CHAPTER 7: Discrete Functions: Sequences and Series

NOTE: Answers are given to the same number of decimal points as the numbers in each question.

Getting Started, p. 414

1. a) The slope-intercept form of a line is $y = mx + b$, where m is the slope and b is the y-intercept. For this line, $m = -\frac{2}{5}$ and $b = 8$, so the equation is $y = -\frac{2}{5}x + 8$

b) The point-slope form of a line is $y = m(x - x_1) + y_1$, where (x_1, y_1) is a point on the line. For this line, $m = -9$ and (5, 4) is a point on this line, so the equation is

$$y = -9(x - 5) + 4$$
$$= -9x + 49$$

c) The points (5, 0) and (0, −7) are on this line, so $m = \frac{0 - (-7)}{5 - 0} = \frac{7}{5}$. The point-slope form is

$$y = \frac{7}{5}(x - 5) + 0$$
$$= \frac{7}{5}x - 7$$

d) The points (−12, 17) and (5, −17) are on this line, so $m = \frac{17 - (-17)}{-12 - 5} = -\frac{34}{17} = -2$.

The point-slope form is

$$y = -2(x - 5) + (-17)$$
$$= -2x + 10 - 17$$
$$= -2x - 7$$

2. a) $g(-2) = 3(-2)^2 + (-2) - 4$

$$= 6$$

b) $f\left(\frac{3}{4}\right) = \left(\frac{4}{5}\right)\left(\frac{3}{4}\right) + \frac{7}{10}$

$$= \frac{13}{10}$$

c) $g(\sqrt{6}) = 4(\sqrt{6})^2 - 24 = 0$

d) $h\left(\frac{1}{3}\right) = 64^{\frac{1}{3}}$

$$= 4$$

3. a) The 1st differences are 5, 5, 5, 5. The 1st differences are constant, so the relation is linear.

b) The 1st differences are 6, 12, 24, 48. The 1st differences are not constant, so the relation is not linear. The 2nd differences are 6, 12, 24. The 2nd differences are not constant, so the relation is not quadratic either. This relation is neither linear nor quadratic.

c) The 1st differences are 2, 6, 10, 14. The 1st differences are not constant, so the relation is not linear. The 2nd differences are 4, 4, 4. The 2nd differences are constant, so the relation is quadratic.

4. a)

$$2x - 3 = 7$$
$$2x = 10$$
$$x = 5$$

b)

$$5x + 8 = 2x - 7$$
$$5x = 2x - 15$$
$$3x = -15$$
$$x = -5$$

c)

$$5(3x - 2) + 7x - 4 = 2(4x + 8) - 2x + 3$$
$$15x - 10 + 7x - 4 = 8x + 16 - 2x + 3$$
$$22x - 14 = 6x + 19$$
$$16x - 14 = 19$$
$$16x = 33$$
$$x = \frac{33}{16}$$

d)

$$-8x + \frac{3}{4} = \frac{1}{3}x - 12$$
$$-\frac{25}{3}x + \frac{3}{4} = -12$$
$$-\frac{25}{3}x = -\frac{51}{4}$$
$$x = \frac{153}{100}$$

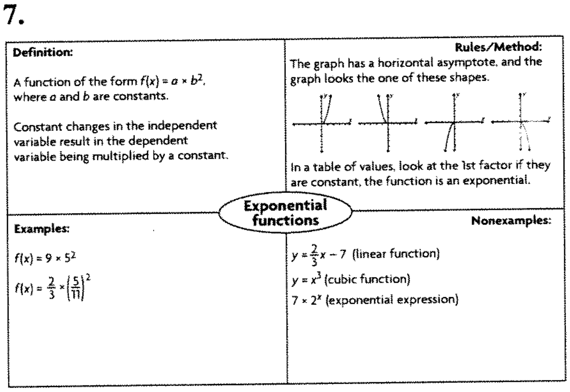
5. Since the half-life of the material is 100 years, and the starting amount is 2.3 kg, the amount remaining, in kg, after t years can be modelled by $M(t) = 2.3\left(\frac{1}{2}\right)^{\frac{t}{100}}$. So, the amount remaining after 1000 years is

$$M(1000) = 2.3\left(\frac{1}{2}\right)^{\frac{1000}{100}}$$
$$= 2.3\left(\frac{1}{2}\right)^{10}$$
$$= 2.3\left(\frac{1}{1024}\right)$$
$$\doteq 0.022 \text{ kg or } 2.2 \text{ g}$$

6. Since the percentage of the pond covered by lily pads doubles every week and the percentage initially covered by lily pads is 0.1, the percentage of the pond covered by lily pads after t weeks can be modelled by $P(t) = 0.1(2)^t$. So, the amount covered after nine weeks is

$$P(9) = 0.1(2)^9$$
$$= 0.1(512)$$
$$= 51.2\%$$

7.



7.1 Arithmetic Sequences, pp. 424–425

1. a) This sequence is arithmetic. The common difference between any pair of consecutive terms is $5 - 1 = 4$.

b) This sequence is not arithmetic, since the differences between consecutive terms alternate between 4 and 6.

c) This sequence is not arithmetic, since the differences between consecutive terms are not constant.

d) This sequence is arithmetic. The common difference between any pair of consecutive terms is $48 - 59 = -11$.

2. a) The general term for any arithmetic sequence is $t_n = a + (n - 1)d$. The common difference d of this arithmetic sequence is $42 - 28 = 14$, and the first term a is 28. The general term for this sequence is

$$t_n = 28 + 14(n - 1)$$
$$= 28 + 14n - 14$$
$$= 14n + 14$$

The recursive formula for an arithmetic sequence is $t_1 = a = 28$,

$$t_n = t_{n-1} + d$$
$$= t_{n-1} + 14, \text{ where } n > 1.$$

b) For this arithmetic sequence, $d = 49 - 53 = -4$, and $a = 53$. The general term for this sequence is

$$t_n = 53 + (-4)(n - 1)$$
$$= 53 - 4n + 4$$
$$= 57 - 4n$$

The recursive formula for this sequence is $t_1 = 53$,

$$t_n = t_{n-1} + (-4)$$
$$= t_{n-1} - 4, \text{ where } n > 1.$$

c) For this arithmetic sequence, $d = (-111)-(-1) = -110$, and $a = -1$. The general term for this sequence is

$$t_n = (-1) + (-110)(n - 1)$$
$$= -1 - 110n + 110$$
$$= 109 - 110n$$

The recursive formula for this sequence is $t_1 = -1$,

$$t_n = t_{n-1} + (-110)$$
$$= t_{n-1} - 110, \text{ where } n > 1.$$

3. $t_{11} - t_{10} = 41 - 29 = 12$. So, $d = 12$. Since this sequence is arithmetic, $t_{12} - t_{11} = d = 12$ as well. So,

$$t_{12} - 41 = 12$$
$$t_{12} = 53$$

4. The common difference d of this arithmetic sequence is $102 - 85 = 17$, and the first term a is 85. The general term for this sequence is

$$t_n = a + (n - 1)d$$
$$= 85 + 17(n - 1)$$
$$= 85 + 17n - 17$$
$$= 17n + 68$$

The 15th term is

$$t_{15} = 17(15) + 68$$
$$= 255 + 68$$
$$= 323$$

5. a) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $11 - 8 = 3$.

ii) For this sequence, $d = 3$ and $a = 8$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 8 + 3(n - 1)$
$= 8 + 3n - 3$
$= 3n + 5$
The recursive formula for this sequence is $t_1 = 8$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 3$, where $n > 1$.
b) i) This sequence is not arithmetic, since the differences between consecutive terms are not constant.
ii) N/A
c) i) This sequence is not arithmetic, since the differences between consecutive terms are not constant.
ii) N/A
d) i) This sequence is not arithmetic, since the differences between consecutive terms are not constant.
ii) N/A
e) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $34 - 23 = 11$.
ii) For this sequence, $d = 11$ and $a = 23$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 23 + 11(n - 1)$
$= 23 + 11n - 11$
$= 11n + 12$
The recursive formula for this sequence is $t_1 = 23$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 11$, where $n > 1$.
f) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$.
ii) For this sequence, $d = \frac{1}{6}$ and $a = \frac{1}{6}$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= \frac{1}{6} + \frac{1}{6}(n - 1)$
$= \frac{1}{6} + \frac{1}{6}n - \frac{1}{6}$
$= \frac{1}{6}n$
The recursive formula for this sequence is $t_1 = \frac{1}{6}$,
$t_n = t_{n-1} + d$
$= t_{n-1} + \frac{1}{6}$, where $n > 1$.
6. a) Since consecutive terms increase by 8, $d = 8$. The recursive formula for this sequence is $t_1 = 19$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 8$, where $n > 1$.
The general term for this sequence is
$t_n = a + (n - 1)d$
$= 19 + 8(n - 1)$
$= 19 + 8n - 8$
$= 8n + 11$
b) Since consecutive terms decrease by 5, $d = -5$. The recursive formula for this sequence is
$t_1 = 4$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 5$, where $n > 1$.
The general term for this sequence is
$t_n = a + (n - 1)d$
$= 4 + (-5)(n - 1)$
$= 4 - 5n + 5$
$= 9 - 5n$
c) Since the first term is 21 and the second term is 26, $d = 26 - 21 = 5$. The recursive formula for this sequence is $t_1 = 21$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 5$, where $n > 1$.
The general term for this sequence is
$t_n = a + (n - 1)d$
$= 21 + 5(n - 1)$
$= 21 + 5n - 5$
$= 5n + 16$
d) Since consecutive terms decrease by 12, $d = -12$. Find the first term a. The general term of this sequence is
$t_n = a + (n - 1)d$
$= a + (-12)(n - 1)$
$= a - 12n + 12$
Since $t_4 = 35$, solve for a:
$t_4 = a - 12(4) + 12$
$35 = a - 36$
$a = 71$.
The recursive formula for this sequence is $t_1 = 71$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 12$, where $n > 1$.
The general term of this sequence is
$t_n = a - 12n + 12$
$t_n = 71 - 12n + 12$
$= 83 - 12n$

7. a) i) This recursive formula defines an arithmetic sequence: For every $n > 1$, $t_n = t_{n-1} + d$ for the constant $d = 14$.

ii) $t_1 = 13$

$t_2 = t_1 + 14$
$= 13 + 14$
$= 27$

$t_3 = t_2 + 14$
$= 27 + 14$
$= 41$

$t_4 = t_3 + 14$
$= 41 + 14$
$= 55$

$t_5 = t_4 + 14$
$= 55 + 14$
$= 69$

The common difference d is 14.

b) i) This recursive formula does not define an arithmetic sequence;

$t_1 = 5$

$t_2 = 3t_1$
$= 3(5)$
$= 15$

$t_3 = 3t_2$
$= 3(15)$
$= 45$

$t_2 - t_1 = 10$, but $t_3 - t_2 = 30$. So, the difference between consecutive terms is not constant.

ii) N/A

c) i) This recursive formula does not define an arithmetic sequence;

$t_1 = 4$

$t_2 = t_1 + 2 - 1$
$= 4 + 1$
$= 5$

$t_3 = t_2 + 3 - 1$
$= 5 + 2$
$= 7$

$t_2 - t_1 = 1$, but $t_3 - t_2 = 2$. So the difference between consecutive terms is not constant.

ii) N/A

d) i) This recursive formula does not appear to define an arithmetic sequence. Computing the first few terms:

$t_1 = 1$

$t_2 = 2t_1 - 2 + 2$
$= 2 - 2 + 2$
$= 2$

$t_3 = 2t_2 - 3 + 2$
$= 4 - 3 + 2$
$= 3$

$t_4 = 2t_3 - 4 + 2$
$= 6 - 4 + 2$
$= 4$

$t_5 = 2t_4 - 5 + 2$
$= 8 - 5 + 2$
$= 5$

So, this looks like an arithmetic sequence with general term $t_n = n$. This can be checked. Firstly, for $n = 1$, this general term certainly holds. And, if $t_{n-1} = n - 1$, then

$t_n = 2t_{n-1} - n + 2$
$= 2(n - 1) - n + 2$
$= 2n - 2 - n + 2$
$= n$

ii) The first five terms are 1, 2, 3, 4, 5. The common difference d is $2 - 1 = 1$.

8. a) i) The common difference d of this sequence is $40 - 35 = 5$ and the first term a is 35. The general term for this sequence is

$t_n = a + (n - 1)d$
$= 35 + 5(n - 1)$
$= 35 + 5n - 5$
$= 5n + 30$

ii) The recursive formula for this sequence is

$t_1 = 35$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 5$, where $n > 1$.

iii) By substituting in $n = 11$,

$t_{11} = 5(11) + 30$
$= 85$

b) i) The common difference d of this sequence is $20 - 31 = -11$ and the first term a is 31. The general term for this sequence is

$t_n = a + (n - 1)d$
$= 31 + (-11)(n - 1)$
$= 31 - 11n + 11$
$= 42 - 11n$

ii) The recursive formula for this sequence is

$t_1 = 31$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 11$, where $n > 1$.

iii) By substituting $n = 11$,

$t_{11} = 42 - 11(11)$
$= -79$

c) i) The common difference d of this sequence is $(-41)-(-29) = -12$, and the first term a is -29. The general term for this sequence is

$t_n = a + (n - 1)d$
$= -29 + (-12)(n - 1)$

$= -29 - 12n + 12$
$= -17 - 12n$

ii) The recursive formula for this sequence is
$t_1 = -29,$
$t_n = t_{n-1} + d$
$= t_{n-1} - 12$, where $n > 1$.

iii) By substituting $n = 11$,
$t_{11} = -17 - 12(11)$
$= -149$

d) i) The common difference d of this sequence is $11 - 11 = 0$, and the first term a is 11. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 11 + 0(n - 1)$
$= 11$

ii) The recursive formula for this sequence is
$t_1 = 11,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 0$
$= t_{n-1}$, where $n > 1$.

iii) By substituting $n = 11$, $t_{11} = 11$.

e) i) The common difference d of this sequence is $\frac{6}{5} - 1 = \frac{1}{5}$, and the first term a is 1. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 1 + \frac{1}{5}(n - 1)$
$= 1 + \frac{1}{5}n - \frac{1}{5}$
$= \frac{1}{5}n + \frac{4}{5}$

ii) The recursive formula for this sequence is
$t_1 = 1,$
$t_n = t_{n-1} + d$
$= t_{n-1} + \frac{1}{5}$, where $n > 1$.

iii) By substituting $n = 11$,
$t_{11} = \frac{1}{5}(11) + \frac{4}{5}$
$= 3$

f) i) The common difference d of this sequence is $0.57 - 0.4 = 0.17$, and the first term a is 0.4. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 0.4 + (0.17)(n - 1)$
$= 0.4 + 0.17n - 0.17$
$= 0.17n + 0.23$

ii) The recursive formula for this sequence is
$t_1 = 0.4,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 0.17$, where $n > 1$.

iii) By substituting $n = 11$,
$t_{11} = 0.23 + 0.17(11)$
$= 2.1$

9. a) i) This general term defines an arithmetic sequence, since it is a linear function.

ii) $t_1 = 8 - 2(1)$
$= 6$
$t_2 = 8 - 2(2)$
$= 4$
$t_3 = 8 - 2(3)$
$= 2$
$t_4 = 8 - 2(4)$
$= 0$
$t_5 = 8 - 2(5)$
$= -2$

The common difference d of this sequence is $4 - 6 = -2$.

b) i) This general term does not define an arithmetic sequence, since it is not a linear function.

ii) N/A

c) i) This general term defines an arithmetic sequence, since it is a linear function.

ii) $t_1 = \frac{1}{4}(1) + \frac{1}{2}$
$= \frac{3}{4}$
$t_2 = \frac{1}{4}(2) + \frac{1}{2}$
$= 1$
$t_3 = \frac{1}{4}(3) + \frac{1}{2}$
$= \frac{5}{4}$
$t_4 = \frac{1}{4}(4) + \frac{1}{2}$
$= \frac{3}{2}$
$t_5 = \frac{1}{4}(5) + \frac{1}{2}$
$= \frac{7}{4}$

The common difference d of this sequence is
$1 - \frac{3}{4} = \frac{1}{4}$

d) i) This general term does not define an arithmetic sequence, since it is not a linear function.
ii) N/A

10. a) The number of seats in the nth row of the opera house can be modelled by an arithmetic function. The first term a of this sequence is 27, the number of seats in the first row. The common difference d is $34 - 27 = 7$. So, the general term of this arithmetic sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 27 + 7(n - 1) \\ &= 27 + 7n - 7 \\ &= 7n + 20 \end{aligned}$$

The number of seats in the 10th row is

$$\begin{aligned} t_{10} &= 7(10) + 20 \\ &= 90 \end{aligned}$$

b) Call n the number of rows in the opera house. Then the final row is the nth row, and has t_n seats. So,

$$\begin{aligned} t_n &= 7n + 20 \\ 181 &= 7n + 20 \\ 161 &= 7n \\ 23 &= n \end{aligned}$$

There are 23 rows of seats in the opera house.

11. Since Janice's wage increases by \$0.15/hr each month, after t months it will have been raised by \$0.15t dollars per hour. Janice's wage after t months is then \$9.25 + \$0.15t dollars per hour. The smallest value of t is needed because this is at least twice her starting wage, or \$18.50.

$$\begin{aligned} \$9.25 + \$0.15t &\geq \$18.50 \\ \$0.15t &\geq \$9.25 \\ t &\geq 61\tfrac{2}{3} \end{aligned}$$

So, since t must be a whole number, the smallest such value of t is 62, and so her wage becomes double her starting wage 62 months after starting the job.

12. Phil's account gains 3.5% of \$5000, or $0.035(5000) = \$175$, each year. So, the amount of money in his account, in dollars, after t years is $\$5000 + 175t$. The smallest value of t is needed because this is at least \$7800.

$$\begin{aligned} \$5000 + 175t &\geq \$7800 \\ 175t &\geq 2800 \\ t &\geq 16 \end{aligned}$$

Phil will have \$7800 in his account after 16 years.

