt{3}}{3}$$

$$\sin \theta = -\frac{3}{2\sqrt{3}}$$

Since $0° \leq \theta \leq 360°$, I had to use the Cartesian plane to determine θ. Cosecant is the reciprocal of sine. I found the reciprocal ratio by switching r and y. Since r is always positive, y must be -3 in this case. There were two cases where a point on the terminal arm has a negative y-coordinate: one in quadrant 3 and the other in quadrant 4.



I used my calculator to evaluate $\frac{-3}{2\sqrt{3}}$. Then I took the inverse sine of the result to determine the angle.

One angle is $-60°$, which is equivalent to $360° + (-60°) = 300°$ in quadrant 4.

The angle $-60°$ corresponds to a related acute angle of 60° of clockwise rotation and has its terminal arm in quadrant 4. I added 360° to $-60°$ to get the equivalent angle using a counterclockwise rotation.

In quadrant 3, the angle is $180° + 60° = 240°$.

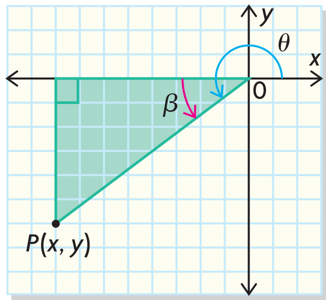
The angle in quadrant 3 must have a related acute angle of 60° as well. So I added 180° to 60° to determine the principal angle.

Given $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$, θ can be either 240° or 300°.

In Summary

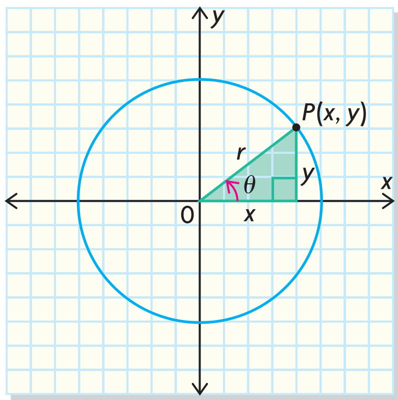
Key Idea

- The trigonometric ratios for any principal angle, θ, in standard position, where $0° \leq \theta \leq 360°$, can be determined by finding the related acute angle, β, using coordinates of any point $P(x, y)$ that lies on the terminal arm of the angle.



Need to Know

- For any point $P(x, y)$ in the Cartesian plane, the trigonometric ratios for angles in standard position can be expressed in terms of x, y, and r.



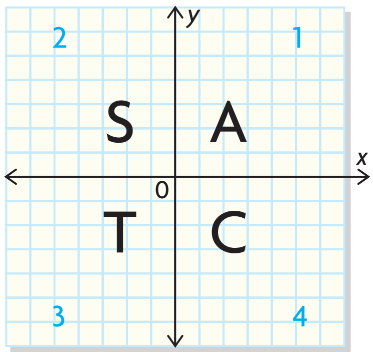
$r^2 = x^2 + y^2$ from the Pythagorean theorem and $r > 0$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

(continued)

- The CAST rule is an easy way to remember which primary trigonometric ratios are positive in which quadrant. Since *r* is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point.
  - In quadrant 1, **A**ll (A) ratios are positive because both *x* and *y* are positive.
  - In quadrant 2, only **S**ine (S) is positive, since *x* is negative and *y* is positive.
  - In quadrant 3, only **T**angent (T) is positive because both *x* and *y* are negative.
  - In quadrant 4, only **C**osine (C) is positive, since *x* is positive and *y* is negative.



CHECK Your Understanding

1. For each trigonometric ratio, use a sketch to determine in which quadrant the terminal arm of the principal angle lies, the value of the related acute angle β, and the sign of the ratio.

a) $\sin 315°$ **b)** $\tan 110°$ **c)** $\cos 285°$ **d)** $\tan 225°$

2. Each point lies on the terminal arm of angle θ in standard position.

i) Draw a sketch of each angle θ.
ii) Determine the value of r to the nearest tenth.
iii) Determine the primary trigonometric ratios for angle θ.
iv) Calculate the value of θ to the nearest degree.

a) $(5, 11)$ **b)** $(-8, 3)$ **c)** $(-5, -8)$ **d)** $(6, -8)$

3. Use the method in Example 3 to determine the primary trigonometric ratios for each given angle.

a) $180°$ **b)** $270°$ **c)** $360°$

4. Use the related acute angle to state an equivalent expression.

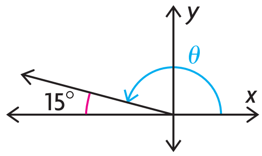
a) $\sin 160°$ **b)** $\cos 300°$ **c)** $\tan 110°$ **d)** $\sin 350°$

PRACTISING

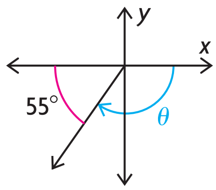
5. **i)** For each angle θ, predict which primary trigonometric ratios are positive.

ii) Determine the primary trigonometric ratios to the nearest hundredth.

a)



b)



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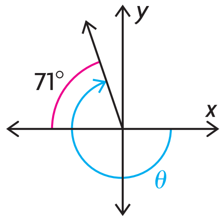

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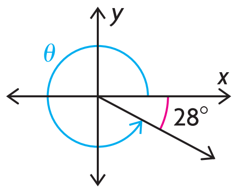


6. Angle θ is a principal angle that lies in quadrant 2 such that $0° \leq \theta \leq 360°$.

K Given each trigonometric ratio,

i) determine the exact values of x, y, and r

ii) sketch angle θ in standard position

iii) determine the principal angle θ and the related acute angle β to the nearest degree

a) $\sin \theta = \frac{1}{3}$

b) $\cot \theta = -\frac{4}{3}$

c) $\cos \theta = -\frac{1}{4}$

d) $\csc \theta = 2.5$

e) $\tan \theta = -1.1$

f) $\sec \theta = -3.5$

7. For each trigonometric ratio in question 6, determine the smallest negative angle that has the same ratio.

8. Use each trigonometric ratio to determine all values of θ, to the nearest degree if $0° \leq \theta \leq 360°$.

a) $\sin \theta = 0.4815$

b) $\tan \theta = -0.1623$

c) $\cos \theta = -0.8722$

d) $\cot \theta = 8.1516$

e) $\csc \theta = -2.3424$

f) $\sec \theta = 0$

9. Given angle θ, where $0° \leq \theta \leq 360°$, determine two possible values of θ where each ratio would be true. Sketch both principal angles.

a) $\cos \theta = 0.6951$
b) $\tan \theta = -0.7571$
c) $\sin \theta = 0.3154$
d) $\cos \theta = -0.2882$
e) $\tan \theta = 2.3151$
f) $\sin \theta = -0.7503$

10. Given each point $P(x, y)$ lying on the terminal arm of angle θ,

i) state the value of θ, using both a counterclockwise and a clockwise rotation
ii) determine the primary trigonometric ratios

a) $P(-1, -1)$
b) $P(0, -1)$
c) $P(-1, 0)$
d) $P(1, 0)$

11. **A** Dennis doesn't like using x, y, and r to investigate angles. He says that he is going to continue using adjacent, opposite, and hypotenuse to evaluate trigonometric ratios for any angle θ. Explain the weaknesses of his strategy.

12. Given $\cos \theta = -\frac{5}{12}$, where $0° \leq \theta \leq 360°$,

a) in which quadrant could the terminal arm of θ lie?
b) determine all possible primary trigonometric ratios for θ.
c) evaluate all possible values of θ to the nearest degree.

13. **T** Given angle α, where $0° \leq \alpha < 360°$, $\cos \alpha$ is equal to a unique value. Determine the value of α to the nearest degree. Justify your answer.

14. **C** How does knowing the coordinates of a point P in the Cartesian plane help you determine the trigonometric ratios associated with the angle formed by the x-axis and a ray drawn from the origin to P? Use an example in your explanation.

Extending

15. Given angle θ, where $0° \leq \theta \leq 360°$, solve for θ to the nearest degree.

a) $\cos 2\theta = 0.6420$
b) $\sin(\theta + 20°) = 0.2045$
c) $\tan(90° - 2\theta) = 1.6443$

16. When you use the inverse trigonometric functions on a calculator, it is important to interpret the calculator result to avoid inaccurate values of θ. Using these trigonometric ratios, describe what errors might result.

a) $\sin \theta = -0.8$
b) $\cos \theta = -0.75$

17. Use sketches to explain why each statement is true.

a) $2 \sin 32° \neq \sin 64°$
b) $\sin 20° + \sin 40° \neq \sin 60°$
c) $\tan 75° \neq 3 \tan 25°$

5 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

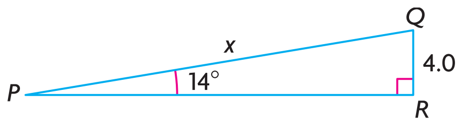
- See Lesson 5.1, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 1 to 5.

Q: Given any right triangle, how would you use a trigonometric ratio to determine an unknown side or angle?

A: You can use either a primary trigonometric ratio or a reciprocal trigonometric ratio. The ratio in which the unknown is in the numerator makes the equation easier to solve.

EXAMPLE

Determine x to the nearest tenth of a unit.



$$\csc 14° = \frac{x}{4.0} \quad \text{or} \quad \sin 14° = \frac{4.0}{x}$$

$$4.0 \csc 14° = x \qquad x \sin 14° = 4.0$$

$$16.5 \doteq x \qquad x = \frac{4.0}{\sin 14°}$$

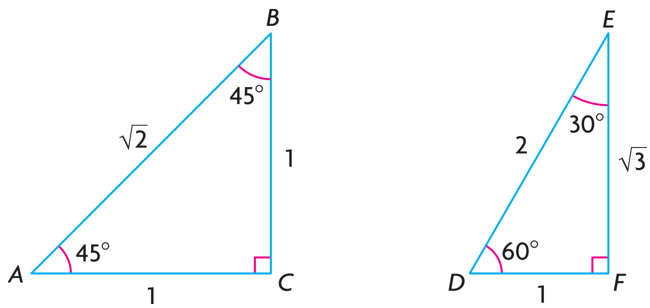
$$x \doteq 16.5$$

Study Aid

- See Lesson 5.2, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 6 and 7.

Q: What is significant about the trigonometric ratios for 45°–45°–90° and 30°–60°–90° right triangles?

A: The trigonometric ratios for 30°, 45°, and 60° can be determined exactly without using a calculator.



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{\sqrt{3}}{3} \doteq 0.5774$
45°	$\frac{\sqrt{2}}{2} \doteq 0.7071$	$\frac{\sqrt{2}}{2} \doteq 0.7071$	1
60°	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{1}{2} = 0.5$	$\sqrt{3} \doteq 1.7321$

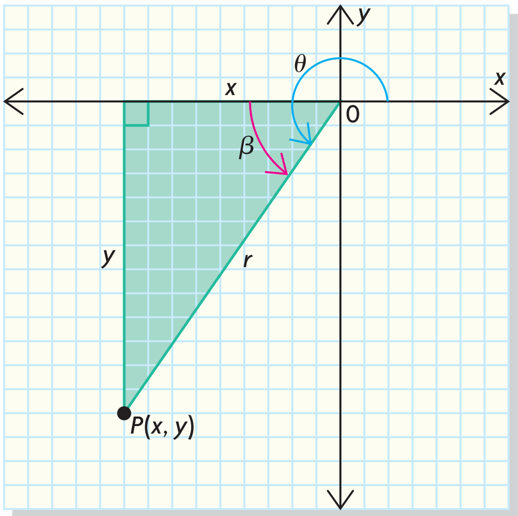
Q: How can you determine the trigonometric ratios for any angle θ, where $0° \leq \theta \leq 360°$?

A: Any angle in standard position in the Cartesian plane can be defined using the point $P(x, y)$, provided that P lies on the terminal arm of the angle. The trigonometric ratios can then be expressed in terms of x, y, and r, where r is the distance from the origin to P.

$$r^2 = x^2 + y^2 \text{ from the Pythagorean theorem and } r > 0$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$



Study *Aid*

- See Lesson 5.4, Examples 1 to 4.
- Try Mid-Chapter Review Questions 9 to 13.