13. a) For this arithmetic sequence, the common difference d is $9 - 7 = 2$, and the first term a is 7.
The general term for this arithmetic sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 7 + 2(n - 1) \\ &= 7 + 2n - 2 \\ &= 2n + 5 \end{aligned}$$

If n denotes the number of terms,

$$\begin{aligned} 63 = t_n &= 2n + 5 \\ 58 &= 2n \\ n &= 29 \end{aligned}$$

There are 29 terms in this arithmetic sequence.

b) For this arithmetic sequence, the common difference d is $-25 - (-20) = -5$, and the first term a is -20.
The general term for this arithmetic sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= -20 + (-5)(n - 1) \\ &= -20 - 5n + 5 \\ &= -5n - 15 \end{aligned}$$

If n denotes the number of terms,

$$\begin{aligned} -205 = t_n &= -5n - 15 \\ -190 &= -5n \\ n &= 38 \end{aligned}$$

There are 38 terms in this arithmetic sequence.

c) For this arithmetic sequence, the common difference d is $27 - 31 = -4$, and the first term a is 31.
The general term for this arithmetic sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 31 + (-4)(n - 1) \\ &= 31 - 4n + 4 \\ &= 35 - 4n \end{aligned}$$

If n denotes the number of terms,

$$\begin{aligned} -25 = t_n &= 35 - 4n \\ -60 &= -4n \\ n &= 15 \end{aligned}$$

There are 15 terms in this arithmetic sequence.

d) For this arithmetic sequence, the common difference d is $16 - 9 = 7$, and the first term a is 9.
The general term for this arithmetic sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 9 + 7(n - 1) \\ &= 9 + 7n - 7 \\ &= 7n + 2 \end{aligned}$$

If n denotes the number of terms,

$$\begin{aligned} 100 = t_n &= 7n + 2 \\ 98 &= 7n \\ n &= 14 \end{aligned}$$

There are 14 terms in this arithmetic sequence.

e) For this arithmetic sequence, the common difference d is , and the first term a is -33.
The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= -33 + 7(n - 1)$
$= -33 + 7n - 7$
$= 7n - 40$
If n denotes the number of terms,
$86 = t_n = 7n - 40$
$126 = 7n$
$n = 18$
There are 18 terms in this arithmetic sequence.
f) For this arithmetic sequence, the common difference d is $19 - 28 = -9$, and the first term a is 28.
The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= 28 + (-9)(n - 1)$
$= 28 - 9n + 9$
$= 37 - 9n$
If n denotes the number of terms,
$-44 = t_n = 37 - 9n$
$-81 = -9n$
$n = 9$
There are 9 terms in this arithmetic sequence.
14. To get from t_4 to t_8, you must add d four times. So, from t_4 and t_8, you can find d by taking $t_8 - t_4$ and dividing by 4. To get from t_8 to t_{100}, you must add d 92 times. So, from t_8 and d, you can find t_{100} by adding $92d$ to t_8.
15. To get from t_{50} to t_{93}, you must add d 43 times. So,
$t_{93} - t_{50} = 43d$
$539 - 238 = 43d$
$301 = 43d$
$d = 7$
The general term for this sequence is
$t_n = a + (n - 1)d$
$= a + 7(n - 1)$
$= a + 7n - 7$
$t_{50} = 238$, so solve for a:
$t_{50} = 238 = a + 7(50) - 7$
$238 = a + 343$
$a = -105$
So, substitute this value of a into our general term:
$t_n = -105 + 7n - 7$
$= -7n - 112$
16. a) For example, take the first term to be 20 and the second term to be 50. Then, $a = 20$ and $d = 50 - 20 = 30$, and the sequence begins 20, 50, 80, . . . or, take the first term to be 50 and the second term to be 20. Then, $a = 50$ and $d = 20 - 50 = -30$, and the sequence begins 50, 20, -10, ... Finally, take the second term to be 20. The third term cannot be taken to be 50, or d will be $50 - 20 = 30$ again. So, take the fourth term to be 50. To get from the second term to the fourth term, you must add d twice. Therefore, $2d = 50 - 20 = 30$, and so $d = 15$. The first term is then 15 less than the second term,
so $a = 20 - 15 = 5$, and the sequence begins 5, 20, 35, ...
b) If 20 and 50 are terms of an arithmetic sequence with common difference d, then there is a nonzero integer m so that
$md = 50 - 20 = 30$. So, d must equal $\frac{30}{m}$ for some nonzero integer m.
17. 13 is the first term of this sequence, so $t_1 = 13$. Let's say that $t_m = 37$ and $t_k = 73$ for integers m and k greater than 1. To get from t_1 to t_m, one must add d $m - 1$ times. Therefore,
$(m - 1)d = 37 - 13$
$(m - 1)d = 24$
Similarly,
$(k - 1)d = 73 - 13$
$(k - 1)d = 60$
$m - 1$ and $k - 1$ are both positive integers. The only values of d which can be multiplied by positive integers to get 24 and 60 are 1, 2, 3, 4, 6, and 12. For each possible value of d, find the 100th term of the arithmetic sequence. The general term of this arithmetic sequence is
$t_n = a + (n - 1)d$
$= 13 + (n - 1)d$
So, $t_{100} = 13 + (100 - 1)d$
$= 13 + 99d$
Calculate this for each possible value of d.
If $d = 1$, then $t_{100} = 13 + 99(1) = 112$. If $d = 2$, then $t_{100} = 13 + 99(2) = 211$. If $d = 3$, then $t_{100} = 13 + 99(3) = 310$. If $d = 4$, then

$t_{100} = 13 + 99(4) = 409$. If $d = 6$, then
$t_{100} = 13 + 99(6) = 607$. If $d = 12$, then
$t_{100} = 13 + 99(12) = 1201$.

18. The general term of our original arithmetic sequence is
$t_n = a + (n - 1)d$.
The nth term of the new sequence is the t_nth term of the original sequence. So, the general term for the new sequence is

$$\begin{aligned} s_n &= a + (t_n - 1)d \\ &= a + (a + (n - 1)d)d \\ &= a + (a + nd - d)d \\ &= a + ad + nd^2 - d^2 \\ &= (a + ad) + (n - 1)d^2 \end{aligned}$$

The new sequence is therefore also arithmetic, with first term $a + ad$ and common difference d^2.

7.2 Geometric Sequences, pp. 430–432

1. a) This sequence is not geometric, since the ratio between consecutive terms is not constant.
b) This sequence is geometric. The ratio r between any pair of consecutive terms is $\frac{15}{5} = 3$.
c) This sequence is not geometric, since the ratio between consecutive terms is not constant.
d) This sequence is geometric. The ratio r between any pair of consecutive terms is $\frac{3000}{6000} = \frac{1}{2}$.

2. a) The common ratio r of this geometric sequence is $\frac{36}{9} = 4$, and the first term a is 9. The general term for this sequence is $t_n = 9 \times 4^{n-1}$
The recursive formula for a geometric sequence is
$t_1 = a = 9$,
$$\begin{aligned} t_n &= rt_{n-1} \\ &= 4t_{n-1}, \text{ where } n > 1. \end{aligned}$$
b) For this geometric sequence, $a = 625$ and $r = \frac{1250}{625} = 2$. The general term for this sequence is $t_n = 625 \times 2^{n-1}$
The recursive formula for this geometric sequence is $t_1 = 625$, $t_n = 2t_{n-1}$, where $n > 1$.

c) For this geometric sequence, $a = 10\,125$ and $r = \frac{6750}{10\,125} = \frac{2}{3}$. The general term for this sequence is $t_n = 625 \times \left(\frac{2}{3}\right)^{n-1}$
The recursive formula for this geometric sequence is $t_1 = 10\,125$, $t_n = \frac{2}{3}t_{n-1}$, where $n > 1$.

3. $\frac{t_{32}}{t_{31}} = \frac{1107}{123} = 9$. So, $r = 9$. Since this sequence is geometric, $\frac{t_{33}}{t_{32}} = r = 9$ as well.

So, $\frac{t_{33}}{1107} = 9$
$t_{33} = 9963$

4. The common ratio r of this geometric sequence is $\frac{302\,330\,880}{1\,813\,985\,280} = \frac{1}{6}$, and the first term a is 1 813 985 280. The general term for this sequence is
$$\begin{aligned} t_n &= ar^{n-1} \\ &= 1\,813\,985\,280\left(\frac{1}{6}\right)^{n-1} \end{aligned}$$
The 10th term is
$$\begin{aligned} t_{10} &= 1\,813\,985\,280\left(\frac{1}{6}\right)^{(10-1)} \\ &= 1\,813\,985\,280\left(\frac{1}{6}\right)^{9} \\ &= 180 \end{aligned}$$

5. a) i) This sequence is geometric. The ratio r between any pair of consecutive terms is $\frac{24}{12} = 2$.
ii) For this sequence, $a = 12$ and $r = 2$. The general term for this sequence is
$$\begin{aligned} t_n &= ar^{n-1} \\ &= 12 \times 2^{n-1} \end{aligned}$$
The recursive formula for this sequence is
$t_1 = 12$,
$$\begin{aligned} t_n &= rt_{n-1} \\ &= 2t_{n-1} \end{aligned}$$
b) i) This sequence is not geometric, since the ratio between consecutive terms is not constant.
ii) N/A
c) i) This sequence is not geometric, since the ratio between consecutive terms is not constant.
ii) N/A
d) i) This sequence is geometric. The ratio r between any pair of consecutive terms is
$-\frac{15}{5} = -3$.

ii) For this sequence, $a = 5$ and $r = -3$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 5 \times (-3)^{n-1}$
The recursive formula for this sequence is
$t_1 = 5,$
$t_n = rt_{n-1}$
$= -3t_{n-1}$
e) i) This sequence is not geometric, since the ratio between consecutive terms is not constant.
ii) N/A
f) i) This sequence is geometric. The ratio r between any pair of consecutive terms is
$\frac{50}{125} = \frac{2}{5}$
ii) For this sequence, $a = 125$ and $r = \frac{2}{5}$.
The general term for this sequence is
$t_n = ar^{n-1}$
$= 125 \times \left(\frac{2}{5}\right)^{n-1}$
The recursive formula for this sequence is
$t_1 = 125,$
$t_n = rt_{n-1}$
$= \frac{2}{5}t_{n-1}.$
6. a) i) The common ratio r of this sequence is $\frac{20}{4} = 5$, and the first term a is 4. The general term for this sequence is
$t_n = ar^{n-1}$
$= 4 \times 5^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = 4,$
$t_n = rt_{n-1}$
$= 5t_{n-1}$, where $n > 1$.
iii) By substituting $n = 6$, notice that
$t_6 = 4 \times 5^{6-1}$
$= 12\,500.$
b) i) The common ratio r of this sequence is $\frac{-22}{-11} = 2$, and the first term a is -11.
The general term for this sequence is
$t_n = ar^{n-1}$
$= (-11) \times 2^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = -11,$
$t_n = rt_{n-1}$
$= 2t_{n-1}$, where $n > 1$.
iii) By substituting $n = 6$, notice that
$t_6 = (-11) \times 2^{6-1}$
$= -352.$
c) i) The common ratio r of this sequence is $-\frac{60}{15} = -4$, and the first term a is 15. The general term for this sequence is
$t_n = ar^{n-1}$
$= 15 \times (-4)^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = 15,$
$t_n = rt_{n-1}$
$= -4t_{n-1}$, where $n > 1$.
iii) By substituting $n = 6$, notice that
$t_6 = 15 \times (-4)^{6-1}$
$= -15\,360.$
d) i) The common ratio r of this sequence is $\frac{448}{896} = \frac{1}{2}$, and the first term a is 896.
The general term for this sequence is
$t_n = ar^{n-1}$
$= 896 \times \left(\frac{1}{2}\right)^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = 896,$
$t_n = rt_{n-1}$
$= \frac{1}{2}t_{n-1}$, where $n > 1$.
iii) By substituting $n = 6$, notice that
$t_6 = 896 \times \left(\frac{1}{2}\right)^{6-1}$
$= 28.$
e) i) The common ratio r of this sequence is $\frac{2}{6} = \frac{1}{3}$, and the first term a is 6. The general term for this sequence is
$t_n = ar^{n-1}$
$= 6 \times \left(\frac{1}{3}\right)^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = 6,$
$t_n = rt_{n-1}$
$= \frac{1}{3}t_{n-1}$, where $n > 1$.
iii) By substituting $n = 6$, notice that
$t_6 = 6 \times \left(\frac{1}{3}\right)^{6-1}$
$= \frac{2}{81}.$

f) i) The common ratio r of this sequence is $= \frac{0.2}{1} = 0.2$, and the first term a is 1. The general term for this sequence is

$t_n = ar^{n-1}$
$= 1 \times (0.2)^{n-1}$
$= 0.2^{n-1}$

ii) The recursive formula for this sequence is
$t_1 = 1,$
$t_n = rt_{n-1}$
$= 0.2t_{n-1}$, where $n > 1$.

iii) By substituting $n = 6$, notice that
$t_6 = 0.2^{6-1}$
$= 0.00032$

7. a) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $13 - 9 = 4$.

ii) For this sequence, $d = 4$ and $a = 9$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 9 + 4(n - 1)$
$= 9 + 4n - 4$
$= 4n + 5$

b) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $-\frac{21}{7} = -3$.

ii) For this sequence, $r = -3$ and $a = 7$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 7 \times (-3)^{n-1}$

c) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $-\frac{18}{18} = -1$.

ii) For this sequence, $r = -1$ and $a = 18$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 18 \times (-1)^{n-1}$

d) i) This sequence is neither arithmetic nor geometric, since neither the difference nor ratio of consecutive terms is constant.

ii) N/A

e) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $19 - 29 = -10$.

ii) For this sequence, $d = -10$ and $a = 29$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 29 + (-10)(n - 1)$
$= 29 - 10n + 10$
$= 39 - 10n$

f) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $\frac{96}{128} = \frac{3}{4}$.

ii) For this sequence, $r = \frac{3}{4}$ and $a = 128$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 128 \times \left(\frac{3}{4}\right)^{n-1}$

8. a) For this geometric sequence, $r = 5$ and $a = 19$. The recursive formula for this sequence is
$t_1 = 19,$
$t_n = rt_{n-1}$
$= 5t_{n-1}$, where $n > 1$.
The general term for this sequence is
$t_n = ar^{n-1}$
$= 19 \times 5^{n-1}$

b) For this geometric sequence, $r = -4$ and the first term $a = -9$. The recursive formula for this sequence is
$t_1 = -9,$
$t_n = rt_{n-1}$
$= -4t_{n-1}$, where $n > 1$.
The general term for this sequence is
$t_n = ar^{n-1}$
$= (-9) \times (-4)^{n-1}$

c) Since the first term is 144 and the second term is 36, $r = \frac{36}{144} = \frac{1}{4}$. The first term $a = 144$. The recursive formula for this sequence is
$t_n = 144,$
$t_n = rt_{n-1}$
$= \frac{1}{4}t_{n-1}$, where $n > 1$.
The general term for this sequence is
$t_n = ar^{n-1}$
$= 144 \times \left(\frac{1}{4}\right)^{n-1}$

d) For this geometric sequence, $r = \frac{1}{6}$ and the first term $a = 900$. The recursive formula for this sequence is $t_1 = 900$, $t_n = rt_{n-1}$

$= \frac{1}{6}t_{n-1}$, where $n > 1$.

The general term for this sequence is

$t_n = ar^{n-1}$

$= 900 \times \left(\frac{1}{6}\right)^{n-1}$

9. a) i) This recursive formula does not define a geometric sequence;

$t_1 = 18$

$t_2 = \left(\frac{2}{3}\right)^1 t_1$

$= \left(\frac{2}{3}\right) \times 18$

$= 12$

$t_3 = \left(\frac{2}{3}\right)^2 t_2$

$= \left(\frac{4}{9}\right) \times 12$

$= \frac{16}{3}$

$= \frac{t_2}{t_1} = \frac{2}{3}$, but $\frac{t_3}{t_2} = \frac{4}{9}$. So the ratio between consecutive terms is not constant.

ii) N/A

b) i) This recursive formula defines an geometric sequence: For every $n > 1$, $t_n = rt_{n-1}$ for the constant $r = -3$.

ii) $t_1 = -8$

$t_2 = -3t_1$

$= (-3)(-8)$

$= 24$

$t_3 = -3t_2$

$= (-3)(24)$

$= -72$

$t_4 = -3t_3$

$= (-3)(72)$

$= 216$

$t_5 = -3t_4$

$= (-3)(216)$

$= -648$

The common ratio r is -3.

c) i) This recursive formula defines an geometric sequence: For every $n > 1$, $t_n = rt_{n-1}$ for the constant $r = \frac{1}{3}$.

ii) $t_1 = 123$

$t_2 = \frac{1}{3}t_1$

$= \frac{1}{3} \times 123$

$= 41$

$t_3 = \frac{1}{3}t_2$

$= \frac{1}{3} \times 41$

$= \frac{41}{3}$

$t_4 = \frac{1}{3}t_3$

$= \frac{1}{3} \times \frac{41}{3}$

$= \frac{41}{9}$

$t_5 = \frac{1}{3}t_4$

$= \frac{1}{3} \times \frac{41}{9}$

$= \frac{41}{27}$

The common ratio r is $\frac{1}{3}$.

d) i) Compute the first few terms of this sequence:

$t_1 = 10$

$t_2 = 20$

$t_3 = 4t_1$

$= 4(10)$

$= 40$

$t_4 = 4t_2$

$= 4(20)$

$= 80$

$t_5 = 4t_3$

$= 4(40)$

$= 160$

It looks like this sequence is geometric, with common ratio $r = 2$ and first term $a = 10$. If this were true, then this sequence would have general term

$t_n = ar^{n-1}$

$= 10 \times 2^{n-1}$

Now, check that this is true. It is certainly satisfied for t_1 and t_2:

$t_1 = 10 = 10 \times 2^{1-1}$

$t_2 = 20 = 10 \times 2^{2-1}$

For any $n > 2$, if t_{n-2} is given by this general term, then

$$\begin{aligned} t_n &= 4t_{n-2} \\ &= 4(10 \times 2^{(n-2)-1}) \\ &= 4(10 \times 2^{n-3}) \\ &= 10 \times 2^{n-1} \end{aligned}$$

So, this general term describes t_n, and so t_n is a geometric sequence.

ii) The first five terms are 10, 20, 40, 80, 160. The common ratio is $r = 2$.