Q: How can you determine all possible values of the principal angle θ in the Cartesian plane associated with a given trigonometric ratio?

A: Use the sign of the ratio to help you decide in which quadrant(s) the terminal arm of angle θ could lie. Then sketch the angle(s) in standard position. Use the appropriate inverse trigonometric function on your calculator to determine a value for θ. An angle in standard position is determined by a counterclockwise rotation and is always positive. A negative angle is determined by a clockwise rotation.

Interpret the calculator result in terms of your sketch, and determine the value of any related acute angle β. Use this value of β to determine all possible values of the principal angle θ.

Study *Aid*

- See Lesson 5.3 and Lesson 5.4, Example 4.
- Try Mid-Chapter Review Questions 10, 11, and 12.

PRACTICE Questions

Lesson 5.1

1. Evaluate each reciprocal trigonometric ratio to four decimal places.
   a) csc 20°
   b) sec 75°
   c) cot 10°
   d) csc 81°

2. Determine the value of θ to the nearest degree if $0° \leq \theta \leq 90°$.
   a) $\cot \theta = 0.8701$
   b) $\sec \theta = 4.1011$
   c) $\csc \theta = 1.6406$
   d) $\sec \theta = 2.4312$

3. A trigonometric ratio is $\frac{7}{5}$. What ratio could it be, and what angle might it be referring to?

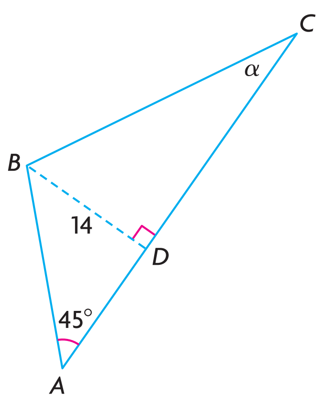
4. Claire is attaching a rope to the top of the mast of her sailboat so that she can lower the sail to the ground to do some repairs. The mast is 8.3 m long, and with her eyes level with the base of the mast, the top forms an angle of 31° with the ground. How much rope does Claire need if 0.5 m of rope is required to tie to the mast? Round your answer to the nearest tenth of a metre.

5. If $\csc \theta < \sec \theta$ and θ is acute, what do you know about θ?

Lesson 5.2

6. Determine the exact value of each trigonometric ratio.
   a) sin 60°
   b) tan 45°
   c) csc 30°
   d) sec 45°

7. Given $\triangle ABC$ as shown,



   a) determine the exact measure of each unknown side if $\sin \alpha = \frac{1}{2}$
   b) determine the exact values of the primary trigonometric ratios for $\angle A$ and $\angle DBC$

Lesson 5.3

8. i) Sketch each angle in standard position. Use the sketch to determine the exact value of the given trigonometric ratio.
   ii) If $0° \leq \theta \leq 360°$, state all values of θ that have the same given trigonometric ratio.
   a) sin 120°
   b) cos 225°
   c) tan 330°
   d) cos 300°

Lesson 5.4

9. $P(-9, 4)$ lies on the terminal arm of an angle in standard position.
   a) Sketch the principal angle θ.
   b) What is the value of the related acute angle β to the nearest degree?
   c) What is the value of the principal angle θ to the nearest degree?

10. Jeff said he found three angles for which $\cos \theta = \frac{4}{5}$. Is that possible if $0° \leq \theta \leq 360°$? Explain.

11. Given $\tan \theta = -\frac{15}{8}$, where $90° \leq \theta \leq 180°$,
    a) state the other five trigonometric ratios as fractions
    b) determine the value of θ to the nearest degree

12. If $\sin \theta = -0.8190$ and $0° \leq \theta \leq 360°$, determine the value of θ to the nearest degree.

13. Angle θ lies in quadrant 2. Without using a calculator, which ratios must be false? Justify your reasoning.
    a) $\cos \theta = 2.3151$
    b) $\tan \theta = 2.3151$
    c) $\sec \theta = 2.3151$
    d) $\csc \theta = 2.3151$
    e) $\cot \theta = 2.3151$
    f) $\sin \theta = 2.3151$

5.5 Trigonometric Identities

GOAL

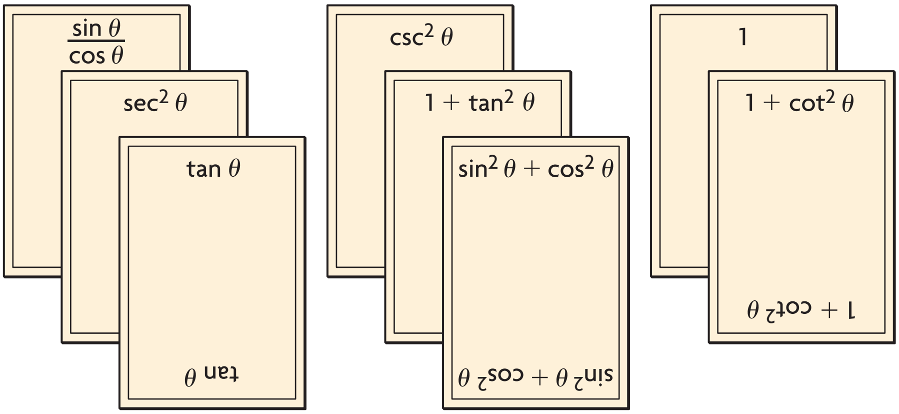
Prove simple trigonometric identities.

LEARN ABOUT the Math

Trident Fish is a game involving a deck of cards, each of which has a mathematical expression written on it. The object of the game is to lay down pairs of equivalent expressions so that each pair forms an identity. Suppose you have the cards shown.

identity

a mathematical statement that is true for all values of the given variables. If the identity involves fractions, the denominators cannot be zero. Any restrictions on a variable must be stated.



? What identities can you form with these cards?

EXAMPLE 1 Proving the quotient identity by rewriting in terms of *x*, *y*, and *r*

Prove the quotient identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for all angles θ, where $0° \leq \theta \leq 360°$.

Jinji's Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{L.S.} = \tan\theta \qquad \text{R.S.} = \frac{\sin\theta}{\cos\theta}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \frac{y}{x} \qquad \text{R.S.} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$$

I wrote $\sin\theta$, $\tan\theta$, and $\cos\theta$ in terms of x, y, and r, since θ can be greater than 90°.

$$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$$

I simplified the right side by multiplying the numerator by the reciprocal of the denominator.

$$= \frac{y}{x}$$

$$= \text{L.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$ for all angles θ, where $0° \le \theta \le 360°$ and $\theta \ne 90°$ or $270°$.

Tan θ is undefined when $\cos\theta = 0$. This occurs when $\theta = 90°$ or $270°$. So θ cannot equal these two values.

EXAMPLE 2 Proving the Pythagorean identity by rewriting in terms of *x*, *y*, and *r*

Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ for all angles θ, where $0° \le \theta \le 360°$.

Lisa's Solution

$$\sin^2\theta + \cos^2\theta = 1$$

$$\text{L.S.} = \sin^2\theta + \cos^2\theta \qquad \text{R.S.} = 1$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

I wrote $\sin\theta$ and $\cos\theta$ in terms of x, y, and r, since θ can be greater than 90°. Then I simplified.

$$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$$

$$= \frac{y^2 + x^2}{r^2}$$

I knew that $r^2 = x^2 + y^2$ from the Pythagorean theorem. I used this equation to further simplify the left side.

$$= \frac{r^2}{r^2}$$

$$= 1$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$$= \text{R.S.}$$

$\therefore \sin^2\theta + \cos^2\theta = 1$ for all angles θ, where $0° \le \theta \le 360°$.

EXAMPLE 3 Proving an identity by using a common denominator

Prove that $1 + \cot^2 \theta = \csc^2 \theta$ for all angles θ between 0° and 360° except 0°, 180°, and 360°.

Pedro's Solution

$1 + \cot^2 \theta = \csc^2 \theta$

L.S. $= 1 + \cot^2 \theta$ R.S. $= \csc^2 \theta$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$= 1 + \left(\frac{\cos \theta}{\sin \theta}\right)^2$ $= \left(\frac{1}{\sin \theta}\right)^2$

$= 1 + \frac{\cos^2 \theta}{\sin^2 \theta}$ $= \frac{1}{\sin^2 \theta}$

I expressed the reciprocal trigonometric ratios in terms of the primary ratios $\sin \theta$ and $\cos \theta$. I knew that $\cot \theta = \frac{1}{\tan \theta}$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$, so $\cot \theta = \frac{\cos \theta}{\sin \theta}$. Since θ can't be 0°, 180°, or 360°, $\sin \theta \neq 0$, I don't have any term that is undefined.

$= \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta}$

On the left side, I expressed 1 as $\frac{\sin^2 \theta}{\sin^2 \theta}$ to get a common denominator of $\sin^2 \theta$.

$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$

I used the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to simplify the numerator.

$= \frac{1}{\sin^2 \theta}$

$=$ R.S.

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore 1 + \cot^2 \theta = \csc^2 \theta$ for all angles θ between 0° and 360° except 0°, 180°, and 360°.

Reflecting

A. What strategy would you use to prove the identity $1 + \tan^2 \theta = \sec^2 \theta$? What restrictions does θ have?

B. When is it important to consider restrictions on θ?

C. How might you create new identities based on Examples 1 and 2?

APPLY the Math

EXAMPLE 4 Proving an identity by factoring

Prove that $\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

Jamal's Solution

$$\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \tan \phi \qquad \text{R.S.} = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

$$= \frac{\sin \phi}{\cos \phi} \qquad = \frac{\sin \phi \cancelto{1}{(1 + \sin \phi)}}{(\cos \phi)\cancelto{1}{(1 + \sin \phi)}}$$

$$= \frac{\sin \phi}{\cos \phi}$$

I knew that $\tan \phi$ could be written as $\frac{\sin \phi}{\cos \phi}$. The right side is more complicated, so I factored out $\sin \phi$ from the numerator. Since $\cos \phi \neq 0$, the denominator will not be 0. I divided the numerator and denominator by the factor $1 + \sin \phi$.

$$= \text{L.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

In Summary

Key Ideas

- A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variable.
- Some trigonometric identities are a result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- Some trigonometric identities that are important to remember are shown below ($0° \leq \theta \leq 360°$).

Identities Based on Definitions	Identities Derived from Relationships	
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$\csc \theta = \frac{1}{\sin \theta}$, where $\sin \theta \neq 0$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$, where $\cos \theta \neq 0$	$\sin^2 \theta + \cos^2 \theta = 1$
$\sec \theta = \frac{1}{\cos \theta}$, where $\cos \theta \neq 0$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$, where $\sin \theta \neq 0$	$1 + \tan^2 \theta = \sec^2 \theta$
$\cot \theta = \frac{1}{\tan \theta}$, where $\tan \theta \neq 0$		$1 + \cot^2 \theta = \csc^2 \theta$

- To prove that a given trigonometric equation is an identity, both sides of the equation need to be shown to be equivalent. This can be done by
  - simplifying the more complicated side until it is identical to the other side or manipulating both sides to get the same expression
  - rewriting all trigonometric ratios in terms of x, y, and r
  - rewriting all expressions involving tangent and the reciprocal trigonometric ratios in terms of sine and cosine
  - applying the Pythagorean identity where appropriate
  - using a common denominator or factoring as required

CHECK *Your Understanding*

1. Prove each identity by writing all trigonometric ratios in terms of x, y, and r. State the restrictions on θ.

a) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

b) $\tan\theta\cos\theta = \sin\theta$

c) $\csc\theta = \dfrac{1}{\sin\theta}$

d) $\cos\theta\sec\theta = 1$

2. Simplify each expression.

a) $(1 - \sin\alpha)(1 + \sin\alpha)$

b) $\dfrac{\tan\alpha}{\sin\alpha}$

c) $\cos^2\alpha + \sin^2\alpha$

d) $\cot\alpha\sin\alpha$

3. Factor each expression.

a) $1 - \cos^2\theta$

b) $\sin^2\theta - \cos^2\theta$

c) $\sin^2\theta - 2\sin\theta + 1$

d) $\cos\theta - \cos^2\theta$

PRACTISING

4. Prove that $\frac{\cos^2\phi}{1 - \sin\phi} = 1 + \sin\phi$, where $\sin\phi \neq 1$, by expressing $\cos^2\phi$ in terms of $\sin\phi$.

5. Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin x}{\tan x} = \cos x$

b) $\dfrac{\tan\theta}{\cos\theta} = \dfrac{\sin\theta}{1 - \sin^2\theta}$

c) $\dfrac{1}{\cos\alpha} + \tan\alpha = \dfrac{1 + \sin\alpha}{\cos\alpha}$

d) $1 - \cos^2\theta = \sin\theta\cos\theta\tan\theta$

6. Mark claimed that $\frac{1}{\cot\theta} = \tan\theta$ is an identity. Marcia let $\theta = 30°$ and found that both sides of the equation worked out to $\frac{1}{\sqrt{3}}$. She said that this proves that the equation is an identity. Is Marcia's reasoning correct? Explain.

7. Simplify each trigonometric expression.

a) $\sin\theta\cot\theta - \sin\theta\cos\theta$

b) $\cos\theta(1 + \sec\theta)(\cos\theta - 1)$

c) $(\sin x + \cos x)(\sin x - \cos x) + 2\cos^2 x$

d) $\dfrac{\csc^2\theta - 3\csc\theta + 2}{\csc^2\theta - 1}$

8. Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin^2\phi}{1 - \cos\phi} = 1 + \cos\phi$

b) $\dfrac{\tan^2\alpha}{1 + \tan^2\alpha} = \sin^2\alpha$

c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

d) $\sin^2\theta + 2\cos^2\theta - 1 = \cos^2\theta$

e) $\sin^4\alpha - \cos^4\alpha = \sin^2\alpha - \cos^2\alpha$

f) $\tan\theta + \dfrac{1}{\tan\theta} = \dfrac{1}{\sin\theta\cos\theta}$

9. **A** Farah claims that if you separate both sides of an equation into two functions and graph them on the same *xy*-axes on a graphing calculator, you can use the result to prove that the equation is an identity.

a) Is her claim correct? Justify your answer.

b) Discuss the limitations of her approach.

10. Is $\csc^2 \theta + \sec^2 \theta = 1$ an identity? Prove that it is true or demonstrate why it is false.

11. **K** Prove that $\sin^2 x\left(1 + \frac{1}{\tan^2 x}\right) = 1$, where $\sin x \neq 0$.

12. **T** Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin^2 \theta + 2\cos \theta - 1}{\sin^2 \theta + 3\cos \theta - 3} = \dfrac{\cos^2 \theta + \cos \theta}{-\sin^2 \theta}$

b) $\sin^2 \alpha - \cos^2 \alpha - \tan^2 \alpha = \dfrac{2\sin^2 \alpha - 2\sin^4 \alpha - 1}{1 - \sin^2 \alpha}$

13. **C** Show how you can create several new identities from the identity $\sin^2 \theta + \cos^2 \theta = 1$ by adding, subtracting, multiplying, or dividing both sides of the equation by the same expression.

Extending

14. **a)** Which equations are not identities? Justify your answers.

b) For those equations that are identities, state any restrictions on the variables.

i) $(1 - \cos^2 x)(1 - \tan^2 x) = \dfrac{\sin^2 x - 2\sin^4 x}{1 - \sin^2 x}$

ii) $1 - 2\cos^2 \phi = \sin^4 \phi - \cos^4 \phi$

iii) $\dfrac{\sin \theta \tan \theta}{\sin \theta + \tan \theta} = \sin \theta \tan \theta$

iv) $\dfrac{1 + 2\sin \beta \cos \beta}{\sin \beta + \cos \beta} = \sin \beta + \cos \beta$

v) $\dfrac{1 - \cos \beta}{\sin \beta} = \dfrac{\sin \beta}{1 + \cos \beta}$

vi) $\dfrac{\sin x}{1 + \cos x} = \csc x - \cot x$

5.6 The Sine Law

YOU WILL NEED

- dynamic geometry software (optional)

Communication *Tip*

To perform a calculation to a high degree of accuracy, save intermediate answers by using the memory keys of your calculator. Round only after the very last calculation.

GOAL

Solve two-dimensional problems by using the sine law.

LEARN ABOUT the Math

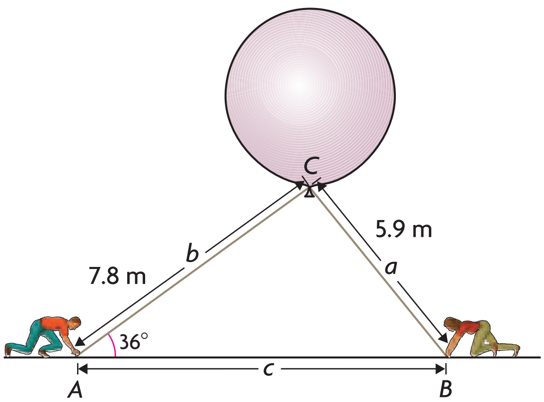
Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long.

? How far, to the nearest tenth of a metre, is Albert from Belle?

EXAMPLE 1 Using the sine law to calculate an unknown length

Determine the distance between Albert and Belle.

Adila's Solution: Assuming that Albert and Belle are on Opposite Sides of the Balloon



From the problem, it is not clear how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on opposite sides of the balloon. I drew a sketch of this situation.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

In $\triangle ABC$, I knew one angle and two sides. To determine AB (side c), I needed to know $\angle C$. So I first calculated $\angle B$ using the **sine law**.

$$\frac{5.9}{\sin 36°} = \frac{7.8}{\sin B}$$

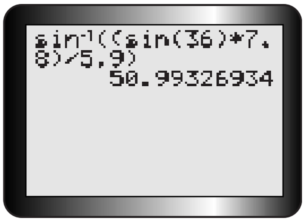
$$(\overset{1}{\cancel{\sin 36°}} \times \sin B)\left(\frac{5.9}{\underset{1}{\cancel{\sin 36°}}}\right) = (\sin 36° \times \overset{1}{\cancel{\sin B}})\left(\frac{7.8}{\underset{1}{\cancel{\sin B}}}\right)$$

To solve for $\angle B$, I multiplied both sides of the equation by the lowest common denominator $(\sin 36° \times \sin B)$ to eliminate the fractions.

$$\frac{(\sin B)(\overset{1}{\cancel{5.9}})}{\underset{1}{\cancel{5.9}}} = \frac{(\sin 36°)(7.8)}{5.9}$$

Then I divided both sides by 5.9 to isolate sin B.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36°)(7.8)}{5.9}\right)$$



I used my calculator to evaluate $\angle B$.

$$\angle B \doteq 51°$$

I rounded to the nearest degree.

$$\angle A + \angle B + \angle C = 180°$$
$$36° + 51° + \angle C = 180°$$
$$\angle C = 180° - (36° + 51°)$$
$$= 93°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Then I used the sine law again to determine c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 93°}$$

$$\frac{5.9}{\sin 36°} \times \sin 93° = \frac{c}{\cancel{\sin 93°}_{1}} \times \overset{1}{\cancel{\sin 93°}}$$

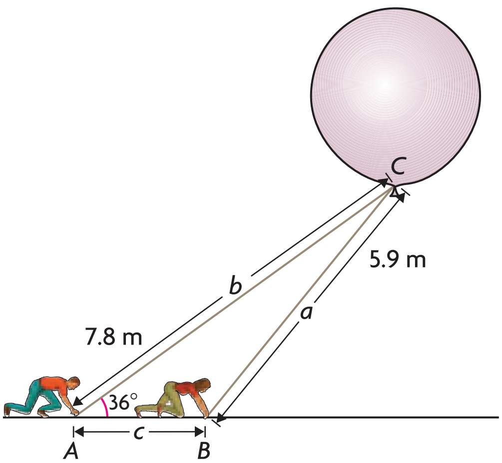
To solve for c, I multiplied both sides of the equation by sin 93°.

$$10.0 \text{ m} \doteq c$$

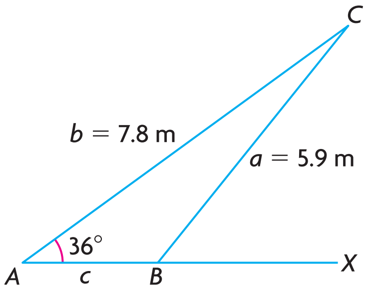
I rounded to the nearest tenth.

If Albert and Belle are on opposite sides of the balloon, they are about 10.0 m apart.

Reuben's Solution: Assuming that Albert and Belle are on the Same Side of the Balloon



The problem did not state how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on the same side of the balloon. I drew a sketch of this situation.



In $\triangle ABC$, I knew one angle and two sides. If I knew $\angle C$, I could determine c using the sine law. First I had to determine $\angle B$ in order to get $\angle C$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

I used the sine law to calculate $\angle B$.

$$\frac{5.9}{\sin 36°} = \frac{7.8}{\sin B}$$

$$(\sin 36° \times \sin B)\left(\frac{5.9}{\sin 36°}\right) = (\sin 36° \times \sin B)\left(\frac{7.8}{\sin B}\right)$$

To solve for $\angle B$, I first multiplied both sides of the equation by the lowest common denominator $(\sin 36° \times \sin B)$ to eliminate the fractions.

$$\frac{(\sin B)(5.9)}{5.9} = \frac{(\sin 36°)(7.8)}{5.9}$$

I divided both sides by 5.9 to isolate $\sin B$.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36°)(7.8)}{5.9}\right)$$

I used my calculator to evaluate $\angle B$.
I rounded to the nearest degree.

$$\angle CBX \doteq 51°$$

51° is the value of the related acute angle $\angle CBX$, but I wanted the obtuse angle $\angle CBA$ in the triangle.

$$\angle CBA = 180° - 51°$$
$$= 129°$$

$$\angle C = 180° - (\angle A + \angle CBA)$$
$$= 180° - (36° + 129°)$$
$$= 15°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

I used the sine law again to calculate side c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 15°}$$

$$\sin 15° \times \frac{5.9}{\sin 36°} = \sin 15° \times \frac{c}{\sin 15°}$$

To solve for c, I multiplied both sides of the equation by sin 15°.

$$2.6 \text{ m} \doteq c$$

I rounded to the nearest tenth.

If Albert and Belle are on the same side of the balloon, they are about 2.6 m apart.

Reflecting

A. Why is the situation in Example 1 called **the ambiguous case of the sine law**?

B. What initial information was given in this problem?

C. What is the relationship between sin B in Adila's solution and sin B in Reuben's solution? Explain why both values of sine are related.

D. Calculate the height of $\triangle ABC$ in both solutions. What do you notice? Compare this value with the length of a and b.

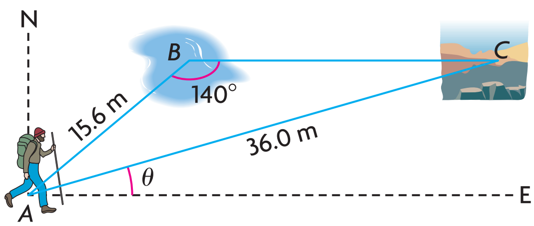
the ambiguous case of the sine law

a situation in which 0, 1, or 2 triangles can be drawn given the information in a problem. This occurs when you know two side lengths and an angle *opposite* one of the sides rather than *between* them (an SSA triangle). If the given angle is acute, 0, 1, or 2 triangles are possible. If the given angle is obtuse, 0 or 1 triangle is possible (see the In Summary box for this lesson).

APPLY the Math

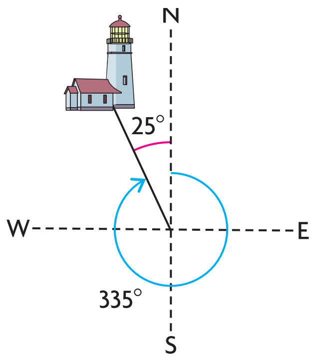
EXAMPLE 2 Using the sine law in the ambiguous case to calculate the only possible angle

Karl's campsite is 15.6 m from a lake and 36.0 m from a scenic lookout as shown. From the lake, the angle formed between the campsite and the lookout is 140°. Karl starts hiking from his campsite to go to the lookout. What is the **bearing** of the lookout from Karl's position ($\angle NAC$)?

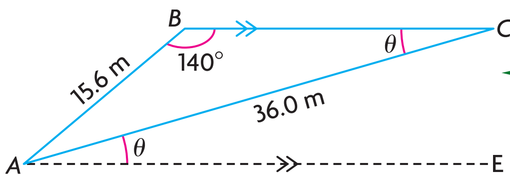


bearing

the direction in which you have to move in order to reach an object. A bearing is a clockwise angle from magnetic north. For example, the bearing of the lighthouse shown is 335°.



Sara's Solution



Based on the given information, this is an SSA triangle. But since the given angle is obtuse, only one situation had to be considered. In $\triangle ABC$, I knew that AE is parallel to BC. Since $\angle C$ and θ are alternate angles between parallel lines, they are equal.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

I needed to calculate an angle, so I set up the sine law equation with the angles in the numerators.

$$\frac{\sin \theta}{15.6} = \frac{\sin 140°}{36.0}$$

$$\overset{1}{\cancel{15.6}} \times \frac{\sin \theta}{\underset{1}{\cancel{15.6}}} = 15.6 \times \frac{\sin 140°}{36.0}$$

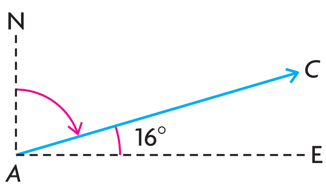
To solve for $\sin \theta$, I multiplied both sides of the equation by 15.6.

$$\theta = \sin^{-1}\left(\frac{(15.6)(\sin 140°)}{36.0}\right)$$

I used the inverse sine function on my calculator to determine $\angle C$ (angle θ).

$$\theta \doteq 16°$$

I rounded to the nearest degree.



In order to state the bearing of the lookout, I needed to know the complementary angle of 16°.

$$\angle NAC = 90° - 16°$$
$$= 74°$$

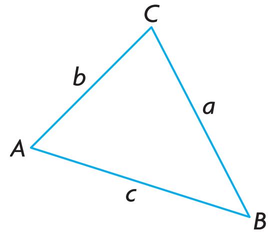
So I subtracted 16° from 90°.

From Karl's campsite, the lookout has a bearing of about 74°.

In Summary

Key Ideas

- The sine law states that in any $\triangle ABC$, the ratios of each side to the sine of its opposite angle are equal.



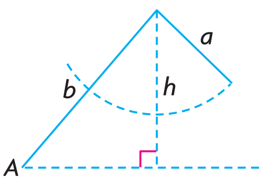
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- Given any triangle, the sine law can be used if you know
  - two sides and one angle opposite a given side (SSA) or
  - two angles and any side (AAS or ASA)
- The ambiguous case arises in a SSA (side, side, angle) triangle. In this situation, depending on the size of the given angle and the lengths of the given sides, the sine law calculation may lead to 0, 1, or 2 solutions.

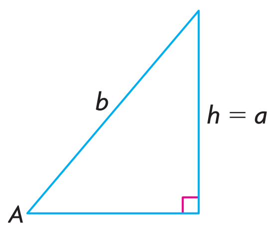
Need to Know

- In the ambiguous case, if $\angle A$, a, and b are given and $\angle A$ is acute, there are four cases to consider. In each case, the height of the triangle is $h = b \sin A$.

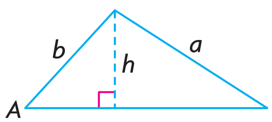
If $\angle A$ is acute and $a < h$, no triangle exists.



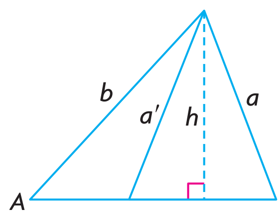
If $\angle A$ is acute and $a = h$, one right triangle exists.



If $\angle A$ is acute and $a > b$, one triangle exists.

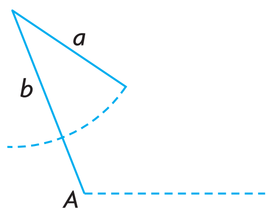


If $\angle A$ is acute and $h < a < b$, two triangles exist.

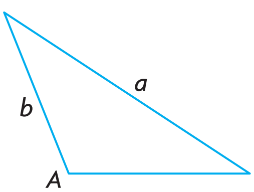


If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are two cases to consider.

If $\angle A$ is obtuse and $a < b$ or $a = b$, no triangle exists.

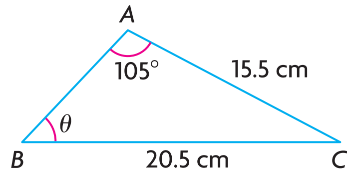


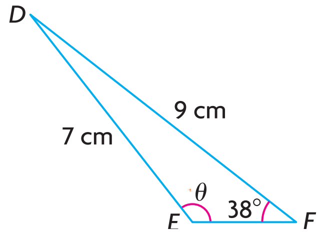
If $\angle A$ is obtuse and $a > b$, one triangle exists.



CHECK *Your Understanding*

1. Determine the measure of angle θ to the nearest degree.

a) 

b) 

2. A triangular plot of land is enclosed by a fence. Two sides of the fence are 9.8 m and 6.6 m long, respectively. The other side forms an angle of 40° with the 9.8 m side.

a) Draw a sketch of the situation.

b) Calculate the height of the triangle to the nearest tenth. Compare it to the given sides.

c) How many lengths are possible for the third side? Explain.

3. Determine whether it is possible to draw a triangle, given each set of information. Sketch all possible triangles where appropriate. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

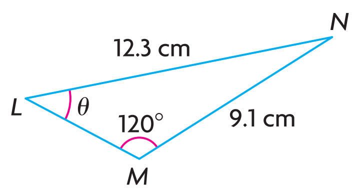
a) $a = 5.2$ cm, $b = 2.8$ cm, $\angle B = 65°$

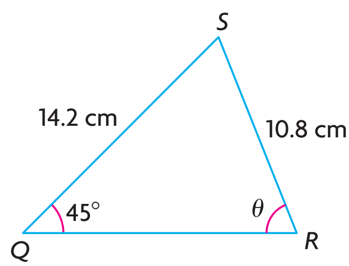
b) $b = 6.7$ cm, $c = 2.1$ cm, $\angle C = 63°$

c) $a = 5.0$ cm, $c = 8.5$ cm, $\angle A = 36°$

PRACTISING

4. Determine the measure of angle θ to the nearest degree.

a) 

b) 

5. **K** Where appropriate, sketch all possible triangles, given each set of information. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $a = 7.2$ mm, $b = 9.3$ mm, $\angle A = 35°$

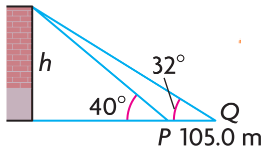
b) $a = 7.3$ m, $b = 14.6$ m, $\angle A = 30°$

c) $a = 1.3$ cm, $b = 2.8$ cm, $\angle A = 33°$

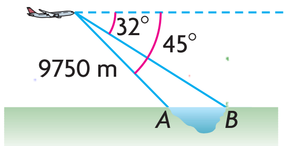
d) $c = 22.2$ cm, $\angle A = 75°$, $\angle B = 43°$

6. The trunk of a leaning tree makes an angle of 12° with the vertical. To prevent the tree from falling over, a 35.0 m rope is attached to the top of the tree and is pegged into level ground some distance away. If the tree is 20.0 m from its base to its top, calculate the angle the rope makes with the ground to the nearest degree.

7. **A** A building of height *h* is observed from two points, *P* and *Q*, that are 105.0 m apart as shown. The angles of elevation at *P* and *Q* are 40° and 32°, respectively. Calculate the height, *h*, to the nearest tenth of a metre.



8. A surveyor in an airplane observes that the angle of depression to two points on the opposite shores of a lake are 32° and 45°, respectively, as shown. What is the width of the lake, to the nearest metre, at those two points?



9. The Pont du Gard near Nîmes, France, is a Roman aqueduct. An observer in a hot-air balloon some distance away from the aqueduct determines that the angle of depression to each end is 54° and 71°, respectively. The closest end of the aqueduct is 270.0 m from the balloon. Calculate the length of the aqueduct to the nearest tenth of a metre.

10. **T** A wind tower at the top of a hill casts a shadow 30 m long along the side of the hill. An observer at the farthest edge of the shadow from the tower estimates the angle of elevation to the top of the tower to be 34°. If the slope of the hill is 13° from the horizontal, how high is the tower to the nearest metre?

11. Carol is flying a kite on level ground, and the string forms an angle of 50° with the ground. Two friends standing some distance from Carol see the kite at angles of elevation of 66° and 35°, respectively. One friend is 11 m from Carol. For each question below, state all possible answers to the nearest metre.

a) How high is the kite above the ground?

b) How long is the string?

c) How far is the other friend from Carol?

12. The Huqiu Tower in China was built in 961 CE. When the tower was first built, its height was 47 m. Since then it has tilted 2.8°, so it is called China's Leaning Tower. There is a specific point on the ground where you can be equidistant from both the top and the bottom of the tower. How far is this point from the base of the tower? Round your answer to the nearest metre.

13. Your neighbour claims that his lot is triangular, with one side 430 m long and the adjacent side 110 m long. The angle opposite one of these sides is 35°. Determine the other side length of this lot to the nearest metre and the interior angles to the nearest degree.

14. **C** In $\triangle LMN$, $\angle L$ is acute. Using a sketch, explain the relationship between $\angle L$, sides l and m, and the height of the triangle for each situation.

a) Only one triangle is possible.

b) Two triangles are possible.

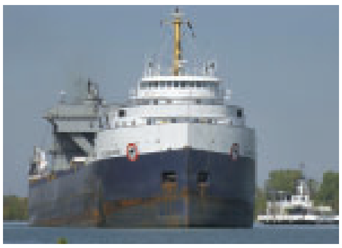
c) No triangle is possible.

Extending

15. A sailor out in a lake sees two lighthouses 11 km apart along the shore and gets bearings of 285° from his present position for lighthouse A and 237° for lighthouse B. From lighthouse B, lighthouse A has a bearing of 45°.

a) How far, to the nearest kilometre, is the sailor from both lighthouses?

b) What is the shortest distance, to the nearest kilometre, from the sailor to the shore?



16. The *Algomarine* is a cargo ship that is 222.5 m long. On the water, small watercraft have the right of way. However, bulk carriers cruise at nearly 30 km/h, so it is best to stay out of their way: If you pass a cargo ship within 40 m, your boat could get swamped! Suppose you spot the *Algomarine* on your starboard (right) side headed your way. The bow and stern of the carrier appear separated by 12°. The captain of the *Algomarine* calls you from the bridge, located at the stern, and says that you are 8° off his bow.

a) How far, to the nearest metre, are you from the stern?

b) Are you in danger of being swamped?

17. The Gerbrandy Tower in the Netherlands is an 80 m high concrete tower, on which a 273.5 m guyed mast is mounted. The lower guy wires form an angle of 36° with the ground and attach to the tower 155 m above ground. The upper guy wires form an angle of 59° with the ground and attach to the mast 350 m above ground. How long are the upper and lower guy wires? Round your answers to the nearest metre.

5.7 The Cosine Law

GOAL

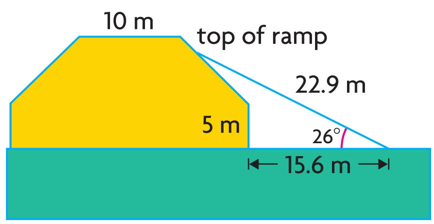
Solve two-dimensional problems by using the cosine law.

YOU WILL NEED

- dynamic geometry software (optional)

LEARN ABOUT the Math

A barn whose cross-section resembles half a regular octagon with a side length of 10 m needs some repairs to its roof. The roofers place a 22.9 m ramp against the side of the building, forming an angle of 26° with the ground. The ramp will be used to transport the materials needed for the repair. The base of the ramp is 15.6 m from the side of the building.

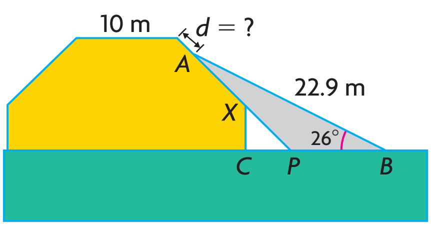


? How far, to the nearest tenth of a metre, is the top of the ramp from the flat roof of the building?