10. a) i) This general term defines a geometric sequence, since

$$\begin{aligned} t_n &= 4^n \\ &= 4 \times 4^{n-1} \end{aligned}$$

ii) $t_1 = 4^1 = 4$

$t_2 = 4^2 = 16$

$t_3 = 4^3 = 64$

$t_4 = 4^4 = 256$

$t_5 = 4^5 = 1024$

The common ratio r is $\frac{16}{4} = 4$.

b) i) This general term does not define a geometric sequence since it is not a discrete exponential function.

ii) N/A

c) i) This general term does not define a geometric sequence since it is not a discrete exponential function.

ii) N/A

d) i) This general term defines a geometric sequence, since

$$\begin{aligned} t_n &= 7 \times (-5)^{n-4} \\ &= 7 \times (-5)^{-3} \times (-5)^{n-1} \\ &= -\frac{7}{125} \times (-5)^{n-1} \end{aligned}$$

ii) $t_1 = 7 \times (-5)^{1-4} = 7 \times (-5)^{-3} = -\frac{7}{125}$

$t_2 = 7 \times (-5)^{2-4} = 7 \times (-5)^{-2} = \frac{7}{25}$

$t_3 = 7 \times (-5)^{3-4} = 7 \times (-5)^{-1} = -\frac{7}{5}$

$t_4 = 7 \times (-5)^{4-4} = 7 \times (-5)^0 = 7$

$t_5 = 7 \times (-5)^{5-4} = 7 \times (-5)^1 = -35$

The common ratio r is $\dfrac{\frac{7}{25}}{-\frac{7}{125}} = -5$.

e) i) This general term does not define a geometric sequence since it is not a discrete exponential function.

ii) N/A

f) i) This general term defines a geometric sequence, since

$$\begin{aligned} t_n &= \frac{11}{13^n} \\ &= 11 \times \left(\frac{1}{13}\right)^n \\ &= \frac{11}{13} \times \left(\frac{1}{13}\right)^{n-1} \end{aligned}$$

ii) $t_1 = \frac{11}{13^1} = \frac{11}{13}$

$t_2 = \frac{11}{13^2} = \frac{11}{169}$

$t_3 = \frac{11}{13^3} = \frac{11}{2197}$

$t_4 = \frac{11}{13^4} = \frac{11}{28\,561}$

$t_5 = \frac{11}{13^5}$

$= \dfrac{11}{371\,293}$

The common ratio r is $\dfrac{\frac{11}{169}}{\frac{11}{13}} = \dfrac{1}{13}$.

11. To get from t_5 to t_8, you must multiply by r three times. Therefore,

$\dfrac{t_8}{t_5} = r^3$

$\dfrac{360}{45} = r^3$

$8 = r^3$

$r = 2$

To get from t_8 to t_{20}, you must multiply by r twelve times. Therefore,

$t_{20} = t_8 \times r^{12}$

$= 360 \times 2^{12}$

$= 1\,474\,560$

12. Call the number of bacteria present at the nth observation $f(n)$. Then, note that $\dfrac{f(2)}{f(1)} = \dfrac{f(3)}{f(2)} = \dfrac{f(4)}{f(3)} = \dfrac{3}{2}$. If this pattern continues, f(n) will be a geometric sequence with common ratio $r = \dfrac{3}{2}$. Since the first term $a = 5120$, the general term for this sequence will be

$f(n) = 5120 \times \left(\dfrac{3}{2}\right)^{n-1}$

The number of bacteria present at the 9th observation is

$f(9) = 5120 \times \left(\dfrac{3}{2}\right)^{9-1}$

$= 131\,220$ bacteria

13. Call the amount of money in dollars in Sam's account in the nth year $f(n)$. Every year, Sam's balance gains interest equal to 8% of itself, and so is multiplied by 1.08. Therefore, $f(n)$ is a geometric sequence, with $r = 1.08$ and $a = \$5000$. The general term for this sequence is

$f(n) = \$5000 \times 1.08^{n-1}$

The amount of money in Sam's account at the end of the 10th year is

$f(10) = \$5000 \times 1.08^{10-1}$

$f(10) \doteq \$10\,794.62$

14. a) Call the percentage of bacteria in your body after n doses of the antibiotic are taken $f(n)$. The percentage of bacteria decreases by 10% with each dose, which is the same as being multiplied by 0.9. Therefore, $f(n)$ is a geometric sequence with $r = 0.9$. The first term a of this sequence is the percentage of bacteria left after the first dose, which is $100(0.9) = 90$. The general term for this sequence is

$f(n) = 90 \times 0.9^{n-1}$

Therefore, the percentage of bacteria left after 4 doses is

$f(4) = 90 \times 0.9^{4-1}$

$= 65.61\%$

b) Determine the smallest value of n so that $f(n)$ is less than or equal to 5.

$f(n) \le 5$

$90 \times 0.9^{n-1} \le 5$

By substituting in integer values of n, check that $90 \times 0.9^{27} \doteq 5.23$ and $90 \times 0.9^{28} \doteq 4.71$. This means that $n = 29$ is the smallest number of doses so that the percentage of bacteria is less than or equal to 5.

15. In order to get from t_5 to t_7, you must multiply by r twice. So, $\dfrac{t_7}{t_5} = r^2$, and you can find r by taking the square root of $\dfrac{t_7}{t_5}$. To get from t_7 to t_{29}, you must multiply by r 22 times. So, you can get t_{29} by multiplying t_7 by r^{22}.

16. a) In stage n, every shaded triangle from stage $n - 1$ is cut into four smaller triangles, three of which are shaded. So, the number of shaded triangles present in stage n is three times the number of shaded triangles present in stage $n - 1$. Model the number of shaded triangles present in stage n as a geometric sequence t_n. The common ratio r of this sequence is 3, and the first term $a = 1$. The general term is

$t_n = ar^{n-1}$

$= 1 \times 3^{n-1}$

$= 3^{n-1}$

The number of shaded triangles present in the sixth stage is

$t_6 = 3^{6-1}$

$= 243$

b) Since the shaded triangles from stage $n - 1$ are broken into four triangles of equal area in stage n, three of which are shaded, the total shaded area is multiplied by $\frac{3}{4}$ in each stage. The shaded area in stage n can be modeled, measured in cm^2, by a geometric sequence t_n. The common ratio r of this sequence is $\frac{3}{4}$ and the first term a is 80. The general term is $t_n = ar^{n-1}$

$$= 80 \times \left(\frac{3}{4}\right)^{n-1}$$

The total shaded area in the 20th stage is

$$t_{20} = 80 \times \left(\frac{3}{4}\right)^{20-1}$$

$$\doteq 0.338 \text{ cm}^2$$

17. Similarities between arithmetic and geometric sequences:
Both can be given by use of a recursive formula.
Both have a general term which depends only on two quantities. (the first term and the common ratio or difference).
For both there is a mathematical operation (subtraction for arithmetic sequences, division for geometric sequences) so that the result of applying this operation to any consecutive pair of terms in the sequence is the same.
Differences between arithmetic and geometric sequences:
The recursive formula for an arithmetic sequence involves addition, and the recursive formula for a geometric sequence involves multiplication.
The general term for an arithmetic sequence involves multiplication by $n - 1$, and the general term for a geometric sequence involves exponentiation by $n - 1$.

18. For this sequence, $r = \frac{1}{2}$ and $a = 1$. The general term for this sequence is

$$t_n = ar^{n-1}$$

$$= 1 \times \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{n-1}$$

Calculate the first four terms:

$$t_1 = \left(\frac{1}{2}\right)^{1-1}$$

$$= \left(\frac{1}{2}\right)^{0}$$

$$= 1$$

$$t_2 = \left(\frac{1}{2}\right)^{2-1}$$

$$= \left(\frac{1}{2}\right)^{1}$$

$$= \frac{1}{2}$$

$$t_3 = \left(\frac{1}{2}\right)^{3-1}$$

$$= \left(\frac{1}{2}\right)^{2}$$

$$= \frac{1}{4}$$

$$t_4 = \left(\frac{1}{2}\right)^{4-1}$$

$$= \left(\frac{1}{2}\right)^{3}$$

$$= \frac{1}{8}$$

The sums of the first 1, 2, 3, and 4 terms are

$$t_1 = 1$$

$$t_1 + t_2 = 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$t_1 + t_2 + t_3 = 1 + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{7}{4}$$

$$t_1 + t_2 + t_3 + t_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{15}{8}$$

These numbers are getting closer to 2, so guess that these sums would get closer to 2 if more terms were added.

19. Notice that 2^1 evenly divides the second term, 2^2 evenly divides the third term, 2^3 evenly divides the fourth term, and so on. In fact,

$t_1 = 3 \times 2^0$
$t_2 = 5 \times 2^1$
$t_3 = 7 \times 2^2$
$t_4 = 9 \times 2^3$
$t_5 = 11 \times 2^4$

This sequence is the product of the arithmetic sequence 3, 5, 7, 9, 11, . . . with the geometric sequence 1, 2, 4, 8, 16, . . . The arithmetic sequence has common difference $d = 3 - 1 = 2$ and first term $a = 3$, so its general term is

$$P_n = a + (n - 1)d$$
$$= 3 + 2(n - 1)$$
$$= 3 + 2n - 2$$
$$= 2n + 1$$

The geometric sequence has common ratio $r = \frac{2}{1} = 2$ and first term $a = 1$, so its general term is

$$q_n = ar^{n-1}$$
$$= 1 \times 2^{n-1}$$
$$= 2^{n-1}$$

The general term for the original sequence is their product:

$$t_n = (2n + 1)2^{n-1}$$

t_{10} can be found by substituting $n = 10$ into the general term.

$$t_{10} = 2^{10-1}(2(10) + 1)$$
$$= 2^9(21)$$
$$= 10\,752$$

20. Yes: for example, the sequence 1, 1, 1, . . . is both arithmetic with common difference $d = 1 - 1 = 0$ and geometric with common ratio $r = \frac{1}{1} = 1$.

21. For example, take the arithmetic sequence 1, 2, 3, . . . with common difference $d = 2 - 1 = 1$. Highlight some of the terms of this sequence: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, . . . The highlighted terms form the geometric sequence 1, 2, 4, 8, 16 with common ratio $r = \frac{2}{1} = 2$. Each highlighted term of the arithmetic sequence determines what the next highlighted term will be.

22. Call t_n the shaded area, in cm^2, added in stage n. Then t_n is the area of an isosceles right triangle. Call s_n the length, in cm, of the sides of this isosceles right triangle. Then, by the Pythagorean theorem, if c is the length of the hypotenuse of this triangle,

$$c^2 = s_n^2 + s_n^2$$
$$= 2s_n^2$$

Take square roots of both sides:

$$c = \sqrt{2s_n^2}$$
$$= \sqrt{2}\sqrt{s_n^2}$$
$$= \sqrt{2}s_n$$

By examining the picture, notice that s_n equals half the length of the hypotenuse of the newly shaded triangle from stage $n - 1$:

$$s_n = \frac{\sqrt{2}s_{n-1}}{2}$$
$$= \frac{\sqrt{2}}{2}s_{n-1}$$

Therefore, s_n is a geometric sequence with common ratio $r = \frac{\sqrt{2}}{2}$. The first term a is the length of the sides of the shaded triangle from stage 1, which is $\frac{1}{2} \times 12 = 6$ cm. So, the general term for s_n is

$$s_n = ar^{n-1}$$
$$= 6 \times \left(\frac{\sqrt{2}}{2}\right)^{n-1}$$

The shaded area t_n added in stage n is the area of a triangle with height and base equal to s_n, and so

$$t_n = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 6 \times \left(\frac{\sqrt{2}}{2}\right)^{n-1}$$
$$\times 6 \times \left(\frac{\sqrt{2}}{2}\right)^{n-1}$$
$$= 18 \times \left(\frac{\sqrt{2}}{2}\right)^{2(n-1)}$$
$$= 18 \times \left(\left(\frac{\sqrt{2}}{2}\right)^2\right)^{n-1}$$
$$= 18 \times \left(\frac{1}{2}\right)^{n-1}$$

The total shaded area in stage 6 is the sum of the shaded areas added in stages 1 through 6.

$$t_1 = 18 \times \left(\frac{1}{2}\right)^{1-1}$$
$$= 18 \times \left(\frac{1}{2}\right)^0$$
$$= 18$$
$$t_2 = 18 \times \left(\frac{1}{2}\right)^{2-1}$$
$$= 18 \times \left(\frac{1}{2}\right)^1$$
$$= 9$$

$t_3 = 18 \times \left(\frac{1}{2}\right)^{3-1}$

$= 18 \times \left(\frac{1}{2}\right)^2$

$= 4.5$

$t_4 = 18 \times \left(\frac{1}{2}\right)^{4-1}$

$= 18 \times \left(\frac{1}{2}\right)^3$

$= 2.25$

$t_5 = 18 \times \left(\frac{1}{2}\right)^{5-1}$

$= 18 \times \left(\frac{1}{2}\right)^4$

$= 1.125$

$t_6 = 18 \times \left(\frac{1}{2}\right)^{6-1}$

$= 18 \times \left(\frac{1}{2}\right)^5$

$= 0.6125$

The total shaded area in stage 6 is then
$t_1 + t_2 + t_3 + t_4 + t_5 + t_6$
$= 18 + 9 + 4.5 + 2.25 + 1.125 + 0.5625$
$= 35.4375\ \text{cm}^2$.

7.3 Creating Rules to Define Sequences, pp. 439–440

1. Sam's recursive formula is accurate for all terms of the sequence given, so it can be assumed that it is true for future terms as well. Sam also noticed that the next six terms of the sequence will be the same as the first six:
$t_7 = t_6 - t_5$
$= (-4) - (-5)$
$= 1$
$t_8 = t_7 - t_6$
$= 1 - (-4)$
$= 5$
And then since each term depends only on the two previous terms, t_9 will be the same as t_3 and $t_{10} = t_4 = -1$.

2. The numerators of these fractions form the arithmetic sequence 1, 2, 3, 4, ... with common difference $d = 1$ and first term $a = 1$. The general term for the numerators is then
$N_n = a + (n - 1)d$
$= 1 + (n - 1)$
$= n$
The denominators of these fractions form the arithmetic sequence 2, 3, 4, 5, ... with common difference $d = 1$ and first term $a = 2$.
The general term for the denominators is then
$D_n = a + (n - 1)d$
$= 2 + (n - 1)$
$= n + 1$
The general term for the original sequence is
$t_n = \frac{N_n}{D_n}$
$= \frac{n}{n+1}$

3. a) Since the first triangle requires three toothpicks, $t_1 = 3$. Each new triangle requires only two toothpicks, since it builds on a toothpick already present. This means that t_n is an arithmetic sequence with common difference $d = 2$, and so the general term is
$t_n = a + (n - 1)d$
$= 3 + 2(n - 1)$
$= 3 + 2n - 2$
$= 2n + 1$

b) The first square requires four toothpicks, so $t_1 = 4$. Each new square requires only three toothpicks, since it builds on a toothpick already present. This means that t_n is an arithmetic sequence with common difference $d = 3$, and so the general term is
$t_n = a + (n - 1)d$
$= 4 + 3(n - 1)$
$= 4 + 3n - 3$
$= 3n + 1$

c) The horizontal toothpicks necessary to make an $n \times n$ grid of squares form $n + 1$ rows with n toothpicks each, and so $n(n + 1) = n^2 + n$ horizontal toothpicks are needed. Similarly, the vertical toothpicks necessary to make an $n \times n$ grid of squares form $n + 1$ columns with n toothpicks each, and so $n(n + 1) = n^2 + n$ vertical toothpicks are needed. This means that the total number of toothpicks necessary to make an $n \times n$ grid is
$t_n = (n^2 + n) + (n^2 + n)$
$= 2n^2 + 2n$

d) By counting the number of toothpicks necessary, you can see that 9 are necessary for a row of 4 triangles and that 40 are necessary for

a 4×4 grid of squares. Substituting $n = 4$ to our formulas from parts a) and c) gives
$2(4) + 1 = 9$ and
$2(4^2) + 2(4) = 40$

4. a) It looks like t_n behaves differently for n odd and n even. Notice that

$t_1 = 0$
$t_3 = -1$
$t_5 = -2$
$t_7 = -3$

and so on. Notice that each of these terms is close to negative one-half times the subscript:

$-\frac{1}{2} = -0.5$
$-\frac{3}{2} = -1.5$
$-\frac{5}{2} = -2.5$
$-\frac{7}{2} = -3.5$

Notice that the value of t_n for odd n is 0.5 more than negative one-half of n, or $-\frac{n}{2} + \frac{1}{2} = \frac{1 - n}{2}$.