EXAMPLE 1 Using the cosine law to calculate an unknown length

Determine the distance from the top of the ramp to the roof by using the **cosine law**.

Tina's Solution

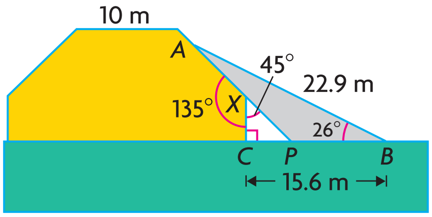


I labelled the top of the ramp A and the bottom of the ramp B. Then I drew a line from A along the sloped part of the building to X and extended the line to the ground at P. I labelled the point where the side of the building touches the ground C.

$$\angle AXC + \angle CXP = 180°$$
$$135° + \angle CXP = 180°$$
$$\angle CXP = 180° - 135°$$
$$= 45°$$

In $\triangle XCP$, $\angle C$ is 90°. Since the octagon is regular, each interior angle is 135°. So $\angle AXC$ is 135°. To determine $\angle CXP$, I subtracted 135° from 180°.

$\therefore$ $\triangle XCP$ is a 45° − 45° − 90° special triangle.



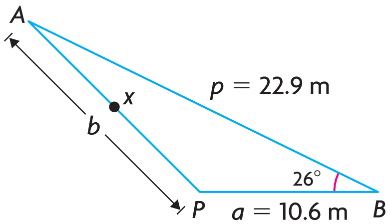
$$CP + PB = 15.6$$

From the given information, I knew that *XC* = 5 m, so *CP* = 5 m, since the triangle is isosceles.

$$5 + PB = 15.6$$

I then subtracted *CP* from *CB* to determine the length of *PB*.

$$PB = 15.6 - 5$$
$$= 10.6 \text{ m}$$



In △*APB*, I knew two side lengths and the contained angle formed by those sides. So I couldn't use the sine law to determine *AP*. I used the cosine law instead.

I substituted the values of *a*, *p*, and ∠*B* into the formula. I calculated *b* by evaluating the right side of the equation and determining its square root.

$$b^2 = a^2 + p^2 - 2ap\cos B$$
$$b^2 = (10.6)^2 + (22.9)^2 - 2(10.6)(22.9)\cos 26°$$
$$b^2 = 200.42 \text{ m}^2$$
$$b = \sqrt{200.42}$$
$$b \doteq 14.16 \text{ m}$$

To determine the distance from the top of the ramp to the roof, I needed to calculate *AX* first. I knew that *XP* is a multiple of $\sqrt{2}$ because △*XCP* is a $45° - 45° - 90°$ special triangle. So I subtracted *XP* from *b* to determine *AX*.

$$XP = 5\sqrt{2}$$
$$AX + XP = b$$
$$AX + 5\sqrt{2} = 14.16$$
$$AX = 14.16 - 5\sqrt{2}$$
$$\doteq 7.09 \text{ m}$$

Then I subtracted *AX* from 10 m to get the distance from the top of the ramp to the roof.

$$\text{required distance} = 10 - AX$$
$$= 10 - 7.09$$
$$\doteq 2.9 \text{ m}$$

The top of the ramp is about 2.9 m from the flat roof of the building.

Reflecting

A. Why did Tina draw line AP on her sketch as part of her solution?

B. Could Tina have used the sine law, instead of the cosine law, to solve the problem? Explain your reasoning.

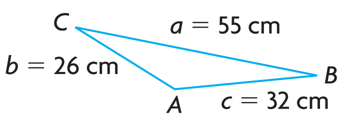
C. The Pythagorean theorem is a special case of the cosine law. What conditions would have to exist in a triangle in order for the cosine law to simplify to the Pythagorean theorem?

APPLY the Math

EXAMPLE 2 Using the cosine law to determine an angle

In $\triangle ABC$, determine $\angle A$ to the nearest degree if $a = 55$ cm, $b = 26$ cm, and $c = 32$ cm.

Claudio's Solution



I drew a diagram of the triangle and labelled all sides.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Since I knew all three sides (SSS) but no angles, I couldn't use the sine law. So I used the cosine law to determine $\angle A$.

$$55^2 = 26^2 + 32^2 - 2(26)(32)\cos A$$

I substituted the values of a, b, and c into the formula.

$$\cos A = \frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}$$

$$\angle A = \cos^{-1}\left(\frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}\right)$$

I used the inverse cosine function on my calculator to determine $\angle A$.

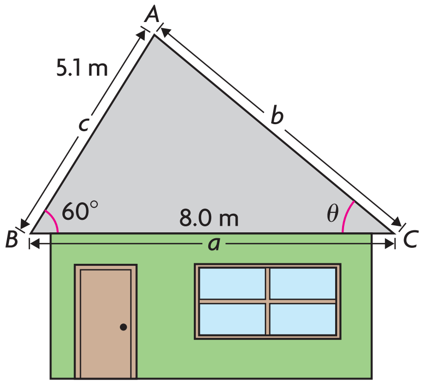
$$\angle A \doteq 143°$$

Given $\triangle ABC$, $\angle A$ is about 143°.

EXAMPLE 3 Solving a problem by using the cosine and the sine laws

Mitchell wants his 8.0 wide house to be heated with a solar hot-water system. The tubes form an array that is 5.1 m long. In order for the system to be effective, the array must be installed on the south side of the roof and the roof needs to be inclined by 60°. If the north side of the roof is inclined more than 40°, the roof will be too steep for Mitchell to install the system himself. Will Mitchell be able to install this system by himself?

Serina's Solution



I drew a sketch of the situation. I wanted to use the sine law to determine angle θ to solve the problem. But before I could do that, I needed to determine the length of side b.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

Since I knew two sides and the angle between them, I couldn't use the sine law to determine b. So I used the cosine law.

$$b^2 = (8.0)^2 + (5.1)^2 - 2(8.0)(5.1)\cos 60°$$

$$b^2 = 49.21 \text{ m}^2$$

$$b = \sqrt{49.21}$$

$$b \doteq 7.0 \text{ m}$$

I substituted the values of a, c, and $\angle B$ into the formula. I calculated b by evaluating the right side of the equation and determining its square root.

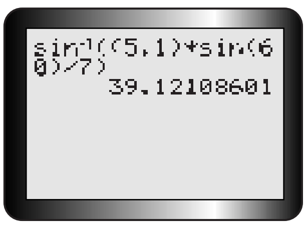
$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{5.1} = \frac{\sin 60°}{7.0}$$

$$\overset{1}{\cancel{5.1}} \times \frac{\sin \theta}{\underset{1}{\cancel{5.1}}} = 5.1 \times \frac{\sin 60°}{7.0}$$

$$\theta = 5.1 \times \frac{\sin 60°}{7.0}$$

I determined $\angle C$ (θ) by using the sine law. Since I needed to solve for an angle, I wrote the sine law with the angles in the numerators. I multiplied both sides of the equation by 5.1 to solve for sin θ.



I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 39°$$

Since Mitchell's roof is inclined about 39° on the north side, he will be able to install the solar hot-water system by himself.

In Summary

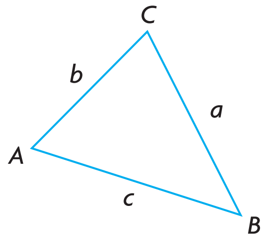
Key Idea

- Given any triangle, the cosine law can be used if you know
  - two sides and the angle contained between those sides (SAS) or
  - all three sides (SSS)

Need to Know

- The cosine law states that in any $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



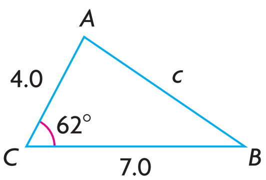
- If $\angle A = 90°$ and $\angle A$ is the contained angle, then the cosine law simplifies to the Pythagorean theorem:

$$a^2 = b^2 + c^2 - 2bc\cos 90°$$
$$a^2 = b^2 + c^2 - 2bc(0)$$
$$a^2 = b^2 + c^2$$

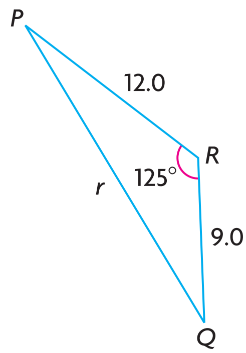
CHECK Your Understanding

1. Determine each unknown side length to the nearest tenth.

a)

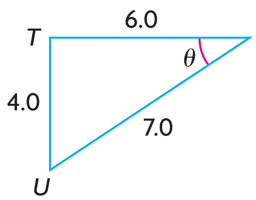


b)



2. For each triangle, determine the value of θ to the nearest degree.

a)



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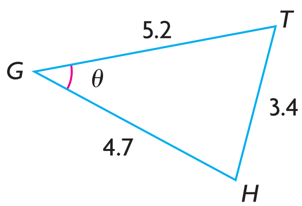
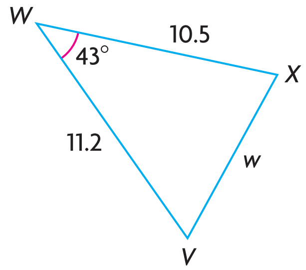
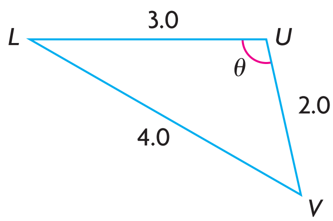


PRACTISING

3. **a)** Determine w to the nearest tenth.



b) Determine the value of θ to the nearest degree.

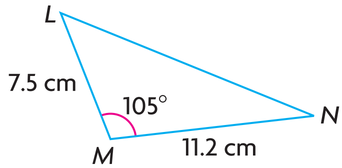


c) In $\triangle ABC$, $a = 11.5$, $b = 8.3$, and $c = 6.6$. Calculate $\angle A$ to the nearest degree.

d) In $\triangle PQR$, $q = 25.1$, $r = 71.3$, and $\cos\ P = \frac{1}{4}$. Calculate p to the nearest tenth.

4. **K** Calculate each unknown angle to the nearest degree and each unknown length to the nearest tenth of a centimetre.

a)



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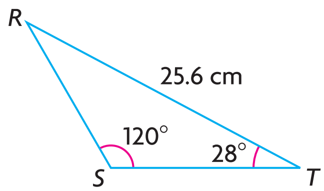

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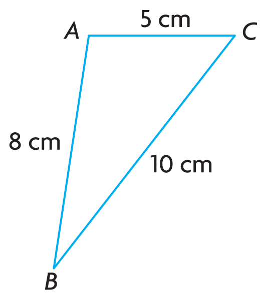

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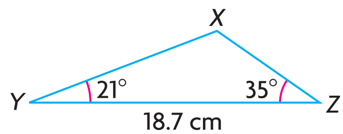
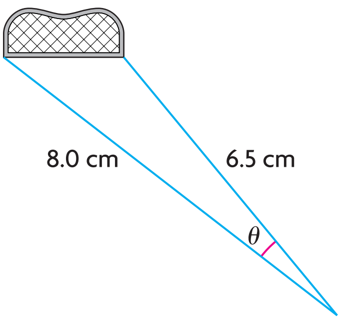
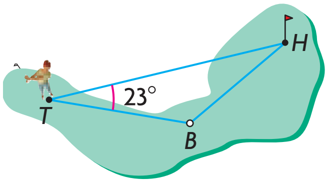


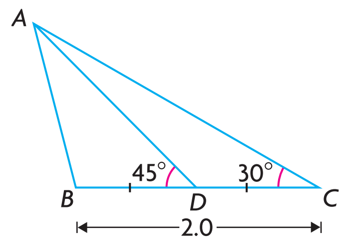


5. **A** The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.



6. While golfing, Sahar hits a tee shot from T toward a hole at H, but the ball veers 23° and lands at B. The scorecard says that H is 270 m from T. If Sahar walks 160 m to the ball (B), how far, to the nearest metre, is the ball from the hole?