Now look at t_n for even n:

$t_2 = 1$
$t_4 = 2$
$t_6 = 3$
$t_8 = 4$

and so on. Each of these terms equals half the subscript. So, the value of t_n for even n is half of n, or $\frac{n}{2}$. To summarize, t_n equals $\frac{1 - n}{2}$ if n if n is odd, and $\frac{n}{2}$ if n is even.

c) $n = 12\,345$ is odd, so

$$t_{12345} = \frac{1 - 12\,345}{2} = -6172$$

5. This sequence seems to be the sum of two sequences with fairly different behaviour.
The first sequence $x, 2x, 3x, \ldots$ appears to be arithmetic with common difference $d = 2x - x = x$ and first term $a = x$.
The general term of this sequence is

$$\begin{aligned} r_n &= a + (n - 1)d \\ &= x + x(n - 1) \\ &= x + nx - x \\ &= nx \end{aligned}$$

The second sequence $\frac{1}{y}, \frac{1}{y^2}, \frac{1}{y^3}, \ldots$ is geometric with common ratio $r = \frac{\frac{1}{y^2}}{\frac{1}{y}} = \frac{1}{y}$ and first term $a = \frac{1}{y}$.
The general term of this sequence is then

$$\begin{aligned} s_n &= ar^{n-1} \\ &= \frac{1}{y} \times \left(\frac{1}{y}\right)^{n-1} \\ &= \left(\frac{1}{y}\right)^n \\ &= \frac{1}{y^n} \end{aligned}$$

The general term for the original sequence is then

$$\begin{aligned} t_n &= r_n + s_n \\ &= nx + \frac{1}{y^n} \end{aligned}$$

6. The numerators of these fractions form the sequence 3, 21, 147, 1029, . . . This sequence is geometric with common ratio $r = \frac{21}{3} = 7$ and first term $a = 3$. The general term for the numerators is

$$\begin{aligned} N_n &= ar^{n-1} \\ &= 3 \times 7^{n-1} \end{aligned}$$

The denominators of these fractions form the sequence 5, 55, 555, 5555, . . . Note that each of these is a string of fives, and so when multiplied by $\frac{9}{5}$ will become a string of nines.When one is added to this quantity, it should arrive at 10 raised to some exponent. Check this: call the denominator of the nth fraction D_n. Then

$$\begin{aligned} \frac{9}{5}D_1 + 1 &= 10 \\ &= 10^1 \end{aligned}$$

$$\begin{aligned} \frac{9}{5}D_2 + 1 &= 100 \\ &= 10^2 \end{aligned}$$

$$\begin{aligned} \frac{9}{5}D_3 + 1 &= 1000 \\ &= 10^3 \end{aligned}$$

and so on. In general,

$$\frac{9}{5}D_n + 1 = 10^n$$

$$\frac{9}{5}D_n = 10^n - 1$$

$$D_n = \frac{5}{9}(10^n - 1)$$

The general term of the original sequence is then

$$t_n = \frac{N_n}{D_n} = \frac{3 \times 7^{n-1}}{\frac{5}{9}(10^n - 1)}$$

7. a) The ratios of consecutive terms of this sequence are close to being constant:

$$\frac{t_2}{t_1} = \frac{9}{4} = 2.25$$

$$\frac{t_3}{t_2} = \frac{19}{9} \doteq 2.111$$

$$\frac{t_4}{t_3} = \frac{39}{19} \doteq 2.053$$

$$\frac{t_5}{t_4} = \frac{79}{39} \doteq 2.026$$

The ratios seem to all be close to 2. It is natural to compare t_n with $2t_{n-1}$ then. $t_1 = 4$. $t_n = 2t_{n-1} + 1$ for all $n > 1$. Using this rule, compute the next three terms:

$$t_6 = 2t_5 + 1 = 2(79) + 1 = 159$$

$$t_7 = 2t_6 + 1 = 2(159) + 1 = 319$$

$$t_8 = 2t_7 + 1 = 2(319) + 1 = 639$$

b) The 1st differences of this sequence are $-1, -2, -3, -4, \ldots$ This is an arithmetic sequence with common difference $d = (-2) - (-1) = -1$, and its next three terms are $-4 + (-1) = -5$, $-5 + (-1) = -6$, and $-6 + (-1) = -7$. Assuming that this pattern continues, then

$$t_6 - t_5 = -5$$
$$t_6 - 90 = -5$$
$$t_6 = 85$$
$$t_7 - t_6 = -6$$
$$t_7 - 85 = -6$$
$$t_7 = 79$$
$$t_8 - t_7 = -7$$
$$t_8 - 79 = -7$$
$$t_8 = 72$$

c) The 1st differences of this sequence are 0, 1, 1, 2, 3, 5, 8, . . . which are almost the same as the terms of the original sequence. In fact,

$$t_3 - t_2 = 2 - 1 = 1 = t_1$$

$$t_4 - t_3 = 3 - 2 = 1 = t_2$$

$$t_5 - t_4 = 5 - 3 = 2 = t_3$$

Assuming that this pattern continues, and that $t_n - t_{n-1} = t_{n-2}$ for all $n > 2$. This formula can be rewritten $t_n = t_{n-1} + t_{n-2}$. Use this to compute the next three terms of the sequence:

$$t_9 = t_8 + t_7 = 21 + 13 = 34$$

$$t_{10} = t_9 + t_8 = 34 + 21 = 55$$

$$t_{11} = t_{10} + t_9 = 55 + 34 = 89$$

d) The 1st differences of this sequence are 2, 5, 2, 12, 2, 26, . . . It appears that every difference $t_n - t_{n-1}$ for even values of n equals 2, and that $t_n - t_{n-1}$ for odd values of n equals t_{n-1}. This means that $t_n = t_{n-1} + 2$ for all even n and

$$t_n = t_{n-1} + t_{n-1} = 2t_{n-1}$$

for all odd n. Use this to compute the next three terms of the sequence:

$$t_8 = t_7 + 2 = 52 + 2 = 54$$

$$t_9 = 2t_8 = 2(54) = 108$$

$$t_{10} = t_9 + 2 = 108 + 2 = 110$$

e) These terms appear to be the perfect cubes, with every other one negative. In other words, $t_n = n^3$ for odd n, and $t_n = -n^3$ for even n. Then,

$t_6 = -6^3$
$= -216$
$t_7 = 7^3$
$= 343$
$t_8 = -8^3$
$= -512$

f) The ratios of consecutive terms of this sequence are close to being constant:

$\frac{t_2}{t_1} = \frac{13}{6}$
$\doteq 2.167$
$\frac{t_3}{t_2} = \frac{27}{13}$
$\doteq 2.077$
$\frac{t_4}{t_3} = \frac{55}{27}$
$\doteq 2.037$

The ratios seem to all be close to 2. It is natural to compare t_n with $2t_{n-1}$ then $T_1 = 6$.

n	t	$2t_n - 1$
	4	—
2	9	8
3	19	18
4	39	38
5	79	78

n	t_n	$2t_n - 1$
1	6	—
2	13	12
3	27	26
4	55	54

$t_n = 2t_{n-1} + 1$ for all $n > 1$. Using this rule, compute the next three terms:

$t_5 = 2t_4 + 1$
$= 2(55) + 1$
$= 111$
$t_6 = 2t_5 + 1$
$= 2(111) + 1$
$= 223$
$t_7 = 2t_6 + 1$
$= 2(223) + 1$
$= 447$

8. Instead of attempting to compute this probably very large number, begin by analyzing smaller cases. Call t_n the number of comparisons necessary to arrange the numbers $n, n - 1, n - 2, \ldots 3, 2, 1$ from lowest to highest. Then compute the first few terms of this sequence.

A bubble sort of the numbers 2, 1 looks like this:
Compare **2, 1** → Switch to give 1, 2.
Compare **1, 2** → Leave as is.
The algorithm would then stop. This means that it takes 2 comparisons to sort the numbers 2, 1, and so that $t_2 = 2$.

A bubble sort of the numbers 3, 2, 1 looks like this:
Compare **3, 2**, 1 ⇒ Switch to give 2, 3, 1.
Compare 2, **3, 1** ⇒ Switch to give 2, 1, 3.
Compare **2, 1**, 3 ⇒ Switch to give 1, 2, 3.
Compare 1, **2, 3** ⇒ Leave as is.
Compare **1, 2**, 3 ⇒ Leave as is.
Compare 1, **2, 3** ⇒ Leave as is.
The algorithm would then stop. This means that it takes 6 comparisons to sort the numbers 3, 2, 1, and so that $t_3 = 6$.

A bubble sort of the numbers 4, 3, 2, 1 looks like this:
Compare **4, 3**, 2, 1 ⇒ Switch to give 3, 4, 2, 1.
Compare 3, **4, 2**, 1 ⇒ Switch to give 3, 2, 4, 1.
Compare 3, 2, **4, 1** ⇒ Switch to give 3, 2, 1, 4.
Compare **3, 2**, 1, 4 ⇒ Switch to give 2, 3, 1, 4.
Compare 2, **3, 1**, 4 ⇒ Switch to give 2, 1, 3, 4.
Compare 2, 1, **3, 4** ⇒ Leave as is.
Compare **2, 1**, 3, 4 ⇒ Switch to give 1, 2, 3, 4.
Compare 1, **2, 3**, 4 ⇒ Leave as is.
Compare 1, 2, **3, 4** ⇒ Leave as is.
Compare **1, 2**, 3, 4 ⇒ Leave as is.
Compare 1, **2, 3**, 4 ⇒ Leave as is.
Compare 1, 2, **3, 4** ⇒ Leave as is.
The algorithm would then stop. This means that it takes 12 comparisons to sort the numbers 4, 3, 2, 1, and so that $t_4 = 12$.

A bubble sort of the numbers 5, 4, 3, 2, 1 looks like this:
Compare **5, 4**, 3, 2, 1 ⇒ Switch to give 4, 5, 3, 2, 1
Compare 4, **5, 3**, 2, 1 ⇒ Switch to give 4, 3, 5, 2, 1
Compare 4, 3, **5, 2**, 1 ⇒ Switch to give 4, 3, 2, 5, 1

Compare 4, 3, 2, **5, 1** $\Rightarrow$ Switch to give 4, 3, 2, 1, 5
Compare **4, 3**, 2, 1, 5 $\Rightarrow$ Switch to give 3, 4, 2, 1, 5
Compare 3, **4, 2**, 1, 5 $\Rightarrow$ Switch to give 3, 2, 4, 1, 5
Compare 3, 2, **4, 1**, 5 $\Rightarrow$ Switch to give 3, 2, 1, 4, 5
Compare 3, 2, 1, **4, 5** $\Rightarrow$ Leave as is.
Compare **3, 2**, 1, 4, 5 $\Rightarrow$ Switch to give 2, 3, 1, 4, 5
Compare 2, **3, 1**, 4, 5 $\Rightarrow$ Switch to give 2, 1, 3, 4, 5
Compare 2, 1, **3, 4**, 5 $\Rightarrow$ Leave as is.
Compare 2, 1, 3, **4, 5** $\Rightarrow$ Leave as is.
Compare **2, 1**, 3, 4, 5 $\Rightarrow$ Switch to give 1, 2, 3, 4, 5
Compare 1, **2, 3**, 4, 5 $\Rightarrow$ Leave as is.
Compare 1, 2, **3, 4**, 5 $\Rightarrow$ Leave as is.
Compare 1, 2, 3, **4, 5** $\Rightarrow$ Leave as is.
Compare **1, 2**, 3, 4, 5 $\Rightarrow$ Leave as is.
Compare 1, **2, 3**, 4, 5 $\Rightarrow$ Leave as is.
Compare 1, 2, **3, 4**, 5 $\Rightarrow$ Leave as is.
Compare 1, 2, 3, **4, 5** $\Rightarrow$ Leave as is.

The algorithm would then stop. This means that it takes 20 comparisons to sort the numbers 5, 4, 3, 2, 1, and so that $t_5 = 20$.

The first four terms of this sequence are $t_2 = 2$, $t_3 = 6$, $t_4 = 12$, $t_5 = 20$. This sequence is neither arithmetic nor geometric, since neither the differences nor ratios of consecutive terms are constant. But by expressing these numbers as products, notice a pattern:

$$\begin{aligned} t_2 &= 2 \\ &= 2 \times 1 \\ t_3 &= 6 \\ &= 3 \times 2 \\ t_4 &= 12 \\ &= 4 \times 3 \\ t_5 &= 20 \\ &= 5 \times 4 \end{aligned}$$

For all of the terms so far,

$$\begin{aligned} t_n &= n(n-1) \\ &= n^2 - n \end{aligned}$$

Assuming that this pattern continues, then the number of comparisons necessary to arrange the numbers 100, 99, 98, . . . , 3, 2, 1 is

$$\begin{aligned} t_{100} &= 100^2 - 100 \\ &= 9900 \end{aligned}$$

9. The ratios of consecutive terms of this sequence are close to being constant:

$$\begin{aligned} \frac{t_2}{t_1} &= \frac{11}{2} \\ &= 5.5 \\ \frac{t_3}{t_2} &= \frac{54}{11} \\ &\doteq 4.909 \\ \frac{t_4}{t_3} &= \frac{271}{54} \\ &\doteq 5.019 \\ \frac{t_5}{t_4} &= \frac{1354}{271} \\ &\doteq 4.996 \end{aligned}$$

The ratios seem to all be close to 5. It is natural to compare t_n with $5t_{n-1}$ then. Notice a pattern:

$$\begin{aligned} t_2 &= 11 \\ &= 5(2) + 1 \\ &= 5t_1 + 1 \\ t_3 &= 54 \\ &= 5(11) - 1 \\ &= 5t_2 - 1 \\ t_4 &= 271 \\ &= 5(54) + 1 \\ &= 5t_3 + 1 \\ t_5 &= 1354 \\ &= 5(271) - 1 \\ &= 5t_4 - 1 \\ t_6 &= 6771 \\ &= 5(1354) + 1 \\ &= 5t_5 + 1 \end{aligned}$$

The pattern seems to be that $t_n = 5t_{n-1} + 1$ for all even n and that $t_n = 5t_{n-1} - 1$ for all odd n. Use this rule to determine the next three terms of the sequence:

$$\begin{aligned} t_8 &= 5t_7 + 1 \\ &= 5(33\,854) + 1 \\ &= 169\,271 \\ t_9 &= 5t_8 - 1 \\ &= 5(169\,271) - 1 \\ &= 846\,354 \\ t_{10} &= 5t_9 + 1 \\ &= 5(846\,354) + 1 \\ &= 4\,231\,771 \end{aligned}$$

10. Start by computing the first few terms of this sequence:

$$t_2 = \frac{5}{2}(t_1 + 1)$$

$= \frac{5}{2}(1 + 1)$
$= 5$
$t_3 = \frac{5}{2}(t_2 + 1)$
$= \frac{5}{2}(5 + 1)$
$= 15$
$t_4 = \frac{5}{2}(t_3 + 1)$
$= \frac{5}{2}(15 + 1)$
$= 40$
$t_5 = \frac{1}{2}t_4$
$= \frac{1}{2}(40)$
$= 20$
$t_6 = \frac{1}{2}t_5$
$= \frac{1}{2}(20)$
$= 10$
$t_7 = \frac{1}{2}t_6$
$= \frac{1}{2}(10)$
$= 5$

And after this, the terms 5, 15, 40, 20, 10 will repeat forever, since each term depends only on the previous one. This means that $t_{1000} = t_{995} = t_{990} = \ldots = t_{10} = t_5$. Since $t_5 = 20$, this means that $t_{1000} = 20$.

11. For example, the sequence 1, 2, 1, 2, 1, 2, . . . is neither arithmetic nor geometric since neither the differences nor ratios of consecutive terms are constant. A rule for generating this sequence is that $t_n = 1$ for odd values of n and $t_n = 2$ for even values of n.

7.4 Exploring Recursive Sequences, p. 443

1. For example, begin with $t_1 = 1$ and $t_2 = 4$. Then, by using the same recursive formula $t_n = t_{n-1} + t_{n-2}$ as is used to define the Fibonacci and Lucas sequences, compute more terms:

$t_3 = t_2 + t_1$
$= 4 + 1$
$= 5$
$t_4 = t_3 + t_2$
$= 5 + 4$
$= 9$
$t_5 = t_4 + t_3$
$= 9 + 5$
$= 14$
$t_6 = t_5 + t_4$
$= 14 + 9$
$= 23$
$t_7 = t_6 + t_5$
$= 23 + 14$
$= 37$
$t_8 = t_7 + t_6$
$= 37 + 23$
$= 60$
$\frac{t_2}{t_1} = \frac{4}{1}$
$= 4$
$\frac{t_3}{t_2} = \frac{5}{4}$
$= 1.25$
$\frac{t_4}{t_3} = \frac{9}{5}$
$= 1.8$
$\frac{t_5}{t_4} = \frac{14}{9}$
$\doteq 1.555$
$\frac{t_6}{t_5} = \frac{23}{14}$
$\doteq 1.648$
$\frac{t_7}{t_6} = \frac{37}{23}$
$\doteq 1.609$
$\frac{t_8}{t_7} = \frac{60}{37}$
$\doteq 1.622$

In fact, continuing on, notice that the ratios of consecutive terms of this sequence approach the same value 1.618... as the same ratios for the Fibonacci and Lucas sequences. In addition, the recursive formula which defines this sequence is the same as the recursive formula defining the Fibonacci and Lucas sequences.

2. Substitute the general term for a geometric sequence into the recursive formula for the Fibonacci and Lucas sequences:

$$t_n = t_{n-1} + t_{n-2}$$
$$ar^n = ar^{n-1} + ar^{n-2}$$
$$r^n = r^{n-1} + r^{n-2}$$
$$r^2 = r + 1$$
$$r^2 - r - 1 = 0$$

Use the quadratic formula to solve this:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

One of these, $\frac{-1 \pm \sqrt{5}}{2} \doteq 1.618$, is the value that the common ratios of consecutive terms of the Fibonacci and Lucas sequences approach.

3. a) $t_1 = 1$

$t_2 = 5$

$$t_3 = t_2 + 2t_1 = 5 + 2(1) = 7$$
$$t_4 = t_3 + 2t_2 = 7 + 2(5) = 17$$
$$t_5 = t_4 + 2t_3 = 17 + 2(7) = 31$$
$$t_6 = t_5 + 2t_4 = 31 + 2(17) = 65$$
$$t_7 = t_6 + 2t_5 = 65 + 2(31) = 127$$
$$t_8 = t_7 + 2t_6 = 127 + 2(65) = 257$$
$$t_9 = t_8 + 2t_7 = 257 + 2(127) = 511$$
$$t_{10} = t_9 + 2t_8 = 511 + 2(257) = 1025$$

b) $\frac{t_2}{t_1} = \frac{5}{1} = 5$

$$\frac{t_3}{t_2} = \frac{7}{5} = 1.4$$
$$\frac{t_4}{t_3} = \frac{17}{7} \doteq 2.571$$
$$\frac{t_5}{t_4} = \frac{31}{17} \doteq 1.824$$
$$\frac{t_6}{t_5} = \frac{65}{31} \doteq 2.096$$
$$\frac{t_7}{t_6} = \frac{127}{65} \doteq 1.954$$
$$\frac{t_8}{t_7} = \frac{257}{127} \doteq 2.024$$
$$\frac{t_9}{t_8} = \frac{511}{257} \doteq 1.988$$
$$\frac{t_{10}}{t_9} = \frac{1025}{511} \doteq 2.006$$

These ratios seem to be approaching 2.

c) Since the ratios seem to be approaching 2, it is natural to compare it to the geometric sequence with first term $a = 1$ and common ratio $r = 2$. This sequence would have general term

$$s_n = ar^{n-1} = 1 \times 2^{n-1} = 2^{n-1}$$

n	t_n	s_n	$s_n + 1$
1	1	1	2
2	5	2	4
3	7	4	8
4	17	8	16
5	31	16	32

$= 2s_6 + 1$

The pattern is that

$$t_n = 2s_n + 1 = 2(2^{n-1}) + 1 = 2^n + 1$$

for all even values of n and that

$$t_n = 2s_n - 1 = 2(2^{n-1}) - 1 = 2^n - 1$$

for all odd values of n.

Mid-Chapter Review, p. 447

1. a) i) The common difference of this arithmetic sequence is $d = 21 - 29 = -8$.
The first term a is 29.
The recursive formula is then $t_1 = 29$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 8$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= 29 + (-8)(n - 1)$
$= 29 - 8n + 8$
$= 37 - 8n$
iii) $t_{10} = 37 - 8(10)$
$= 37 - 80$
$= -43$

b) i) The common difference of this arithmetic sequence is $d = -16 - (-8) = -8$. The first term a is -8.
The recursive formula is then $t_1 = -16$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 8$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= -8 + (-8)(n - 1)$
$= -8 - 8n + 8$
$= -8n$
iii) $t_{10} = -8(10)$
$= -80$

c) i) The common difference of this arithmetic sequence is $d = -9 - (17) = 8$. The first term a is -17.
The recursive formula is then $t_1 = -17$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 8$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= -17 + 8(n - 1)$
$= -17 + 8n - 8$
$= 8n - 25$
iii) $t_{10} = 8(10) - 25$
$= 80 - 25$
$= 55$

d) i) The common difference of this arithmetic sequence is $d = 9.5 - 3.25 = 6.25$. The first term a is 3.25.
The recursive formula is then $t_1 = 3.25$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 6.25$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= 3.25 + 6.25(n - 1)$
$= 3.25 + 6.25n - 6.25$
$= 6.25n - 3$
iii) $t_{10} = 6.25(10) - 3$
$= 6.25 - 3$
$= 59.5$

e) i) The common difference of this arithmetic sequence is $d = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. The first term a is $\frac{1}{2}$.
The recursive formula is then $t_1 = \frac{1}{2}$,
$t_n = t_{n-1} + d$
$= t_{n-1} + \frac{1}{6}$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= \frac{1}{2} + \frac{1}{6}(n - 1)$
$= \frac{1}{2} + \frac{1}{6}n - \frac{1}{6}$
$= \frac{1}{6}n + \frac{1}{3}$
iii) $t_{10} = \frac{1}{6}(10) + \frac{1}{3}$
$= \frac{10}{6} + \frac{1}{3}$
$= 2$

f) i) The common difference of this arithmetic sequence is $d = 3x + 3y - x = 2x + 3y$. The first term a is x.
The recursive formula is then $t_1 = x$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 2x + 3y$ for $n > 1$.
ii) The general term for this arithmetic sequence is
$t_n = a + (n - 1)d$
$= x + (2x + 3y)(n - 1)$
$= x + 2nx - 2x + 3ny - 3y$
$= (2n - 1)x + (3n - 3)y$
iii) $t_{10} = (2(10) - 1)x + (3(10) - 3)y$
$= (20 - 1)x + (30 - 3)y$
$= 19x + 27y$

2. a) For this arithmetic sequence, the common difference d is 11 and the first term a is 17. The recursive formula is then $t_1 = 17$,

$t_n = t_{n-1} + d$
$= t_{n-1} + 11$ for $n > 1$.
The general term is
$t_n = a + (n - 1)d$
$= 17 + 11(n - 1)$
$= 17 + 11n - 11$
$= 11n + 6$

b) For this arithmetic sequence, the common difference d is -7 and the first term a is $t_1 = 38$. The recursive formula is then $t_1 = 38$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 7$ for $n > 1$.
The general term is
$t_n = a + (n - 1)d$
$= 38 + (-7)(n - 1)$
$= 38 - 7n + 7$
$= 45 - 7n$

c) For this arithmetic sequence, the common difference is $d = t_2 - t_1 = 73 - 55 = 18$ and the first term a is 55. The recursive formula is then $t_1 = 55$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 18$ for $n > 1$.
The general term is
$t_n = a + (n - 1)d$
$= 55 + 18(n - 1)$
$= 55 + 18n - 18$
$= 18n + 37$

d) To find the first term of this arithmetic sequence, notice that to get from t_1 to t_3, you must add d twice. This means that
$t_1 = t_3 - 2d$
$= -34 - 2(-38)$
$= -34 + 76$
$= 42$
So, for this sequence, $a = 42$ and $d = -38$. The recursive formula is then $t_1 = 42$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 38$ for $n > 1$.
The general term is
$t_n = a + (n - 1)d$
$= 42 + (-38)(n - 1)$
$= 42 - 38n + 38$
$= 80 - 38n$

e) To get from t_5 to t_7, you must add d twice. This means that
$t_7 - t_5 = 2d$
$57 - 91 = 2d$
$-34 = 2d$
$d = -17$
Then, solve for t_1:
$t_1 = t_5 - 4d$
$= 91 - 4(-17)$
$= 91 + 68$
$= 159$
So, for this sequence, $a = 159$ and $d = -17$. The recursive formula is then $t_1 = 159$,
$t_n = t_{n-1} + d$
$= t_{n-1} - 17$ for $n > 1$.
The general term is
$t_n = a + (n - 1)d$
$= 159 + (-17)(n - 1)$
$= 159 - 17n + 17$
$= 176 - 17n$

3. Call the number of seats in the nth row of the stadium t_n. Then t_n is an arithmetic sequence with $t_{13} = 189$ and $t_{25} = 225$. This means that
$t_{25} - t_{13} = 12d$
$225 - 189 = 12d$
$36 = 12d$
$d = 3$
$t_{55} = t_{25} + 30d$
$= 225 + 30(3)$
$= 225 + 90$
$= 315$

4. a) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $30 - 15 = 15$.
ii) For this sequence, $d = 15$ and $a = 15$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 15 + 15(n - 1)$
$= 15 + 15n - 15$
$= 15n$
The recursive formula for this sequence is $t_1 = 15$,
$t_n = t_{n-1} + d$
$= t_{n-1} + 15$ for $n > 1$.
Find t_6 by substituting $n = 6$ to the general term:
$t_6 = 15(6)$
$= 90$

b) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is
$\frac{320}{640} = \frac{1}{2}$

ii) For this sequence, $r = \frac{1}{2}$ and $a = 640$. The general term for this sequence is

$t_n = ar^{n-1}$
$= 640 \times \left(\frac{1}{2}\right)^{n-1}$
The recursive formula for this sequence is
$t_1 = 640,$
$t_n = rt_{n-1}$
$= \frac{1}{2}t_{n-1}$ for $n > 1$.
Find t_6 by substituting $n = 6$ into the general term:
$t_6 = 640 \times \left(\frac{1}{2}\right)^{6-1}$
$= 640 \times \left(\frac{1}{2}\right)^5$
$= 20$
c) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $-\frac{46}{23} = -2$.
ii) For this sequence, $r = -2$ and $a = 23$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 23 \times (-2)^{n-1}$
The recursive formula for this sequence is $t_1 = 23$,
$t_n = rt_{n-1}$
$= -2t_{n-1}$ for $n > 1$.
Find t_6 by substituting $n = 6$ into the general term:
$t_6 = 23 \times (-2)^{6-1}$
$= 23 \times (-2)^5$
$= -736$
d) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $\frac{900}{3000} = \frac{3}{10}$.
ii) For this sequence, $r = \frac{3}{10}$ and $a = 3000$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 3000 \times \left(\frac{3}{10}\right)^{n-1}$
The recursive formula for this sequence is
$t_1 = 3000,$
$t_n = rt_{n-1}$
$= \frac{3}{10}t_{n-1}$ for $n > 1$.
Find t_6 by substituting $n = 6$ into the general term:
$t_6 = 3000 \times \left(\frac{3}{10}\right)^{6-1}$
$= 3000 \times \left(\frac{3}{10}\right)^5$
$= 7.29$
e) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $5 - 3.8 = 1.2$.
ii) For this sequence, $d = 1.2$ and $a = 3.8$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 3.8 + 1.2(n - 1)$
$= 3.8 + 1.2n - 1.2$
$= 1.2n + 2.6$
The recursive formula for this sequence is
$t_1 = 3.8,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 1.2$ for $n > 1$.
Find t_6 by substituting in $n = 6$ to the general term:
$t_6 = 1.2(6) + 2.6$
$= 7.2 + 2.6$
$= 9.8$
f) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is
$\frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{3}$.
ii) For this sequence, $r = \frac{2}{3}$ and $a = \frac{1}{2}$. The general term for this sequence is
$t_n = ar^{n-1}$
$= \frac{1}{2} \times \left(\frac{2}{3}\right)^{n-1}$
The recursive formula for this sequence is
$t_1 = \frac{1}{2},$
$t_n = rt_{n-1}$
$= \frac{1}{2}t_{n-1}$ for $n > 1$.
Find t_6 by substituting $n = 6$ into the general term:
$t_6 = \frac{1}{2} \times \left(\frac{2}{3}\right)^{6-1}$
$= \frac{1}{2} \times \left(\frac{2}{3}\right)^5$

$= \frac{16}{243}$

5. a) i) The general term of this sequence is a discrete exponential function, so the sequence is geometric.

ii) $t_1 = 5^1$
$= 5$
$t_2 = 5^2$
$= 25$
$t_3 = 5^3$
$= 125$
$t_4 = 5^4$
$= 625$
$t_5 = 5^5$
$= 3125$

b) i) The general term of this sequence can be rewritten as

$t_n = \frac{3}{4^n + 3}$

This is a discrete exponential function, so the sequence is geometric.

ii) $t_1 = \frac{3}{4^1 + 3}$
$= \frac{3}{7}$
$t_2 = \frac{3}{4^2 + 3}$
$= \frac{3}{19}$
$t_3 = \frac{3}{4^3 + 3}$
$= \frac{3}{67}$
$t_4 = \frac{3}{4^4 + 3}$
$= \frac{3}{259}$
$t_5 = \frac{3}{4^5 + 3}$
$= \frac{3}{1027}$

c) i) Since this sequence is defined by a recursive formula of the type $t_n = t_{n-1} + d$, the sequence is arithmetic.

ii) $t_1 = 5$
$t_2 = t_1 - 12$
$= 5 - 12$
$= -7$
$t_3 = t_2 - 12$
$= -7 - 12$
$= -19$
$t_4 = t_3 - 12$
$= -19 - 12$
$= -31$
$t_5 = t_4 - 12$
$= -31 - 12$
$= -43$

d) i) The recursive formula defining this sequence can be rewritten:

$\frac{t_n}{t_{n-1}} = -2$
$t_n = -2t_{n-1}$

Since this formula is of the form $t_n = rt_{n-1}$, the sequence is geometric.

ii) $t_1 = -2$
$t_2 = -2t_1$
$= (-2)(-2)$
$= 4$
$t_3 = -2t_2$
$= (-2)(4)$
$= -8$
$t_4 = -2t_3$
$= (-2)(-8)$
$= 16$
$t_5 = -2t_4$
$= (-2)(16)$
$= -32$

e) i) At first glance, this recursive formula does not seem as if it would define either an arithmetic or geometric sequence. Let's compute the first few terms though.

$t_1 = 8$
$t_2 = 11$
$t_3 = 2t_2 - t_1$
$= 2(11)-8$
$= 14$
$t_4 = 2t_3 - t_2$
$= 2(14)-11$
$= 17$
$t_5 = 2t_4 - t_3$
$= 2(17)-14$
$= 20$

These terms 8, 11, 14, 17, 20, . . ., seem to be an arithmetic sequence. Since this sequence has first term $a = 8$ and common difference $d = 11 - 8 = 3$, its general term appears to be

$t_n = a + (n - 1)d$
$= 8 + 3(n - 1)$

$= 8 + 3n - 3$
$= 3n + 5$
In order to check this. This general term is certainly accurate for $n = 1$ and $n = 2$:
$t_1 = 8$
$= 3(1) + 5$
$t_2 = 11$
$= 3(2) + 5$
Assume that $t_{n-1} = 3(n - 1) + 5$ and $t_{n-2} = 3(n - 2) + 5$, then
$t_n = 2t_{n-1} - t_{n-2}$
$= 2(3(n - 1) + 5) - (3(n - 2) + 5)$
$= 2(3n - 3 + 5) - (3n - 6 + 5)$
$= 2(3n + 2) - (3n - 1)$
$= 6n + 4 - 3n + 1$
$= 3n + 5$
Therefore, this general term is accurate and this sequence is an arithmetic sequence.
ii) The first five terms of this sequence to be 8, 11, 14, 17, 20 were already computed.
6. Call the price of the piece of art, in dollars, after n weeks t_n. The price decreases by 10% each week. Put another way, the price is multiplied by 0.9 each week. This means that the ratio of consecutive terms of the sequence t_n is 0.9, and so that t_n is a geometric sequence with $r = 0.9$. The first term of this sequence is the price of the art in dollars after one week, which is $\$10\,000(0.9) = \$9000$. The general term of this sequence is then
$t_n = ar^{n-1}$
$= \$9000 \times 0.9^{n-1}$
The smallest value of n needs to be found for which t_n is less than or equal to 100.
$t_n \leq 100$
$\$9000 \times 0.9^{n-1} \leq 100$
By substituting integer values of n, check that $\$9000 \times 0.9^{42} \doteq 107.75$ and $90 \times 0.9^{43} \doteq 96.98$. This means that $n = 44$ is the smallest value of n so that $t_n \leq 100$, or that after 44 weeks, you will be able to purchase the piece of art.
7. The 1st differences of this sequence are 8, 20, 38, 62, 92, . . .
The 2nd differences are 12, 18, 24, 30, . . .
It looks like the 2nd differences form an arithmetic sequence with common difference $d = 18 - 12 = 6$.
Assuming that this pattern continues, then the next few 2nd differences will be 36, 42, and 48. This means that the next three 1st differences will be $92 + 36 = 128$, $128 + 42 = 170$, and $170 + 48 = 218$. The next three terms of the original sequence will then be $221 + 128 = 349$, $349 + 170 = 519$, and $519 + 218 = 737$.
8. This sequence is the sum of two sequences with different behaviour. The sequence which begins $x, x^2, x^3, \ldots$, is a geometric sequence with common ratio $r = \frac{x^2}{x} = x$ and first term $a = x$.
Its general term is then
$p_n = ar^{n-1}$
$= x \times x^{n-1}$
$= x^n$
The sequence which begins $y, 2y, 3y, \ldots$, is an arithmetic sequence with common difference $2y - y = y$ and first term $a = y$. Its general term is then
$q_n = a + (n - 1)d$
$= y + y(n - 1)$
$= y + ny - y$
$= ny$
The general term of the original sequence is
$t_n = p_n + q_n$
$= x^n + ny$.
9. a) The first few terms in this sequence are the numbers of unit cubes in the first few large cubes, or 1, 8, 27, . . .
b) The number of unit cubes necessary to make a cube n units tall is $n \times n \times n = n^3$. This means that the next few terms of the sequence are $4^3, 5^3, 6^3, \ldots$, or 64, 125, 216, . . .
c) The general term for this sequence is $t_n = n^3$.
d) The number of unit cubes necessary to build the 15th cube is $t_{15} = 15^3 = 3375$.
10. a) The 1st differences of this sequence are $-1, 3, 2, 5, 7, \ldots$ It looks like $t_n - t_{n-1} = t_{n-2}$ for $n > 2$, or put another way, $t_n = t_{n-1} + t_{n-2}$ for $n > 2$.
This means that
$t_6 = t_5 + t_4$
$= 12 + 7$
$= 19$
$t_7 = t_6 + t_5$
$= 19 + 12$
$= 31$
$t_8 = t_7 + t_6$
$= 31 + 19$
$= 50$

$t_9 = t_8 + t_7$
$= 50 + 31$
$= 81$
$t_{10} = t_9 + t_8$
$= 81 + 50$
$= 131$
$t_{11} = t_{10} + t_9$
$= 131 + 81$
$= 212$
$t_{12} = t_{11} + t_{10}$
$= 212 + 131$
$= 343$
$t_{13} = t_{12} + t_{11}$
$= 343 + 212$
$= 555$
$t_{14} = t_{13} + t_{12}$
$= 555 + 343$
$= 898$
$t_{15} = t_{14} + t_{13}$
$= 898 + 555$
$= 1453$

b) The recursive formula for this sequence is
$t_1 = 3, t_2 = 2,$
$t_n = t_{n-1} + t_{n-2}$ for $n > 2$.

7.5 Arithmetic Series, pp. 452–453

1. a) This is an arithmetic series with $n = 10$, $a = 59$, and $d = 64 - 59 = 5$. The sum is therefore

$$S_{10} = \frac{n[2a + (n - 1)d]}{2}$$
$$= \frac{10[2(59) + 5(10-1)]}{2}$$
$$= 815$$

b) This is an arithmetic series with $n = 10$, $a = 31$, and $d = 23 - 31 = -8$. The sum is therefore

$$S_{10} = \frac{n[2a + (n - 1)d]}{2}$$
$$= \frac{10[2(31) + (-8)(10 - 1)]}{2}$$
$$= -50$$

c) This is an arithmetic series with $n = 10$, $a = -103$, and $d = -110 - (-103) = -7$. The sum is therefore

$$S_{10} = \frac{n[2a + (n - 1)d]}{2}$$
$$= \frac{10[2(-103) + (-7)(10 - 1)]}{2}$$
$$= -1345$$

d) This is an arithmetic series with $n = 10$, $a = -78$, and $d = -56 - (-78) = 22$. The sum is therefore

$$S_{10} = \frac{n[2a + (n - 1)d]}{2}$$
$$= \frac{10[2(-78) + (22)(10 - 1)]}{2}$$
$$= 210$$

2. For this arithmetic series, it is known that $n = 20$, $a = 18$, and $d = 11$. Then, the sum is $18 + 29 + 40 + \ldots t^{20}$

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$S_{20} = \frac{20[2(18) + (11)(20 - 1)]}{2}$$
$$= 2450$$

3. Call the number of bricks in the nth row from the top t_n. Then t_n is an arithmetic sequence, and the total number of bricks in the stack is an arithmetic series with $n = 20$, $t_1 = 5$, and the last term $t_n = 62$.
Then,

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{20} = \frac{20(5 + 62)}{2}$$
$$= 670$$

There are 670 bricks in the stack.

4. a) i) This series is arithmetic, since the difference between any two consecutive terms in the sum is $1 - (-1) = 6$.
ii) This is an arithmetic series with $n = 25$, $a = -5$, and $d = 6$. The sum of the first 25 terms is

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$S_{25} = \frac{25[2(-5) + 6(25 - 1)]}{2}$$
$$= 1675$$

b) i) This series is not arithmetic: the difference between pairs of consecutive terms is not constant.
ii) N/A
c) i) This series is not arithmetic: the difference between pairs of consecutive terms is not constant.

ii) N/A

d) i) This series is arithmetic, since the difference between any two consecutive terms in the sum is $22 - 18 = 4$.

ii) This is an arithmetic series with $n = 25$, $a = 18$, and $d = 4$. The sum of the first 25 terms is

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(18) + 4(25 - 1)]}{2}$$

$$= 1650$$

e) i) This series is arithmetic, since the difference between any two consecutive terms in the sum is $22 - 31 = -9$.

ii) This is an arithmetic series with $n = 25$, $a = 31$, and $d = -9$. The sum of the first 25 terms is

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(31) + (-9)(25 - 1)]}{2}$$

$$= -1925$$

f) i) This series is not arithmetic: the difference between pairs of consecutive terms is not constant.

ii) N/A

5. a) The sequence of the terms of this series is an arithmetic sequence with common difference $d = 21 - 17 = 4$ and first term $a = 37$. The general term is then

$$t_n = a + (n - 1)d$$
$$= 37 + 4(n - 1)$$
$$= 37 + 4n - 4$$
$$= 4n + 33$$

Substitute in $n = 12$:

$$t_{12} = 4(12) + 33$$
$$= 81$$

Since $d = 4$, $t_1 = 37$, and $t_{12} = 81$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$S_{12} = \frac{12(37 + 81)}{2}$$

$$= 708$$

b) The sequence of the terms of this series is an arithmetic sequence with common difference $d = -24 - (-13) = -11$ and first term $a = -13$. The general term is then

$$t_n = a + (n - 1)d$$
$$= -13 + (-11)(n - 1)$$
$$= -13 - 11n + 11$$
$$= -11n - 2$$

Substitute in $n = 12$:

$$t_{12} = -11(12) - 2$$
$$= -134$$

Since $d = -11$, $t_1 = -13$, and $t_{12} = -134$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$S_{12} = \frac{12(-13 - 134)}{2}$$

$$= -882$$

c) The sequence of the terms of this series is an arithmetic sequence with common difference $d = -12 - (-18) = 6$ and first term $a = -18$. The general term is then

$$t_n = a + (n - 1)d$$
$$= -18 + 6(n - 1)$$
$$= -18 + 6n - 6$$
$$= 6n - 24$$

Substitute in $n = 12$:

$$t_{12} = 6(12) - 24$$
$$= 48$$

Since $d = 6$, $t_1 = -18$, and $t_{12} = 48$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$S_{12} = \frac{12(-18 + 48)}{2}$$

$$= 180$$

d) The sequence of the terms of this series is an arithmetic sequence with common difference $d = \frac{7}{10} - \frac{1}{5} = 0.5$ and first term $a = \frac{1}{5} = 0.2$.