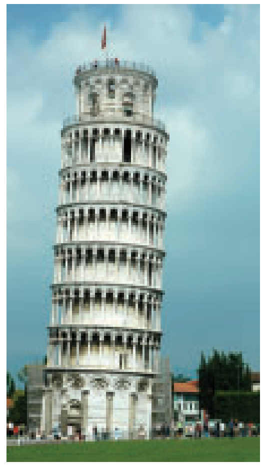
7. Given $\triangle ABC$ at the right, $BC = 2.0$ and D is the midpoint of BC. Determine AB, to the nearest tenth, if $\angle ADB = 45°$ and $\angle ACB = 30°$.



8. Two forest fire towers, A and B, are 20.3 km apart. From tower A, the bearing of tower B is 70°. The ranger in each tower observes a fire and radios the bearing of the fire from the tower. The bearing from tower A is 25° and from tower B is 345°. How far, to the nearest tenth of a kilometre, is the fire from each tower?

9. **T** Two roads intersect at an angle of 15°. Darryl is standing on one of the roads 270 m from the intersection.
   a) Create a question that requires using the sine law to solve it. Include a complete solution and a sketch.
   b) Create a question that requires using the cosine law to solve it. Include a complete solution and a sketch.

10. The Leaning Tower of Pisa is 55.9 m tall and leans 5.5° from the vertical. If its shadow is 90.0 m long, what is the distance from the top of the tower to the top edge of its shadow? Assume that the ground around the tower is level. Round your answer to the nearest metre.



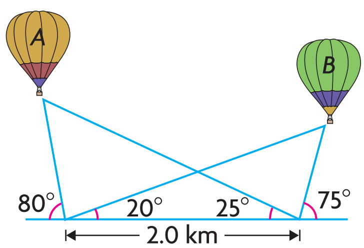
11. **C** The side lengths and the interior angles of any triangle can be determined by using the cosine law, the sine law, or a combination of both. Sketch a triangle and state the minimum information required to use
   a) the cosine law
   b) both laws
   Under each sketch, use the algebraic representation of the law to show how to determine all unknown quantities.

Extending

12. The interior angles of a triangle are 120°, 40°, and 20°. The longest side is 10 cm longer than the shortest side. Determine the perimeter of the triangle to the nearest centimetre.

13. For each situation, determine all unknown side lengths to the nearest tenth of a centimetre and/or all unknown interior angles to the nearest degree. If more than one solution is possible, state all possible answers.
   a) A triangle has exactly one angle measuring 45° and sides measuring 5.0 cm, 7.4 cm, and 10.0 cm.
   b) An isosceles triangle has at least one interior angle of 70° and at least one side of length 11.5 cm.

14. Two hot-air balloons are moored to level ground below, each at a different location. An observer at each location determines the angle of elevation to the opposite balloon as shown at the right. The observers are 2.0 km apart.
   a) What is the distance separating the balloons, to the nearest tenth of a kilometre?
   b) Determine the difference in height (above the ground) between the two balloons. Round your answer to the nearest metre.



5.8 Solving Three-Dimensional Problems by Using Trigonometry

YOU WILL NEED

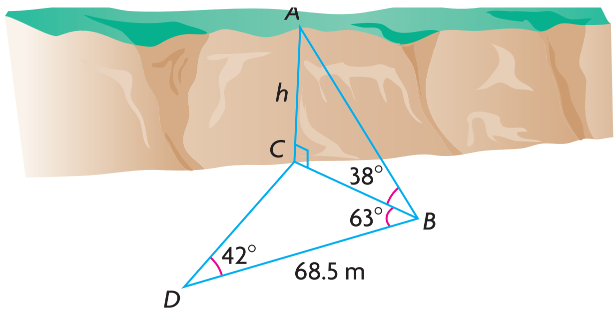
- dynamic geometry software (optional)

GOAL

Solve three-dimensional problems by using trigonometry.

LEARN ABOUT the Math

From point B, Manny uses a clinometer to determine the angle of elevation to the top of a cliff as 38°. From point D, 68.5 m away from Manny, Joe estimates the angle between the base of the cliff, himself, and Manny to be 42°, while Manny estimates the angle between the base of the cliff, himself, and his friend Joe to be 63°.



? What is the height of the cliff to the nearest tenth of a metre?

EXAMPLE 1 Solving a three-dimensional problem by using the sine law

Calculate the height of the cliff to the nearest tenth of a metre.

Matt's Solution

In $\triangle DBC$:

$\angle C = 180° - (63° + 42°)$

$= 75°$

BC is in $\triangle ABC$. In $\triangle ABC$, I don't have enough information to calculate h, but *BC* is also in $\triangle DBC$.

In $\triangle DBC$, I knew two angles and a side length. Before I could calculate *BC*, I needed to determine $\angle C$. I used the fact that the sum of all three interior angles is 180°.

$$\frac{BC}{\sin D} = \frac{BD}{\sin C}$$

Using $\triangle DBC$ and the value of $\angle C$, I used the sine law to calculate BC.

$$\frac{BC}{\sin 42°} = \frac{68.5}{\sin 75°}$$

$$\overset{1}{\cancel{\sin 42°}} \times \frac{BC}{\underset{1}{\cancel{\sin 42°}}} = \sin 42° \times \frac{68.5}{\sin 75°}$$

To solve for BC, I multiplied both sides of the equation by sin 42°.

$$BC = \sin 42° \times \frac{68.5}{\sin 75°}$$

I used my calculator to evaluate.

$$BC \doteq 47.45 \text{ m}$$

$$\tan 38° = \frac{h}{BC}$$

$$\tan 38° = \frac{h}{47.45}$$

Then I used $\triangle ABC$ to calculate h. I knew that $\triangle ABC$ is a right triangle and that h is opposite the 38° angle while BC is adjacent to it. So I used tangent.

$$\tan 38° \times 47.45 = \frac{h}{\underset{1}{\cancel{47.45}}} \times \overset{1}{\cancel{47.45}}$$

To evaluate h, I multiplied both sides of the equation by 47.45.

$$37.1 \text{ m} \doteq h$$

The height of the cliff is about 37.1 m.

Reflecting

A. Was the given diagram necessary to help Matt solve the problem? Explain.

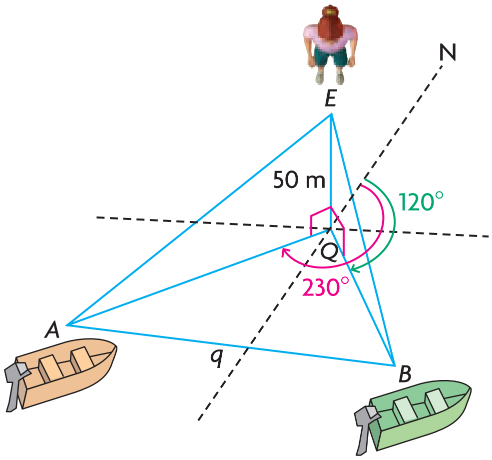
B. Why did Matt begin working with $\triangle DBC$ instead of $\triangle ABC$?

C. What strategies might Matt use to check whether his answer is reasonable?

APPLY the Math

EXAMPLE 2 Solving a three-dimensional problem by using the sine law

Emma is on a 50 m high bridge and sees two boats anchored below. From her position, boat A has a bearing of 230° and boat B has a bearing of 120°. Emma estimates the angles of depression to be 38° for boat A and 35° for boat B. How far apart are the boats to the nearest metre?



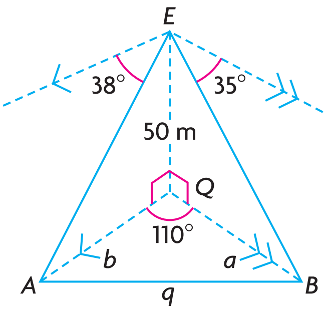
Kelly's Solution

In $\triangle AQB$:

$\angle Q = 230° - 120°$

$= 110°$

In $\triangle AQB$, I knew that the value of $\angle Q$ is equal to the difference of the bearings of boats A and B. So I subtracted 120° from 230°.

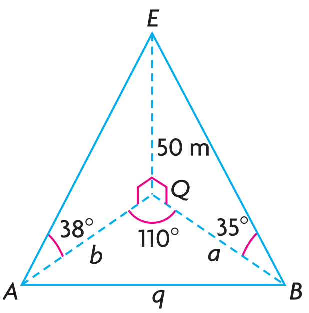


In $\triangle AQB$, I knew only one angle and no side lengths. In order to calculate q, I needed to determine AQ (side b) and BQ (side a) first.

I drew a sketch that included the angles of depression. I used these angles to determine the values of $\angle EAQ$ and $\angle EBQ$.

$\angle EAQ = 38°$ $\angle EBQ = 35°$

The angle of depression to A is measured from a line parallel to AQ. So $\angle EAQ$ is equal to 38°. Using the same reasoning, I determined that $\angle EBQ$ is equal to 35°.



I included the values of $\angle EAQ$ and $\angle EBQ$ in my sketch.

In $\triangle AEQ$:

$$\tan 38° = \frac{50}{b}$$

$$b = \frac{50}{\tan 38°}$$

$$b \doteq 64.0 \text{ m}$$

In $\triangle BEQ$:

$$\tan 35° = \frac{50}{a}$$

$$a = \frac{50}{\tan 35°}$$

$$a \doteq 71.4 \text{ m}$$

Since $\triangle AEQ$ and $\triangle BEQ$ are right triangles, I expressed AQ in terms of tan 38° and BQ in terms of tan 35°. Then I solved for b and a.

$$q^2 = b^2 + a^2 - 2ba \cos 110°$$

In $\triangle AQB$, I now knew two side lengths and the angle between those sides. So I used the cosine law to calculate q.

$$q^2 = (64.0)^2 + (71.4)^2 - 2(64.0)(71.4)\cos 110°$$

$$q = \sqrt{12\,320.6}$$

$$q \doteq 111 \text{ m}$$

I substituted the values of b and a into the equation and evaluated q.

The boats are about 111 m apart.

In Summary

Key Ideas

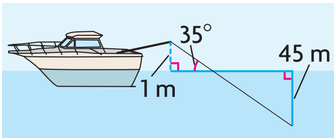
- Three-dimensional problems involving triangles can be solved using some combination of these approaches:
  - trigonometric ratios
  - the Pythagorean theorem
  - the sine law
  - the cosine law
- The approach you use depends on the given information and what you are required to find.

Need to Know

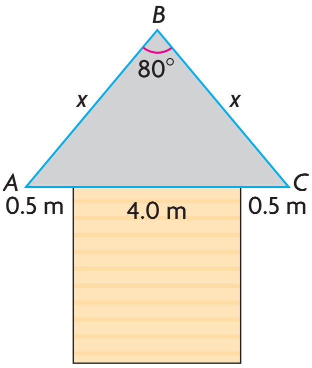
- When solving problems, always start with a sketch of the given information. Determine any unknown angles by using any geometric facts that apply, such as facts about parallel lines, interior angles in a triangle, and so on. Revise your sketch so that it includes any new information that you determined. Then use trigonometry to solve the original problem.
- In right triangles, use the primary or reciprocal trigonometric ratios.
- In all other triangles, use the sine law and/or the cosine law.

Given Information	Required to Find	Use
SSA	angle	sine law
ASA or AAS	side	sine law
SAS	side	cosine law
SSS	side	cosine law

CHECK Your Understanding



1. Morana is trolling for salmon in Lake Ontario. She sets the fishing rod so that its tip is 1 m above water and the line forms an angle of 35° with the water's surface. She knows that there are fish at a depth of 45 m. Describe the steps you would use to calculate the length of line she must let out.

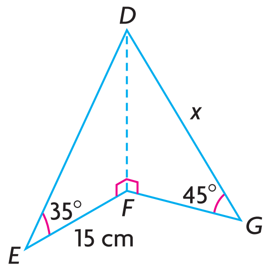


2. Josh is building a garden shed that is 4.0 m wide. The two sides of the roof are equal in length and must meet at an angle of 80°. There will be a 0.5 m overhang on each side of the shed. Josh wants to determine the length of each side of the roof.
   a) Should he use the sine law or the cosine law? Explain.
   b) How could Josh use the primary trigonometric ratios to calculate x? Explain.

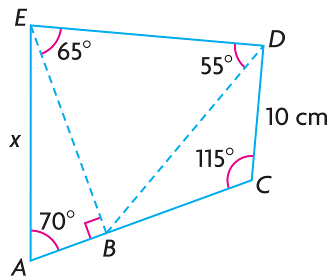
PRACTISING

3. Determine the value of x to the nearest centimetre and θ to the nearest **K** degree. Explain your reasoning for each step of your solution.