The general term is then

$$t_n = a + (n - 1)d$$
$$= 0.2 + 0.5(n - 1)$$
$$= 0.2 + 0.5n - 0.5$$
$$= 0.5n - 0.3$$

Substitute in $n = 12$:

$$t_{12} = 0.5(12) - 0.3$$
$$= 5.7$$

Since $d = 0.5$, $t_1 = 0.2$, and $t_{12} = 5.7$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$n = \frac{12(0.2 + 5.7)}{2}$$

$$= 35.4$$

e) The sequence of the terms of this series is an arithmetic sequence with common difference $d = 4.31 - 3.19 = 1.12$ and first term $a = 3.19$. The general term is then

$$t_n = a + (n - 1)d$$
$$= 3.19 + 1.12(n - 1)$$
$$= 3.19 + 1.12n - 1.12$$
$$= 1.12n + 2.07$$

Substitute in $n = 12$:

$$t_{12} = 1.12(12) + 2.07$$
$$= 15.51$$

Since $d = 1.12$, $t_1 = 3.19$, and $t_{12} = 15.51$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{12} = \frac{12(3.19 + 15.51)}{2}$$
$$= 112.2$$

f) The sequence of the terms of this series is an arithmetic sequence with common difference $d = 2p + 2q - p = p + 2q$ and first term $a = p$. The general term is then

$$t_n = a + (n - 1)d$$
$$= p + (p + 2q)(n - 1)$$
$$= p + pn - p + 2qn - 2q$$
$$= (p + 2q)n - 2q$$

Substitute in $n = 12$:

$$t_{12} = 12(p + 2q) - 2q$$
$$= 12p + 24q - 2q$$
$$= 12p + 22q$$

Since $d = p + 2q$, $t_1 = p$, and $t_{12} = 12p + 22q$, calculate S_{12}:

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$n = \frac{12(p + (12p + 22q))}{2}$$
$$= \frac{12(13p + 22q)}{2}$$
$$= 6(13p + 22q)$$
$$= 78p + 132q$$

6. a) For this arithmetic series, $n = 20$, $a = 8$, and $d = 5$. So,

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$n = \frac{20[2(8) + 5(20 - 1)]}{2}$$
$$= 1110$$

b) For this arithmetic series, $n = 20$, $t_1 = 31$, and $t_{20} = 109$. So,

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$n = \frac{20(31 + 109)}{2}$$
$$= 1400$$

c) For this arithmetic series, $n = 20$,

$$d = t_2 - t_1$$
$$= 37 - 53$$
$$= -16,$$

and $a = 53$. So,

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$n = \frac{20[2(53) + (-16)(20 - 1)]}{2}$$
$$= -1980$$

d) For this arithmetic series, $n = 20$ and $d = 17$. Compute t_1 from t_4: to get from t_1 to t_4, one must add d three times. So,

$$t_1 = t_4 - 3d$$
$$= 18 - 3(17)$$
$$= -33$$

So,

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$n = \frac{20[2(-33) + (17)(20 - 1)]}{2}$$
$$= 2570$$

e) For this arithmetic series, $n = 20$ and $d = -3$. Compute t_1 from t_{15}: to get from t_1 to t_{15}, one must add d fourteen times. So,

$$t_1 = t_{15} - 14d$$
$$= 107 - 14(-3)$$
$$= 149$$

So,

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$n = \frac{20[2(149) + (-3)(20 - 1)]}{2}$$
$$= 2410$$

f) For this arithmetic series, $n = 20$. Compute d from t_7 and t_{13}: to get from t_7 to t_{13}, one must add d six times. So,

$$t_{13} - t_7 = 6d$$
$$109 - 43 = 6d$$
$$66 = 6d$$
$$d = 11$$

Compute t_1 from t_7: to get from t_1 to t_7, one must add d six times. So,

$$\begin{aligned} t_1 &= t_7 - 6d \\ &= 43 - 6(11) \\ &= -23 \end{aligned}$$

This means that

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$n = \frac{20[2(-23) + (11)(20 - 1)]}{2}$$

$$= 1630$$

7. a) The first and last terms of this series, so compute the sum once the number of terms n is known. The common difference of the sequence of terms of this series is $6 - 1 = 5$. So, the general term for this sequence is

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 1 + 5(n - 1) \\ &= 1 + 5n - 5 \\ &= 5n - 4 \end{aligned}$$

$t_n = 96$

$$\begin{aligned} t_n &= 96 \\ 5n - 4 &= 96 \\ 5n &= 100 \\ n &= 20 \end{aligned}$$

The sum of this series is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$n = \frac{20(1 + 96)}{2}$$

$$= 970$$

b) The common difference of the sequence of terms of this series is $37 - 24 = 13$. So,

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 24 + 13(n - 1) \\ &= 24 + 13n - 13 \\ &= 13n + 11 \end{aligned}$$

$t_n = 349$

$$\begin{aligned} t_n &= 349 \\ 13n + 11 &= 349 \\ 13n &= 338 \\ n &= 26 \end{aligned}$$

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$n = \frac{26(24 + 349)}{2}$$

$$= 4849$$

c) The common difference of the sequence of terms of this series is $77 - 85 = -8$. So,

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 85 + (-8)(n - 1) \\ &= 85 - 8n + 8 \\ &= 93 - 8n \end{aligned}$$

$t_n = -99$

$$\begin{aligned} t_n &= -99 \\ 93 - 8n &= -99 \\ -8n &= -192 \\ n &= 24 \end{aligned}$$

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$n = \frac{24(85 + (-99))}{2}$$

$$= -168$$

d) The common difference of the sequence of terms of this series is $8 - 5 = 3$. So,

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 5 + 3(n - 1) \\ &= 5 + 3n - 3 \\ &= 3n + 2 \end{aligned}$$

$t_n = 2135.$

$$\begin{aligned} t_n &= 2135 \\ 3n + 2 &= 2135 \\ 3n &= 2133 \\ n &= 711 \end{aligned}$$

The sum of this series is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$

$$n = \frac{711(5 + 2135)}{2}$$

$$= 760\ 770$$

e) The common difference of the sequence of terms of this series is $-38 - (-31) = -7$. So,

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= -31 + (-7)(n - 1) \\ &= -31 - 7n + 7 \\ &= -7n - 24 \end{aligned}$$

$t_n = 136$

$$\begin{aligned} t_n &= -136 \\ -7n - 24 &= -136 \\ -7n &= -112 \\ n &= 16 \end{aligned}$$

$$Sn = \frac{n(t_1 + t_n)}{2}$$

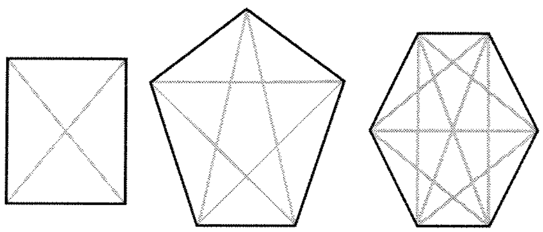
$$n = \frac{16(-31 + (-136))}{2}$$

$$= -1336$$

f) The common difference of the sequence of terms of this series is $-57 - (-63) = 6$. So,

$t_n = a + (n - 1)d$
$= -63 + 6(n - 1)$
$= -63 + 6n - 6$
$= 6n - 69$
$t_n = 63$
$t_n = 63$
$6n - 69 = 63$
$6n = 132$
$n = 22$
$Sn = \frac{n(t_1 + t_n)}{2}$
$n = \frac{22(-63 + 63)}{2}$
$= 0$

8. a) Call the number of diagonals of a regular m-sided polygon d_m.



The above picture shows m-sided polygons for $m = 4, 5,$ and 6, along with their diagonals. Notice that by counting that $d_4 = 2$, $d_5 = 5$, and $d_6 = 9$. The 1st differences of this sequence so far are 3 and 4. Assuming that the 1st differences continue to increase by 1, then we could write d_m as the sum of an arithmetic series:

$d_4 = 2$
$d_5 = d_4 + 3$
$= 2 + 3$
$d_6 = d_5 + 4$
$= 2 + 3 + 4$
$d_7 = d_6 + 5$
$= 2 + 3 + 4 + 5$

and so on. This means that for any $m > 3$, $d_m = 2 + 3 + \ldots + (m - 2)$. To calculate this sum, it must be known how many terms are in the series. The terms of this series are an arithmetic sequence with first term $a = 2$ and common difference $d = 1$, and so the general term is

$t_n = a + (n - 1)$
$= 2 + n - 1$
$= n + 1$

To see how many terms are in the series d_m, solve for n when $t_n = m - 2$:

$t_n = m - 2$
$n + 1 = m - 2$
$n = m - 3$

The number of terms of d_m is then $m - 3$, and so the sum of the series is

$Sn = \frac{n(t_1 + t_n)}{2}$
$n = \frac{(m - 3)(t_1 + t_{m-3})}{2}$
$= \frac{(m - 3)(2 + (m - 2))}{2}$
$= \frac{m(m - 3)}{2}$

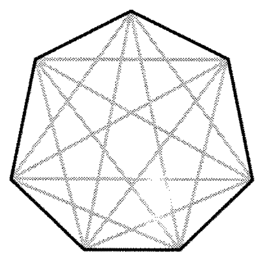
The formula is then

$d_m = \frac{m(m - 3)}{2}$ for $m > 3$.

b) According to our formula, the number of diagonals of a regular heptagon should be

$d_7 = \frac{7(7 - 3)}{2}$
$= 14.$

Check this by drawing a picture:



The heptagon shown here does in fact have 14 diagonals, so our formula works for $m = 7$.

9. To solve this problem, it is easiest to figure out how much each of Joe's investments will be worth at the start of the fifth year. The investment that he made at the start of the fifth year has earned no interest yet, and so is worth \$1000. The investment that he made at the start of the fourth year has earned simple interest once. The amount of interest that it earned is 6.3% of \$1000, or $0.063 \times \$1000 = \$63$. So, the \$1000 investment that Joe made at the beginning of the fourth year is worth $\$1000 + \$63 = \$1063$ at the beginning of the fifth year. Similarly, the investments that Joe made at the beginnings of the third, second, and first years are now worth
$\$1000 + 2(\$63) = \$1126$,
$\$1000 + 3(\$63) = \$1189$, and

$\$1000 + 4(\$63) = \$1252$ respectively. The total value in dollars of Joe's investments at the beginning of the fifth year is $\$1000 + \$1063 + \$1126 + \$1189 + \$1252$, which is an arithmetic series with 5 terms. Its sum is then

$$S_5 = \frac{n(t_1 + t_n)}{2}$$
$$= \frac{5(1000 + 1252)}{2}$$
$$= \$5630$$

The total value of Joe's investments at the beginning of the fifth year is \$5630.

10. Chandra fell 4.9 m in the 1st second, $4.9 + 9.8 = 14.7$ m in the 2nd second, $14.7 + 9.8 = 24.3$ m in the 3rd second, and so on. The total distance, in metres, that she fell in 15 s is the sum of the first 15 terms of the arithmetic series $4.9 + 14.7 + 24.3 + \ldots$ where consecutive terms differ by $d = 9.8$. This sum is

$$S_{15} = \frac{n[2a + (n - 1)d]}{2}$$
$$= \frac{15[2(4.9) + 14(9.78)]}{2}$$
$$= 1102.5$$

Chandra fell 1102.5 m in the 15 seconds before opening her parachute.

11. Let the number of toys that Jamal assembles on the nth day of work n called t_n. Then, since each day Jamal assembles 3 more toys than the previous day, t_n is an arithmetic sequence with common difference $d = 3$. Since Jamal assembled 137 toys on the 20th day of work, $t_{20} = 137$. To get from t_1 to t_{20}, you must add d 19 times, so

$$t_{20} = t_1 + 19d$$
$$137 = t_1 + 19(3)$$
$$137 = t_1 + 57$$
$$t_1 = 80$$

The total number of toys that Jamal assembled in his first 20 days is the sum of the arithmetic series $t_1 + t_2 + \ldots + t_{20}$. For this series, $n = 20$, $a = 80$, and $d = 3$. The sum is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$n = \frac{20(80 + 137)}{2}$$
$$= 2170$$

Jamal assembled 2170 toys in his first 20 days.

12. The number of seconds given for level 1 is 200, and the number of seconds given for level 20 is 105. Since a fixed number of seconds is subtracted for each new level, the sequence t_n defined as the amount of time given for level n is an arithmetic sequence. This means that the total amount of time, in seconds, given for the first 20 levels is the arithmetic series $200 + \ldots + 105$ with 20 terms, or

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$n = \frac{20(200 + 105)}{2}$$
$$= 3050$$

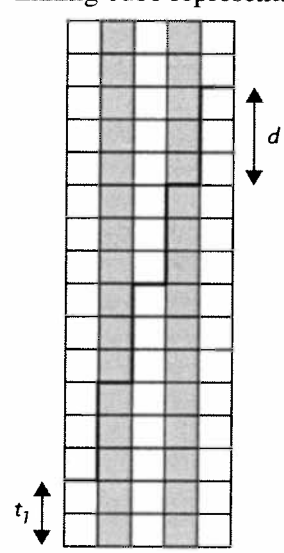
The total amount of time given for the first 20 levels is 3050 s.

13. Call the distance, in km, that Sara runs in the nth week t_n. Then, $t_1 = 5(5) = 25$, and $t_2 = 5(7) = 49$. Every week, Sara runs 2 km more per day than in the previous week, so her total distance is 10 km more than it was in the previous week. This means that t_n is an arithmetic sequence, and so the total distance, in km, that Sara runs in the first 10 weeks is the sum of the first 10 terms of the arithmetic series $35 + 49 + \ldots$ or

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$n = \frac{10[2(25) + 9(10)]}{2}$$
$$= 700$$

So, Sara ran 700 km in the first 10 weeks.

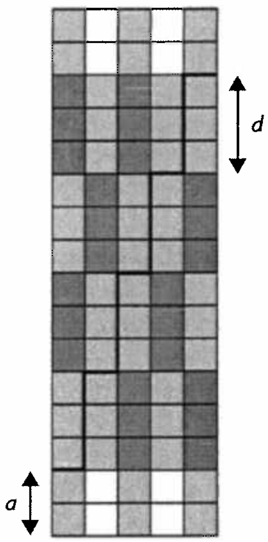
14. Put together two copies of the first type of linking cube representation to get a rectangle:



If the number of cubes in the portion under the jagged line is equal to the sum of the arithmetic series $t_1 + t_2 + \ldots + t_n$ (the first column has t_1 cubes, the second has t_2 cubes, until the nth column, with t_n cubes) then the entire rectangle has a width of n and height of $t_1 + t_n$. The number of cubes in the rectangle is then $n(t_1 + t_n)$. Since the number of cubes in the rectangle is twice the sum of the series,

$$t_1 + t_2 + \ldots + t_n = \frac{n(t_1 + t_n)}{2}.$$

Also, put together two copies of the second type of linking cube representation to get a rectangle:



If the number of cubes in the portion under the jagged line is equal to the sum of the arithmetic series $a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$, (the first column has a cubes, the second has $a + d$ cubes, until the nth column, with $a + (n - 1)$ cubes) then the entire rectangle has a width of n and height of $a + (a + (n - 1)d) = 2a + (n - 1)d$. The number of cubes in the rectangle is then $n[2a + (n - 1)d]$. Since the number of cubes in the rectangle is twice the sum of the series, $a + (a + d) + \ldots + (a + (n - 1)d) = \dfrac{n[2a + (n - 1)d]}{2}$

15. Since the sum of the first 20 terms of the series is 710,

$$S_{20} = 710$$
$$\frac{n[2a + (n - 1)d]}{2} = 710$$
$$\frac{20[2a + 19d]}{2} = 710$$
$$20a + 190d = 710$$

Since the 10th term of the series is 34,

$$t_{10} = 34$$
$$a + (n - 1)d = 34$$
$$a + 9d = 34$$

Use these equations to solve for a and d. Solve the second equation for a:

$$a + 9d = 34$$
$$a = 34 - 9d$$

Then substitute this formula for a into the first equation:

$$20a + 190d = 710$$
$$20(34 - 9d) + 190d = 710$$
$$680 - 180d + 190d = 710$$
$$680 - 10d = 710$$
$$10d = 30$$
$$d = 3$$

Substitute this into the formula for a:

$$a = 34 - 9d$$
$$= 34 - 9(3)$$
$$= 7$$

The 25th term of this arithmetic series is then

$$t_{25} = a + (n - 1)d$$
$$= 7 + 3(25 - 1)$$
$$= 79$$

16. The common difference d of the terms of this arithmetic series is $4 - 1 = 3$. If the number of terms in this series is n, then

$$S_n = 1001$$
$$\frac{n[2a + (n - 1)d]}{2} = 1001$$
$$\frac{n[2(1) + 3(n - 1)]}{2} = 1001$$
$$\frac{n[2 + 3n - 3]}{2} = 1001$$
$$\frac{n[3n - 1]}{2} = 1001$$
$$3n^2 - n = 2002$$
$$3n^2 - n - 2002 = 0$$
$$(3n + 77)(n - 26) = 0$$

The possible values for n are then 26 and $-\dfrac{77}{3}$.

Since n is the number of terms of a series, it must be positive, so there are 26 terms in the series.

7.6 Geometric Series, pp. 459–461

1. a) This is a geometric series with $n = 7$, $a = 6$, and $r = \frac{18}{6} = 3$. Its sum is then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{6(3^7 - 1)}{3 - 1}$$
$$= 6558$$

b) This is a geometric series with $n = 7$, $a = 100$, and $r = \frac{50}{100} = 0.5$. Its sum is then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{100(0.5^7 - 1)}{0.5 - 1}$$
$$= 198.4375$$

c) This is a geometric series with $n = 7$, $a = 8$, and $r = \frac{-24}{8} = -3$. Its sum is then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{8((-3)^7 - 1)}{(-3) - 1}$$
$$= 4376$$

d) This is a geometric series with $n = 7$, $a = \frac{1}{3}$, and $r = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$. Its sum is then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{\frac{1}{3}\left(\left(\frac{1}{2}\right)^7 - 1\right)}{\frac{1}{2} - 1}$$
$$= \frac{-\frac{127}{384}}{-\frac{1}{2}}$$
$$= \frac{127}{192}$$

2. This geometric series has $n = 6$, $a = 11$, and $r = 4$. Its sum is then

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{11(4^6 - 1)}{4 - 1}$$
$$= 15\,015$$

3. a) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{30}{6} = 5$ and first term $a = 6$. The general term is then

$$t_n = ar^{n-1}$$
$$= 6 \times 5^{n-1}$$

Substitute in $n = 6$:

$$t_6 = 6 \times 5^{6-1}$$
$$= 6 \times 5^5$$
$$= 18\,750$$

Since $r = 5$ and $a = 6$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{6(5^6 - 1)}{5 - 1}$$
$$= 23\,436$$

b) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{-33}{-11} = 3$ and first term $a = -11$. The general term is then

$$t_n = ar^{n-1}$$
$$= (-11) \times 3^{n-1}$$

Substitute in $n = 6$:

$$t_6 = (-11) \times 3^{6-1}$$
$$= (-11) \times 3^5$$
$$= -2673$$

Since $r = 3$ and $a = -11$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{-11(3^6 - 1)}{3 - 1}$$
$$= -4004$$

c) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{4\,200\,000}{21\,000\,000} = 0.2$ and first term $a = 21\,000\,000$. The general term is then

$$t_n = ar^{n-1}$$
$$= 21\,000\,000 \times 0.2^{n-1}$$

Substitute in $n = 6$:

$t_6 = 21\,000\,000 \times 0.2^{6-1}$
$= 21\,000\,000 \times 0.2^5$
$= 6720$

Since $r = 0.2$ and $a = 6720$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{21\,000\,000(0.2^6 - 1)}{0.2 - 1}$$
$$= 26\,248\,320$$

d) The sequence of the terms of this series is a geometric sequence with common ratio

$$r = \frac{\frac{8}{15}}{\frac{4}{5}} = \frac{2}{3} \text{ and first term } a = \frac{4}{5}.$$

The general term is then

$$t_n = ar^{n-1}$$
$$= \frac{4}{5} \times \left(\frac{2}{3}\right)^{n-1}$$

Substitute in $n = 6$:

$$t_6 = \frac{4}{5} \times \left(\frac{2}{3}\right)^{6-1}$$
$$= \frac{4}{5} \times \left(\frac{2}{3}\right)^5$$
$$= \frac{128}{1215}$$

Since $r = \frac{2}{3}$ and $a = \frac{4}{5}$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{\frac{4}{5}\left(\left(\frac{2}{3}\right)^6 - 1\right)}{\frac{2}{3} - 1}$$
$$= \frac{-\frac{532}{729}}{-\frac{1}{3}}$$
$$= \frac{532}{243}$$

e) The sequence of the terms of this series is a geometric sequence with common ratio

$$r = -\frac{7.14}{3.4} = -2.1 \text{ and first term } a = 3.4.$$

The general term is then

$$t_n = ar^{n-1}$$
$$= 3.4 \times (-2.1)^{n-1}$$

Substitute in $n = 6$:

$$t_6 = 3.4 \times (-2.1)^{6-1}$$
$$= 3.4 \times (-2.1)^5$$
$$= -138.859$$

Since $r = -2.1$ and $a = 3.4$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{3.4((-2.1)^6 - 1)}{(-2.1) - 1}$$
$$= -92.969$$

f) The sequence of the terms of this series is a geometric sequence with common ratio

$$r = \frac{3x^2}{1} = 3x^2 \text{ and first term } a = 1.$$

The general term is then

$$t_n = ar^{n-1}$$
$$= 1 \times (3x^2)^{n-1}$$
$$= (3x^2)^{n-1}$$

Substitute in $n = 6$:

$$t_6 = (3x^2)^{6-1}$$
$$= (3x^2)^5$$
$$= 243x^{10}.$$

Since $r = 3x^2$ and $a = 1$, calculate S_6:

$$S_6 = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1((3x^2)^6 - 1)}{3x^2 - 1}$$
$$= \frac{729x^{12} - 1}{3x^2 - 1}$$

4. a) i) This series is arithmetic; the difference of any two consecutive terms is $d = 15 - 10 = 5$.

ii) N/A

b) i) This series is geometric; the ratio of any two consecutive terms is $r = \frac{21}{7} = 3$.

ii) For this geometric series, $n = 8$, $r = 3$, and $a = 7$. Its sum is then

$$S_8 = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{7(3^8 - 1)}{3 - 1}$$
$$= 22\,960$$

c) i) This series is geometric; the ratio of any two consecutive terms is $r = -\frac{512}{2048} = -\frac{1}{4}$.

ii) For this geometric series, $n = 8$, $r = -\frac{1}{4}$, and $a = 2048$. Its sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{2048\left(\left(-\frac{1}{4}\right)^8 - 1\right)}{\left(-\frac{1}{4}\right) - 1}$$

$$= \frac{-\frac{65\,535}{32}}{-\frac{5}{4}} = \frac{13\,107}{8}$$

d) i) This sequence is neither arithmetic nor geometric: neither the ratio nor the difference of consecutive terms is constant.

ii) N/A

e) i) This series is geometric; the ratio of any two consecutive terms is $r = \frac{1.21}{1.1} = 1.1$.

ii) For this geometric series, $n = 8$, $r = 1.1$, and $a = 1.1$. Its sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{1.1(1.1^8 - 1)}{1.1 - 1}$$

$$\doteq 12.579$$

f) i) This series is arithmetic; the difference of any two consecutive terms is $d = 63 - 81 = -18$.

ii) N/A

5. a) Since $n = 7$, $a = 13$, and $r = 5$, the sum of this series is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{13(5^7 - 1)}{5 - 1}$$

$$= 253\,903$$

b) To find the common ratio r of the terms of this series. To get from t_1 to t_7, you must multiply by r six times. In other words,

$$t_7 = r^6 t_1$$
$$704 = 11r^6$$
$$64 = r^6$$
$$r = \pm\sqrt[6]{64}$$
$$= \pm 2$$

Therefore, for this series, $n = 7$, $a = 11$, and $r = \pm 2$. The two possibilities for the sum of this series are then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{11(2^7 - 1)}{2 - 1}$$

$$= 1397$$

or

$$S_7 = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{11((-2)^7 - 1)}{(-2) - 1}$$

$$= 473$$

c) If $t_2 = 30$ and $t_1 = 120$, then the common ratio of the terms of this series is $r = \frac{30}{120} = 0.25$. So, $n = 7$, $a = 120$, and $r = 0.25$. The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{120((0.25)^7 - 1)}{(0.25) - 1}$$

$$\doteq 159.990$$

d) Since the terms increase by a factor of 3, $r = 3$. To get from t_1 to t_3, you must multiply by r twice, so

$$t_3 = r^2 t_1$$
$$18 = 3^2(t_1)$$
$$t_1 = 2$$

$n = 7$, $a = 2$, and $r = 3$. So, the sum of this series is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$n = \frac{2(3^7 - 1)}{3 - 1}$$

$$= 2186$$

e) Since the terms decrease by a factor of $\frac{2}{3}$, $r = \frac{2}{3}$. To get from t_1 to t_8, you must multiply by r seven times, so

$$t_8 = r^7 t_1$$

$$1024 = \left(\frac{2}{3}\right)^7 t_1$$

$$1024 = \frac{128}{2187} t_1$$

$$t_1 = 17\,496$$

For this series, $n = 7$, $t_1 = 17\,496$, $t_8 = 1024$, and $r = \frac{2}{3}$. The sum of the series is then

$$Sn = \frac{t_{n+1} - t_1}{r - 1}$$
$$n = \frac{1024 - 17\,496}{\frac{2}{3} - 1}$$
$$= 49\,416$$

f) To get from t_5 to t_8, you must multiply by r three times. So,

$$t_8 = t_5 r^3$$
$$-40 = 5r^3$$
$$-8 = r^3$$
$$r = \sqrt[3]{-8}$$
$$= -2$$

To get from t_1 to t_5, you must multiply by r four times. So,

$$t_5 = t_1 r^4$$
$$5 = t_1(-2)^4$$
$$5 = 16t_1$$
$$t_1 = \frac{5}{16}$$

$n = 7$, $a = \frac{5}{16}$, and $r = -2$. This means that the sum of this series is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{\frac{5}{16}((-2)^7 - 1)}{(-2) - 1}$$
$$= \frac{215}{16}$$

6. a) The common ratio of the terms of this series is $r = \frac{6}{1} = 6$, and the first term a is 1.
The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= 1 \times 6^{n-1}$$
$$= 6^{n-1}$$

Check how many terms are in the series by setting $t_n = 279\,936$ and solving for n:

$$t_n = 279\,936$$
$$6^{n-1} = 279\,936$$
$$n - 1 = 8$$
$$n = 9$$

For this series, $n = 9$, $a = 1$, and $r = 6$.
The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{1(6^9 - 1)}{6 - 1}$$
$$= 335\,923$$

b) The common ratio of the terms of this series is
$r = \frac{480}{960} = \frac{1}{2}$, and the first term a is 960.
The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= 960 \times \left(\frac{1}{2}\right)^{n-1}$$

Check how many terms are in the series by setting $t_n = 15$ and solving for n:

$$t_n = 15$$
$$960 \times \left(\frac{1}{2}\right)^{n-1} = 15$$
$$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$$
$$n - 1 = 6$$
$$n = 7$$

For this series, $n = 7$, $a = 960$, and $r = \frac{1}{2}$. The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{960\left(\left(\frac{1}{2}\right)^7 - 1\right)}{\frac{1}{2} - 1}$$
$$= 1905$$

c) The common ratio of the terms of this series is
$r = -\frac{51}{17} = -3$, and the first term a is 17.
The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= 17 \times (-3)^{n-1}$$

Check how many terms are in the series by setting $t_n = 334\,611$ and solving for n:

$$t_n = 334\,611$$
$$17 \times (-3)^{n-1} = 334\,611$$
$$(-3)^{n-1} = 19\,683$$

$n - 1 = 9$
$n = 10$
For this series, $n = 10$, $a = 17$, and $r = -3$. The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{17((-3)^9 - 1)}{(-3)-1}$$
$$= -250\,954$$

d) The common ratio of the terms of this series is $r = \frac{3600}{24\,000} = 0.15$, and the first term a is 24 000. The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= 24\,000 \times 0.15^{n-1}$$

Check how many terms are in the series by setting $t_n = 1.8225$ and solving for n:

$$t_n = 1.8225$$
$$24\,000 \times 0.15^{n-1} = 1.8225$$
$$0.15^{n-1} = 0.000\,075\,937\,5$$
$$n - 1 = 5$$
$$n = 6$$

For this series, $n = 6$, $a = 24000$, and $r = 0.15$. The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{24\,000(0.15^6 - 1)}{0.15 - 1}$$
$$= 28\,234.9725$$

e) The common ratio of the terms of this series is $r = -\frac{24}{6} = -4$, and the first term a is -6.

The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= (-6) \times (-4)^{n-1}$$

Check how many terms are in the series by setting $t_n = 98\,304$ and solving for n:

$$t_n = 98\,304$$
$$(-6) \times (-4)^{n-1} = 98\,304$$
$$(-4)^{n-1} = -16\,384$$
$$n - 1 = 7$$
$$n = 8$$

For this series, $n = 8$, $a = -6$, and $r = -4$. The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{(-6)((-4)^8 - 1)}{(-4)-1}$$
$$= 78\,642$$

f) The common ratio of the terms of this series is $r = \frac{2}{4} = \frac{1}{2}$, and the first term a is 4.

The general term for the geometric sequence of terms of this series is then

$$t_n = ar^{n-1}$$
$$= 4 \times \left(\frac{1}{2}\right)^{n-1}$$

Check how many terms are in the series by setting $t_n = \frac{1}{1024}$ and solving for n:

$$t_n = \frac{1}{1024}$$
$$4 \times \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1024}$$
$$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{4096}$$
$$n - 1 = 12$$
$$n = 13$$

For this series, $n = 13$, $a = 4$, and $r = \frac{1}{2}$.

The sum of this series is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{4\left(\left(\frac{1}{2}\right)^{13} - 1\right)}{\frac{1}{2} - 1}$$
$$= \frac{8191}{1024}$$

7. The total distance travelled by the ball until hitting the ground for the fifth time, in metres, is 3 + 2(height ball reaches after first bounce) + 2(height ball reaches after second bounce) + 2(height ball reaches after third bounce) + 2 (height ball reaches after fourth bounce). Since the height that the ball reaches after each bounce is 60% of the previous height, rewrite this as

$3 + 2(0.6 \times 3) + 2(0.6^2 \times 3) + 2(0.6^3 \times 3) + 2(0.6^4 \times 3)$, or as
$3 + 2(1.8 + 1.08 + .648 + .3888)$.

Calculate the sum of the geometric series 1.8 + 1.08 + 0.648 + 0.3888; for this series, $n = 4$, $a = 1.8$, and $r = 0.6$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{1.8(0.6^4 - 1)}{0.6 - 1}$$
$$= 3.9168$$

The total distance travelled by the ball, in metres, is then $3 + 2(3.9168) = 10.8336$ metres.

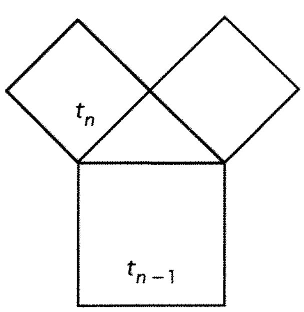
8. If the terms of a geometric series have common ratio $r = 1$, then each term of the series equals the previous term, so the series is just $a + a + a + \ldots + a$. If this series has n terms, then its sum is $S_n = na$.

9. The number of segments drawn in stage 1 is 1. The number of new segments drawn in stage 2 is 2. From the pictures, notice that the number of new segments drawn in each stage is twice the number of new segments drawn in the previous stage. This means that the total number of segments needed to draw stage 20 is the sum of the first 20 terms of the geometric series $1 + 2 + 4 + \ldots$ with common ratio $r = 2$. For this series, $n = 20$, $a = 1$, and $r = 2$, so the sum is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{1(2^{20} - 1)}{2 - 1}$$
$$= 1\,048\,575$$

Stage 20 requires 1 048 575 line segments.

10. Let's say that the side length, in metres, of any new square drawn in stage n is called t_n. Then, $t_1 = 1$. Use the following picture to determine the relation between t_n and t_{n-1} for $n > 1$:



The isosceles right triangle in the centre of the picture has legs with length t_n and hypotenuse with length t_{n-1}. By the Pythagorean theorem,

$$t_{n-1}^2 = t_n^2 + t_n^2$$
$$t_{n-1}^2 = 2t_n^2$$
$$t_n^2 = 0.5t_{n-1}^2$$
$$t_n = \sqrt{0.5t_{n-1}^2}$$
$$= \sqrt{0.5}\sqrt{t_{n-1}^2}$$
$$= \sqrt{0.5}t_{n-1}$$

This means that t_n is a geometric sequence with $a = 1$ and $r = \sqrt{0.5}$. The general term is then

$$t_n = ar^{n-1}$$
$$= 1 \times \sqrt{0.5^{n-1}}$$
$$= \sqrt{0.5^{n-1}}$$

Say that the number of new squares drawn in stage n is called s_n. Then $s_1 = 1$ and from the picture, notice that every new square drawn in stage $n - 1$ yields two new squares in stage n, and so $s_n = 2s_{n-1}$ for $n > 1$. This means that s_n is a geometric sequence with $a = 1$ and $r = 2$. The general term is then

$$s_n = ar^{n-1}$$
$$= 1 \times 2^{n-1}$$
$$= 2^{n-1}$$

During stage n, each square drawn in stage $n - 1$ will have a right triangle and two smaller squares added to it as illustrated above. The area of this added portion, in square meters, is

$$2t_n^2 + 0.5t_n^2 = 2.5t_n^2$$
$$= 2.5(\sqrt{0.5^{n-1}})^2$$
$$= 2.5 \times 0.5^{n-1}$$

This much area will be added during stage n for every square added during stage $n - 1$. The number of such squares is $s_{n-1} = 2^{n-2}$. Therefore, the total new area in stage n is

$$(2.5 \times 0.5^{n-1}) \times 2^{n-2} = 2.5 \times 0.5 \times 0.5^{n-2} \times 2^{n-2}$$
$$= 1.25 \times (0.5 \times 2)^{n-2}$$
$$= 1.25 \text{ m}^2$$

This means that the total area of the tree at the 10th stage is

$$1 + 1.25 + 1.25 + \ldots + 1.25 = 1 + 9(1.25)$$
$$= 12.25 \text{ m}^2$$

11. The number of employees in the first level of the phone tree is 5. The number of employees in each successive level is 3 times the number in the previous level, so that the total number of employees in the company is the sum of the first seven terms of the geometric series

$5 + (3 \times 5) + (3^2 \times 5) + \dots$

For this series, $a = 5$ and $r = 3$, so this sum is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{5(3^7 - 1)}{3 - 1}$$
$$= 5465$$

The company has 5465 employees.