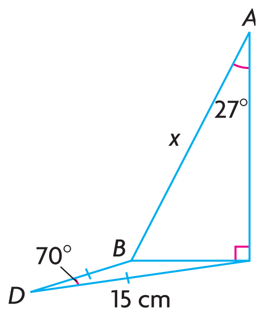
a)



c)



b)



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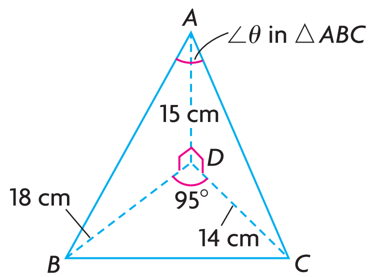
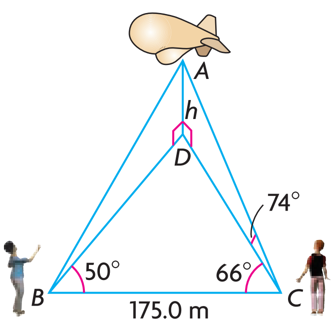




4. As a project, a group of students was asked to determine the altitude, h, of a promotional blimp. The students' measurements are shown in the sketch at the left.
   a) Determine h to the nearest tenth of a metre. Explain each of your steps.
   b) Is there another way to solve this problem? Explain.

5. While Travis and Bob were flying a hot-air balloon from Beamsville to Vineland in southwestern Ontario, they decided to calculate the straight-line distance, to the nearest metre, between the two towns.

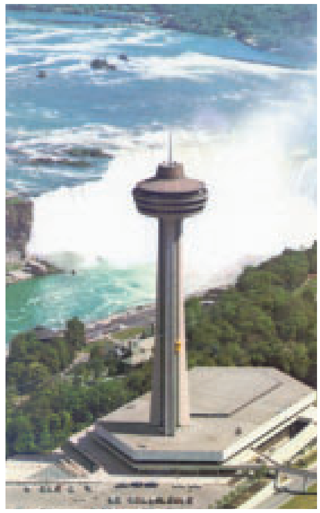
- From an altitude of 226 m, they simultaneously measured the angle of depression to Beamsville as 2° and to Vineland as 3°.
- They measured the angle between the lines of sight to the two towns as 80°.

Is there enough information to calculate the distance between the two towns? Justify your reasoning with calculations.

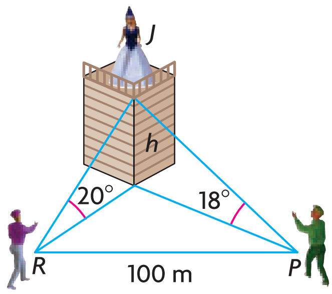
6. **A** The observation deck of the Skylon Tower in Niagara Falls, Ontario, is 166 m above the Niagara River. A tourist in the observation deck notices two boats on the water. From the tourist's position,

- the bearing of boat *A* is 180° at an angle of depression of 40°
- the bearing of boat *B* is 250° at an angle of depression of 34°

Calculate the distance between the two boats to the nearest metre.



7. Suppose Romeo is serenading Juliet while she is on her balcony. Romeo is facing north and sees the balcony at an angle of elevation of 20°. Paris, Juliet's other suitor, is observing the situation and is facing west. Paris sees the balcony at an angle of elevation of 18°. Romeo and Paris are 100 m apart as shown. Determine the height of Juliet's balcony above the ground, to the nearest metre.



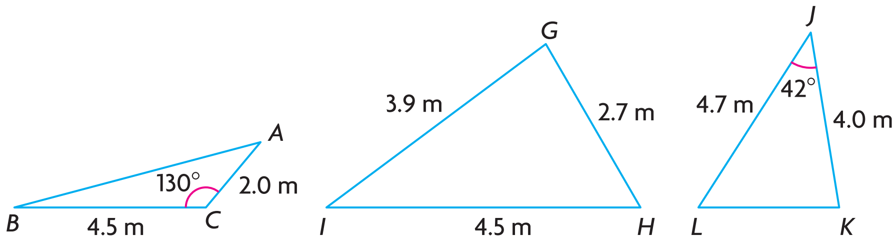
8. A coast guard helicopter hovers between an island and a damaged sailboat.

- From the island, the angle of elevation to the helicopter is 73°.
- From the helicopter, the island and the sailboat are 40° apart.
- A police rescue boat heading toward the sailboat is 800 m away from the scene of the accident. From this position, the angle between the island and the sailboat is 35°.
- At the same moment, an observer on the island notices that the sailboat and police rescue boat are 68° apart.

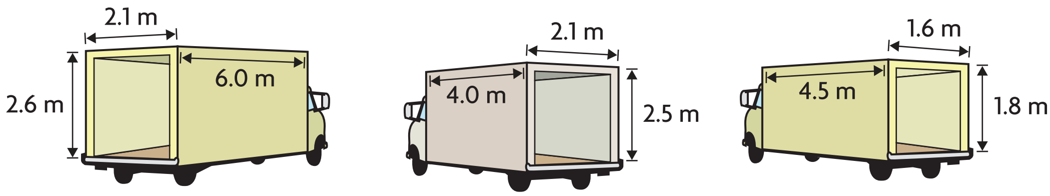
Explain how you would calculate the straight-line distance, to the nearest metre, from the helicopter to the sailboat. Justify your reasoning with calculations.

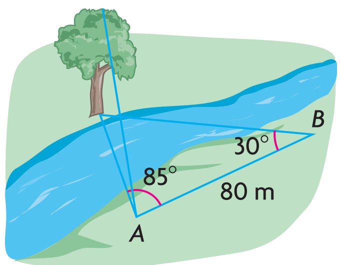
9. Brit and Tara are standing 13.5 m apart on a dock when they observe a sailboat moving parallel to the dock. When the boat is equidistant between both girls, the angle of elevation to the top of its 8.0 m mast is 51° for both observers. Describe how you would calculate the angle, to the nearest degree, between Tara and the boat as viewed from Brit's position. Justify your reasoning with calculations.

10. **T** In setting up for an outdoor concert, a stage platform has been dismantled into three triangular pieces as shown.



There are three vehicles available to transport the pieces. In order to prevent damaging the platform, each piece must fit exactly inside the vehicle. Explain how you would match each piece of the platform to the best-suited vehicle. Justify your reasoning with calculations.





11. Bert wants to calculate the height of a tree on the opposite bank of a river. To do this, he lays out a baseline 80 m long and measures the angles as shown at the left. The angle of elevation from *A* to the top of the tree is 28°. Explain if this information helps Bert to calculate the height of the tree to the nearest metre. Justify your reasoning with calculations.

12. **C** Chandra's homework question reads like this:
Bill and Chris live at different intersections on the same street, which runs north to south. When both of them stand at their front doors, they see a hot-air balloon toward the east at angles of elevation of 41° and 55°, respectively. Calculate the distance between the two friends.

a) Chandra says she doesn't have enough information to answer the question. Evaluate Chandra's statement. Justify your reasoning with calculations.

b) What additional information, if any, would you need to solve the problem? Justify your answer.

Extending

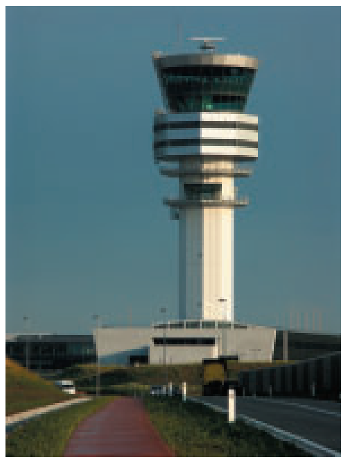
13. Two roads intersect at 34°. Two cars leave the intersection on different roads at speeds of 80 km/h and 100 km/h. After 2 h, a traffic helicopter that is above and between the two cars takes readings on them. The angle of depression to the slower car is 20°, and the straight-line distance from the helicopter to that car is 100 km. Assume that both cars are travelling at constant speed.

a) Calculate the straight-line distance, to the nearest kilometre, from the helicopter to the faster car. Explain your reasoning for each step of your solution.

b) Determine the altitude of the helicopter to the nearest kilometre.

14. Simone is facing north at the entrance of a tunnel through a mountain. She notices that a 1515 m high mountain in the distance has a bearing of 270° and its peak appears at an angle of elevation of 35°. After she exits the tunnel, the same mountain has a bearing of 258° and its peak appears at an angle of elevation of 31°. Assuming that the tunnel is perfectly level and straight, how long is it to the nearest metre?

15. An airport radar operator locates two planes flying toward the airport. The first plane, *P*, is 120 km from the airport, *A*, at a bearing of 70° and with an altitude of 2.7 km. The other plane, *Q*, is 180 km away on a bearing of 125° and with an altitude of 1.8 km. Calculate the distance between the two planes to the nearest tenth of a kilometre.



16. Mario is standing at ground level exactly at the corner where two exterior walls of his apartment building meet. From Mario's position, his apartment window on the north side of the building appears 44.5 m away at an angle of elevation of 55°. Mario notices that his friend Thomas's window on the west side of the building appears 71.0 m away at an angle of elevation of 34°.

a) If a rope were pulled taut from one window to the other, around the outside of the building, how long, to the nearest tenth of a metre, would the rope need to be? Explain your reasoning.

b) What is the straight-line distance through the building between the two windows? Round your answer to the nearest tenth of a metre.

5 Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 5.5, Examples 1 to 4.
- Try Chapter Review Questions 6 and 7.

Q: What steps would you follow to prove a trigonometric identity?

A: A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variable. You may rewrite the trigonometric ratios in terms of x, y, and r and then simplify, or you may rewrite each side of the equation in terms of sine and cosine and then use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, where appropriate. If a trigonometric ratio is in the denominator of a fraction, there are restrictions on the variable because the denominator cannot equal zero.

For example, the solution below is one way to prove that $\tan^2 \theta + 1 = \sec^2 \theta$ is an identity.

EXAMPLE

$\tan^2 \theta + 1 = \sec^2 \theta$

$$\text{L.S.} = \tan^2 \theta + 1 \qquad \text{R.S.} = \sec^2 \theta$$

First separate both sides of the equation.

$$= \left(\frac{\sin \theta}{\cos \theta}\right)^2 + 1 \qquad = \left(\frac{1}{\cos \theta}\right)^2$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} + 1 \qquad = \frac{1}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta}$$

Write $\tan \theta$ as $\frac{\sin \theta}{\cos \theta}$. The left side of the equation is more complicated, so simplify it. Find a common denominator. Then use the Pythagorean identity. Since the denominator cannot equal 0, there is a restriction on θ, so $\cos \theta \neq 0$.

$$= \frac{1}{\cos^2 \theta}$$

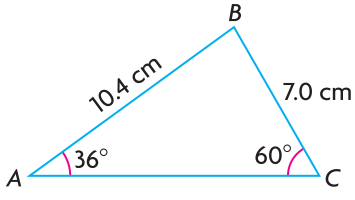
$$= \text{R.S.}$$

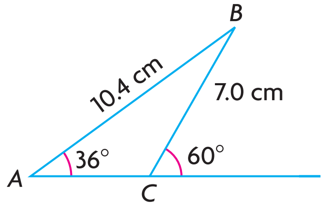
$\therefore \tan^2 \theta + 1 = \sec^2 \theta$ for all angles θ, where $\cos \theta \neq 0$.

Q: How do you know when you are dealing with the ambiguous case of the sine law?

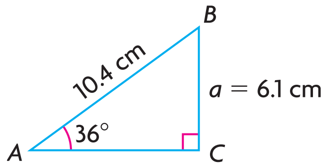
A: The ambiguous case of the sine law refers to the situation where 0, 1, or 2 triangles are possible given the information in a problem. This situation occurs when you know two side lengths and an angle (SSA).

For example, given $\triangle ABC$, where $\angle A = 36°$, $a = 7.0$ cm, and $c = 10.4$ cm, there are two possible triangles:





If $a = 6.1$ cm, then $\triangle ABC$ is a right triangle and 6.1 cm is the shortest possible length for a:



If $a < 6.1$ cm, a triangle cannot be drawn.

Study Aid

- See Lesson 5.6, Examples 1 and 2.
- Try Chapter Review Questions 8 and 9.

Q: How do you decide when to use the sine law or the cosine law to solve a problem?

A: Given any triangle, if you know two sides and the angle between those sides, or all three sides, use the cosine law. If you know an angle opposite a side, use the sine law.

Study Aid

- See Lesson 5.7, Examples 1 and 2.
- Try Chapter Review Questions 10 and 11.

Q: What approaches are helpful in solving two- and three-dimensional trigonometric problems?

A: Always start with a sketch of the given information because the sketch will help you determine whether the Pythagorean theorem, the sine law, or the cosine law is the best method to use. If you have right triangles, use the Pythagorean theorem and/or trigonometric ratios. If you know three sides or two sides and the contained angle in an oblique triangle, use the cosine law. For all other cases, use the sine law.

Study Aid

- See Lesson 5.8, Examples 1, 2, and 3.
- Try Chapter Review Questions 12 and 13.

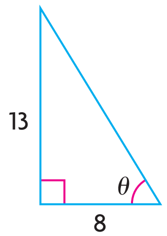
PRACTICE Questions

Lesson 5.1

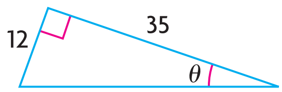
1. i) For each triangle, state the reciprocal trigonometric ratios for angle θ.

ii) Calculate the value of θ to the nearest degree.

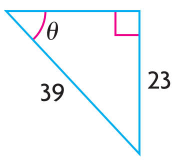
a)



b)



c)



Lesson 5.2

2. Determine the exact value of each trigonometric expression. Express your answers in simplified radical form.

a) $(\sin 45°)(\cos 45°) + (\sin 30°)(\cos 60°)$

b) $(1 - \tan 45°)(\sin 30°)(\cos 30°)(\tan 60°)$

c) $\tan 30° + 2(\sin 45°)(\cos 60°)$

Lesson 5.3

3. i) State the sign of each trigonometric ratio. Use a calculator to determine the value of each ratio.

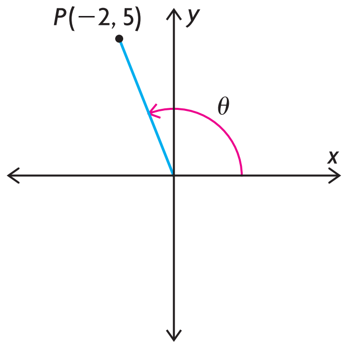
ii) For each trigonometric ratio, determine the principal angle and, where appropriate, the related acute angle. Then sketch another angle that has the equivalent ratio. Label the principal angle and the related acute angle on your sketch.

a) $\tan 18°$ **b)** $\sin 205°$ **c)** $\cos(-55°)$

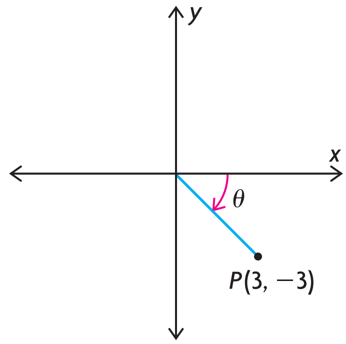
Lesson 5.4

4. For each sketch, state the primary trigonometric ratios associated with angle θ. Express your answers in simplified radical form.

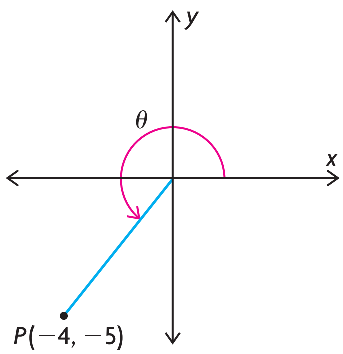
a)



b)



c)



5. Given $\cos \phi = \frac{-7}{\sqrt{53}}$, where $0° \leq \phi \leq 360°$,

a) in which quadrant(s) does the terminal arm of angle ϕ lie? Justify your answer.

b) state the other five trigonometric ratios for angle ϕ.

c) calculate the value of the principal angle ϕ to the nearest degree.

Lesson 5.5

6. Determine whether the equation $\cos \beta \cot \beta = \frac{1}{\sin \beta} - \sin \beta$ is an identity. State any restrictions on angle β.

7. Prove each identity. State any restrictions on the variables if all angles vary from 0° to 360°.

a) $\tan \alpha \cos \alpha = \sin \alpha$

b) $\frac{1}{\cot \phi} = \sin \phi \sec \phi$

c) $1 - \cos^2 x = \frac{\sin x \cos x}{\cot x}$

d) $\sec \theta \cos \theta + \sec \theta \sin \theta = 1 + \tan \theta$

Lesson 5.6

8. Determine whether it is possible to draw a triangle given each set of information. Sketch all possible triangles where appropriate. Calculate, then label, all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $b = 3.0$ cm, $c = 5.5$ cm, $\angle B = 30°$

b) $b = 12.2$ cm, $c = 8.2$ cm, $\angle C = 34°$

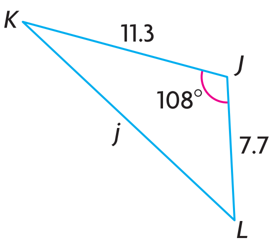
c) $a = 11.1$ cm, $c = 5.2$ cm, $\angle C = 33°$

9. Two forest fire stations, P and Q, are 20.0 km apart. A ranger at station Q sees a fire 15.0 km away. If the angle between the line PQ and the line from P to the fire is 25°, how far, to the nearest tenth of a kilometre, is station P from the fire?

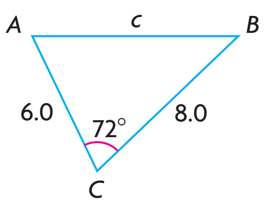
Lesson 5.7

10. Determine each unknown side length to the nearest tenth.

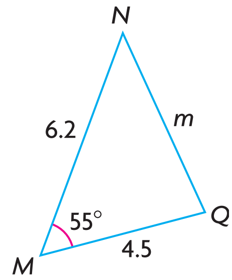
a)



b)



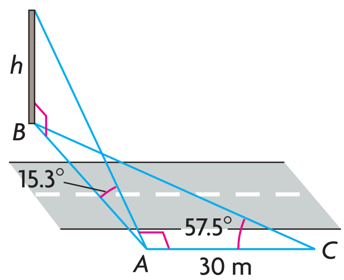
c)



11. Two spotlights, one blue and the other white, are placed 6.0 m apart on a track on the ceiling of a ballroom. A stationary observer standing on the ballroom floor notices that the angle of elevation is 45° to the blue spotlight and 70° to the white one. How high, to the nearest tenth of a metre, is the ceiling of the ballroom?

Lesson 5.8

12. To determine the height of a pole across a road, Justin takes two measurements. He stands at point A directly across from the base of the pole and determines that the angle of elevation to the top of the pole is 15.3°. He then walks 30 m parallel to the freeway to point C, where he sees that the base of the pole and point A are 57.5° apart. From point A, the base of the pole and point C are 90.0° apart. Calculate the height of the pole to the nearest metre.



13. While standing at the left corner of the schoolyard in front of her school, Suzie estimates that the front face is 8.9 m wide and 4.7 m high. From her position, Suzie is 12.0 m from the base of the right exterior wall. She determines that the left and right exterior walls appear to be 39° apart. From her position, what is the angle of elevation, to the nearest degree, to the top of the left exterior wall?

5 Chapter Self-Test

1. i) For each point, sketch the angle in standard position to determine all six trigonometric ratios.

ii) Determine the value of the principal angle and the related acute angle, where appropriate, to the nearest degree.

a) $P(-3, 0)$ **b)** $S(-8, -6)$

2. Given angle θ, where $0° \leq \theta \leq 360°$, determine all possible angles for θ.

a) $\sin \theta = -\frac{1}{2}$

b) $\cos \theta = \frac{\sqrt{3}}{2}$

c) $\cot \theta = -1$

d) $\sec \theta = -2$

3. Given $\cos \theta = -\frac{5}{13}$, where the terminal arm of angle θ lies in quadrant 2, evaluate each trigonometric expression.

a) $\sin \theta \cos \theta$ **b)** $\cot \theta \tan \theta$

4. i) Prove each identity. Use a different method for parts (a) and (b). State any restrictions on the variables.

ii) Explain why these identities are called Pythagorean identities.

a) $\tan^2 \phi + 1 = \sec^2 \phi$ **b)** $1 + \cot^2 \alpha = \csc^2 \alpha$

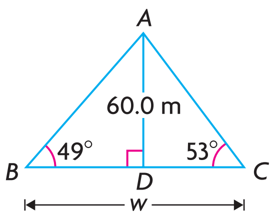
5. a) Sketch a triangle of your own choice and label the sides and angles.

b) State all forms of the cosine law that apply to your triangle.

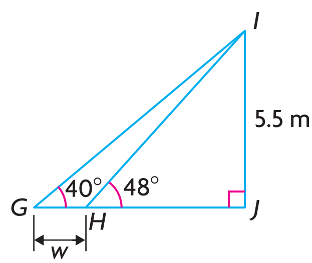
c) State all forms of the sine law that apply to your triangle.

6. For each triangle, calculate the value of w to the nearest tenth of a metre.

a)



b)



7. Given each set of information, determine how many triangles can be drawn. Calculate, then label, all side lengths to the nearest tenth and all interior angles to the nearest degree, where appropriate.

a) $a = 1.5$ cm, $b = 2.8$ cm, and $\angle A = 41°$

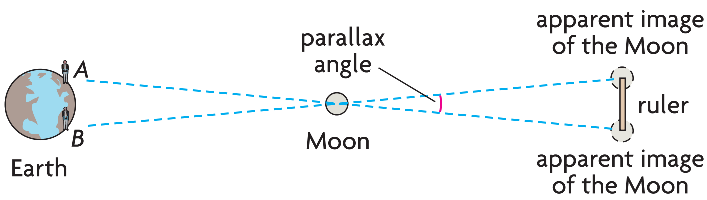
b) $a = 2.1$ cm, $c = 6.1$ cm, and $\angle A = 20°$

8. To estimate the amount of usable lumber in a tree, Chitra must first estimate the height of the tree. From points A and B on the ground, she determined that the angles of elevation for a certain tree were 41° and 52°, respectively. The angle formed at the base of the tree between points A and B is 90°, and A and B are 30 m apart. If the tree is perpendicular to the ground, what is its height to the nearest metre?

5 Chapter Task

Parallax

Parallax is the apparent displacement of an object when it is viewed from two different positions.



Astronomers measure the parallax of celestial bodies to determine how far those bodies are from Earth.

On October 28, 2004, three astronomers (Peter Cleary, Pete Lawrence, and Gerardo Addiègo) each at a different location on Earth, took a digital photo of the Moon during a lunar eclipse at exactly the same time. The data related to these photos is shown.

	Shortest Distance on Earth's Surface Between Two Locations	Parallax Angle
AB (Montréal, Canada to Selsey, UK)	5 220 km	0.7153°
AC (Montréal, Canada to Montevideo, Uruguay)	9 121 km	1.189°
BC (Selsey, UK to Montevideo, Uruguay)	10 967 km	1.384°

? What is the most accurate method to determine the distance between the Moon and Earth, from the given data?

A. Sketch a triangle with the Moon and locations *A* and *B* as the vertices. Label all the given angles and distances. What kind of triangle do you have?

B. Determine all unknown sides to the nearest kilometre and angles to the nearest thousandth of a degree. How far, to the nearest kilometre, is the Moon from either Montréal or Selsey?

C. Repeat parts A and B for locations *B* and *C*, and for *A* and *C*.

D. On October 28, 2004, the Moon was about 391 811 km from Earth (surface to surface). Calculate the relative error, to the nearest tenth of a percent, for all three distances you calculated.

E. Which of your results is most accurate? What factors contribute most to the error in this experiment?

Task Checklist

- ✔ Did you draw the correct sketches?
- ✔ Did you show your work?
- ✔ Did you provide appropriate reasoning?
- ✔ Did you explain your thinking clearly?