12. The geometric series

$t_1 + t_2 + \dots + 5\,859\,375$ with common ratio $r = \frac{5}{3}$ can have its terms reversed and be rewritten as $5\,859\,375 + \dots + t_1$, where the common ratio in the new series is the reciprocal of the previous common ratio, or $\frac{1}{\left(\frac{5}{3}\right)} = \frac{3}{5}$. For the series written in this way,

$n = 10$, $a = 5\,859\,375$, and $r = \frac{3}{5}$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{5\,859\,375\left(\left(\frac{3}{5}\right)^7 - 1\right)}{\frac{3}{5} - 1}$$
$$= 14\,559\,864$$

13. Because of condition c), the sequence of cash prizes, in dollars, listed from least to greatest, forms a geometric series with common ratio r an integer greater than 2 and less than 10. By condition b), the first term a of this series is a positive integer. By condition a), the sum of the series is at most \$2 000 000. The goal is then to find values of n, a, and r so that n and a are positive integers and r is an integer greater than 2 and less than 10 so that $S_n = \frac{a(r^n - 1)}{r - 1}$ is at most 2 000 000. There are many possible choices, for example $n = 7$, $a = 1829$, and $r = 3$. In this case, the values of the prizes are \$1829, \$5387, ..., \$1 333 341, and the total value, in dollars, of all prizes is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$n = \frac{1829(3^7 - 1)}{3 - 1}$$
$$= \$1\,999\,097$$

14. (See Table Below)

15.

$$S_3 = 372$$
$$\frac{a(r^n - 1)}{r - 1} = 372$$
$$\frac{12(r^3 - 1)}{r - 1} = 372$$
$$12(r^3 - 1) = 372(r - 1)$$
$$r^3 - 1 = 31(r - 1)$$
$$(r^2 + r + 1)(r - 1) = 31(r - 1)$$
$$(r^2 + r + 1)(r - 1) - 31(r - 1) = 0$$
$$(r^2 + r - 30)(r - 1) = 0$$
$$(r + 6)(r - 5)(r - 1) = 0$$

Arithmetic	Geometric	Similarities	Differences
$S_n = \frac{n(t_1 + t_n)}{2}$ Write the terms of S_n twice, once forward, then once backward above each other. Then add pairs of terms. The sum of the pairs is constant.	$S_n = \frac{t_{n+1} - t_1}{r - 1}$ where $r \neq 1$ Write the terms of S_n and rS_n above each other. Then subtract pairs of terms. The difference of all middle parts is zero.	Both general formulas involve two "end" terms of the series.	You add with formula for arithmetic series, but subtract for geometric series. You divide by 2 for arithmetic series, but by $r - 1$ for geometric series.

So, $r = -6$, 5, or 1. Check the value of t_5 in each case. If $r = -6$, then

$t_5 = ar^{n-1}$
$= 12 \times (-6)^{5-1}$
$= 15\,552$

If $r = 5$, then

$t_5 = ar^{n-1}$
$= 12 \times 5^{5-1}$
$= 7500$

If $r = 1$, then

$t_5 = ar^{n-1}$
$= 12 \times 1^{-1}$
$= 12$

So, the largest possible value for t_5 is 15 552.

16. To get from t_1 to t_3, you must multiply by r twice. This means that

$t_3 = r^2 t_1$
$92 = 23r^2$
$r^2 = 4$
$r = \pm\sqrt{4}$
$= \pm\sqrt{2}$

The sum of the series is 62 813, so for n equal to the number of terms in the series,

$S_n = 62\,813$

$\dfrac{a(r^n - 1)}{r - 1} = 62\,813$

$\dfrac{23(r^n - 1)}{r - 1} = 62\,813$

Check the possible values of r. If $r = 2$, then

$\dfrac{23(2^n - 1)}{2 - 1} = 62\,813$

$23(2^n - 1) = 62\,813$
$2^n - 1 = 2731$
$2^n = 2732$

This equation is not solvable for n an integer, since $2^{11} = 2048$ and $2^{12} = 4096$.

If $r = -2$, then

$\dfrac{23((-2)^n - 1)}{(-2)-1} = 62\,813$

$23((-2)^n - 1) = (-3)62\,813$
$(-2)^n-1 = -8193$
$(-2)^n = -8192$
$n = 13$

There are 13 terms in the series.

17. Use the formula for the sum of the geo metric series $1 + x + x^2 + \ldots + x^{14}$ with first term $a = 1$ and common ratio $r = x$:

$S_{15} = 1 + x + x^2 + \ldots + x^{14}$

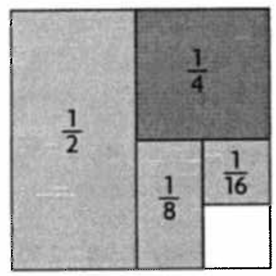
$\dfrac{a(r^n - 1)}{r - 1} = 1 + x + x^2 + \ldots + x^{14}$

$\dfrac{x^{15} - 1}{x - 1} = 1 + x + x^2 + \ldots + x^{14}$

$x^{15} - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{14})$

In fact, the second term $1 + x + x^2 + \ldots + x^{14}$ can be factored further, but doing so is fairly difficult. The full factorization of $x^{15} - 1$ is actually $(x - 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1)$.

18. a)



b) You can use the formula for the sum of a geometric series to compute the sum of the first n terms for any positive integer n. For this series,

$a = \dfrac{1}{2}$ and $r = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$, so the sum of the first n terms is

$S_n = \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1}$

$= \dfrac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right)}{-\frac{1}{2}}$

$= 1 - \left(\dfrac{1}{2}\right)^n$

c) In this case, it does make sense. The sum of the first n terms is $S_n = 1 - \left(\dfrac{1}{2}\right)^n$, and as n gets larger, this quantity gets closer and closer to 1. So, the sum of this infinite geometric series is 1.

7.7 Pascal's Triangle and Binomial Expansions, p. 466

1. The first three entries of the 13th row of Pascal's triangle are $1 + 12 = 13$, $12 + 66 = 78$, and $66 + 220 = 286$.

2. a) The 5th row of Pascal's triangle is 1, 5, 10, 10, 5, 1, so $(x + 2)^5$

$= 1(x)^5 + 5(x)^4(2)^1 + 10(x)^3(2)^2 + 10(x)^2(2)^3 + 5(x)^1(2)^4 + 1(2)^5$

$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

b) The 6th row of Pascal's triangle is 1, 6, 15, 20, 15, 6, 1, so $(x - 1)^6$

$= 1(x)^6 + 6(x)^5(-1)^1 + 15(x^4)(-1)^2 + 20(x)^3(-1)^3 + 15(x)^2(-1)^4 + 6(x)^1(-1)^5 + (-1)^6$

$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

c) The 3rd row of Pascal's triangle is 1, 3, 3, 1, so $(2x - 3)^3$

$= 1(2x)^3 + 3(2x)^2(-3)^1 + 3(2x)^1(-3)^2 + 1(-3)^3$

$= 8x^3 - 36x^2 + 54x - 27$

3. a) The first three numbers of the 10th row of Pascal's triangle are 1, 10 (the 10th counting number), and 45 (the 9th triangular number).

So, $(x + 5)^{10} = 1(x)^{10} + 10(x)^9(5)^1 + 45(x)^8(5)^2 + \cdots$
$= x^{10} + 50x^9 + 1125x^8 + \cdots$

b) The first three numbers of the 8th row of Pascal's triangle are 1, 8 (the 8th counting number), and 28 (the 7th triangular number).

So, $(x - 2)^8 = 1(x)^8 + 8(x)^7(-2)^1 + 28(x)^6(-2)^2 + \cdots$
$= x^8 - 16x^7 + 112x^6 + \cdots$

c) The first three numbers of the 9th row of Pascal's triangle are 1, 9 (the 9th counting number), and 36 (the 8th triangular number). So,

$(2x - 7)^9 = 1(2x)^9 + 9(2x)^8(-7)^1 + 36(2x)^7(-7)^2 + \ldots$
$= 512x^9 - 16\,128x^8 + 225\,729x^7 - \ldots$

4. a) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so $(k + 3)^4$

$= 1(k)^4 + 4(k)^3(3)^1 + 6(k)^2(3)^2 + 4(k)^1(3)^3 + 1(3)^4$

$= k^4 + 12k^3 + 54k^2 + 108k + 81$

b) The 6th row of Pascal's triangle is 1, 6, 15, 20, 15, 6, 1, so $(y - 5)^6$

$= 1(y)^6 + 6(y)^5(-5)^1 + 15(y)^4(-5)^2 + 20(y)^3(-5)^3 + 15(y)^2(-5)^4 + 6(y)^1(-5)^5 + 1(-5)^6$

$= y^6 - 30y^5 + 375y^4 - 2500y^3 + 9375y^2 - 18\,750y + 15\,625$

c) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so $(3q - 4)^4$

$= 1(3q)^4 + 4(3q)^3(-4)^1 + 6(3q)^2(-4)^2 + 4(3q)^1(-4)^3 + 1(-4)^4$

$= 81q^4 - 432q^3 + 864q^2 - 768q + 256$

d) The 3rd row of Pascal's triangle is 1, 3, 3, 1, so $(2x + 7y)^3$

$= 1(2x)^3 + 3(2x)^2(7y)^1 + 3(2x)^1(7y)^2 + 1(7y)^3$

$= 8x^3 + 84x^2y + 294xy^2 + 343y^3$

e) The 6th row of Pascal's triangle is 1, 6, 15, 20, 15, 6, 1, so $\left(\sqrt{2}x + \sqrt{3}\right)^6$

$= 1\left(\sqrt{2}x\right)^6 + 6\left(\sqrt{2}x\right)^5\left(\sqrt{3}\right)^1 + 15\left(\sqrt{2}x\right)^4\left(\sqrt{3}\right)^2 + 20\left(\sqrt{2}x\right)^3\left(\sqrt{3}\right)^3 + 15\left(\sqrt{2}x\right)^2\left(\sqrt{3}\right)^4 + 6\left(\sqrt{2}x\right)^1\left(\sqrt{3}\right)^5 + 1\left(\sqrt{3}\right)^6$

$= 8x^6 + 24\sqrt{6}x^5 + 180x^4 + 120\sqrt{6}x^3 + 270x^2 + 54\sqrt{6}x + 27$

f) The 5th row of Pascal's triangle is 1, 5, 10, 10, 5, 1, so $(2z^3 - 3y^2)^5$

$= 1(2z^3)^5 + 5(2z^3)^4(-3y^2)^1 + 10(2z^3)^3(-3y^2)^2 + 10(2z^3)^2(-3y^2)^3 + 5(2z^3)^1(-3y^2)^4 + 1(-3y^2)^5$

$= 32z^{15} - 240z^{12}y^2 + 720z^9y^4 - 1080z^6y^6 + 810z^3y^8 - 243y^{10}$

5. a) The first three numbers of the 13th row of Pascal's triangle are 1, 13 (the 13th counting number), and 78 (the 12th triangular number). So,

$(x - 2)^{13} = 1(x)^{13} + 13(x)^{12}(-2)^1 + 78(x)^{11}(-2)^2 + \cdots$
$= x^{13} - 26x^{12} + 312x^{11} + \cdots$

b) The first three numbers of the 9th row of Pascal's triangle are 1, 9 (the 9th counting number), and 36 (the 8th triangular number). So,

$(3y + 5)^9 = 1(3y)^9 + 9(3y)^8(5)^1 + 36(3y)^7(5)^2 + \cdots$
$= 19\,683y^9 + 295\,245y^8 + 1\,968\,300y^7 + \cdots$

c) The first three numbers of the 11th row of

Pascal's triangle are 1, 11 (the 11th counting number), and 55 (the 10th triangular number). So,

$$(z^5 - z^3)^{11} = 1(z^5)^{11} + 11(z^5)^{10}(-z^3)^1 + 55(z^5)^9(-z^3)^2 + \cdots$$
$$= z^{55} - 11z^{53} + 55z^{51} + \cdots$$

d) The first three numbers of the 10th row of Pascal's triangle are 1, 10 (the 10th counting number), and 45 (the 9th triangular number). So,

$$\left(\sqrt{a} + \sqrt{5}\right)^{10} = 1\left(\sqrt{a}\right)^{10} + 10\left(\sqrt{a}\right)^9\left(\sqrt{5}\right)^1 + 36\left(\sqrt{a}\right)^8\left(\sqrt{5}\right)^2 + \cdots$$
$$= a^5 + 10\sqrt{5}aa^4 + 225a^4 + \cdots$$

e) The first three numbers of the 14th row of Pascal's triangle are 1, 14 (the 14th counting number), and 91 (the 13th triangular number). So,

$$\left(3b^2 + \frac{2}{b}\right)^{14} = 1(3b^2)^{14} + 14(3b^2)^{13}\left(\frac{2}{b}\right)^1 + 91(3b^2)^8\left(\frac{2}{b}\right)^2 + \cdots$$
$$= 4\,782\,969b^{28} - 44\,641\,044b^{25} + 193\,444\,524b^{22} + \cdots$$

f) The first three numbers of the 8th row of Pascal's triangle are 1, 8 (the 8th counting number), and 28 (the 7th triangular number). So,

$$(5x^3 + 3y^2)^8 = 1(5x^3)^8 + 8(5x^3)^7(3y^2)^1 + 28(5x^3)^6(3y^2)^2 + \cdots$$
$$= 390\,625x^{24} + 1\,875\,000x^{21}y^2 + 39\,375\,000x^{18}y^4 + \cdots$$

6. a) Expand 2^n by representing it as a binomial. Let's say that the numbers in the nth row of Pascal's triangle are $t_0, t_1, \ldots, t_n$.

$$2^n = (1 + 1)^n$$
$$= t_0(1)^n + t_1(1)^{n-1}(1)^1 + t_2(1)^{n-1}(1)^2 + \cdots t_n(1)^n$$
$$= t_0 + t_1 + t_2 + \cdots + t_n$$

Notice that the sum of the numbers in the nth row of Pascal's triangle is equal to 2^n.

b) Expand 0^n by representing it as a binomial.

$$0^n = (1 - 1)^n$$
$$= t_0(1)^n + t_1(1)^{n-1}(-1) + t_2(1)^{n-2}(-1)^2 + \cdots + t_n(-1)^n$$
$$= t_0 - t_1 + t_2 - \cdots \pm t_n$$

Notice that if we alternately add and subtract the numbers in the nth row of Pascal's triangle, the result is always 0.

7. $$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}\right)^1\left(1+\sqrt{5}\right)^1 - \left(\frac{1}{2}\right)^1\left(1-\sqrt{5}\right)^1\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{2^1}\left[\left(1+\sqrt{5}\right) - \left(1-\sqrt{5}\right)\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{2} \times 2\sqrt{5}$$
$$= 1$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}\right)^2\left(1+\sqrt{5}\right)^2 - \left(\frac{1}{2}\right)^2(1-\sqrt{5})^2\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{2^2}\left[\left((1)^2 + 2(1)^1\left(\sqrt{5}\right)^1 + \left(\sqrt{5}\right)^2\right) - \left((1)^2 + 2(1)^1\left(-\sqrt{5}\right)^1 + \left(-\sqrt{5}\right)^2\right)\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{4} \times \left[1 + 2\sqrt{5} + 5 - 1 + 2\sqrt{5} - 5\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{4} \times \left[4\sqrt{5}\right]$$
$$= 1$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}\right)^3\left(1+\sqrt{5}\right)^3 - \left(\frac{1}{2}\right)^3\left(1-\sqrt{5}\right)^3\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{2^3}\left[\left((1)^3 + 3(1)^2\left(\sqrt{5}\right)^1 + 3(1)^1\left(\sqrt{5}\right)^2\right) + 1\left(\sqrt{5}\right)^3\right) - ((1)^3 + 3(1)^2\left(-\sqrt{5}\right)^1 + 3(1)^1\left(-\sqrt{5}\right)^2 + 1\left(-\sqrt{5}\right)^3\Big)\Big]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{8} \times \left[1 + 3\sqrt{5} + 15 + 5\sqrt{5} - 1 + 3\sqrt{5} - 15 + 5\sqrt{5}\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{8} \times \left[16\sqrt{5}\right]$$
$$= 2$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^4 - \left(\frac{1-\sqrt{5}}{2}\right)^4\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}\right)^4\left(1+\sqrt{5}\right)^4 - \left(\frac{1}{2}\right)^4\left(1-\sqrt{5}\right)^4\right]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{2^4}\Big[\Big((1)^4 + 4(1)^3\left(\sqrt{5}\right)^1$$
$$+ 6(1)^2\left(\sqrt{5}\right)^2 + 4(1)^1\left(\sqrt{5}\right)^3$$
$$+ 1\left(\sqrt{5}\right)^4\Big) - \Big((1)^4 - 4(1)^3\left(\sqrt{5}\right)^1$$
$$+ 6(1)^2\left(\sqrt{5}\right)^2 - 4(1)^1\left(\sqrt{5}\right)^3 + 1\left(\sqrt{5}\right)^4\Big)\Big]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{16} \times \Big[1 + 4\sqrt{5} + 30 + 20\sqrt{5}$$
$$+ 25 - 1 + 4\sqrt{5} - 30 + 20\sqrt{5} - 25\Big]$$
$$= \frac{1}{\sqrt{5}} \times \frac{1}{16} \times \left[48\sqrt{5}\right]$$
$$= 3$$

These values 1, 1, 2, 3, . . ., form the first four terms of the Fibonacci sequence.

8. The number of ways for Joan to walk to school from her house is equal to the number of different orders that 5 steps east and 5 steps north can be written in, which is the number of orders for 4 steps east and 5 steps north, plus the number of orders for 5 steps east and 4 steps north; this relationship shows that the number of routes for Joan is equal to the 6th number in the 10th row of Pascal's triangle. The 10th row of Pascal's triangle is 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, so the number that is 252.

9. You can rewrite $(x + y + z)^{10}$ as $(x + (y + z))^{10}$, use the formula for expanding a binomial to write this as an expression in which the only quantities left to expand are powers of $(y + z)$, and then use the binomial expansion formula to expand those.

10. The 6th row of Pascal's triangle is 1, 6, 15, 20, 15, 6, 1, so $(3x - 5y)^6$

$$= 1(3x)^6 + 6(3x)^5(-5y)^1 + 15(3x)^4(-5y)^2$$
$$+ 20(3x)^3(-5y)^3 + 15(3x)^2(-5y)^4$$
$$+ 6(3x)^1(-5y)^5 + (-5y)^6$$
$$= 729x^6 - 7290x^5y + 30\,375x^4y^2 - 67\,500x^3y^3$$
$$+ 84\,375x^2y^4 - 56\,250xy^5 + 15\,625y^6$$

11. To expand $(a + b)^n$ for n a positive integer, write the numbers for the nth row of Pascal's triangle. Each term in the expansion is the product of a number from Pascal's triangle, a power of a, and a power of b. As the numbers from Pascal's triangle progress from left to right, the exponent of a decreases from n to zero by ones, and the exponent of b increases from zero to n by ones.

12. The formula for a general cubic relation is $t_n = an^3 + bn^2 + cn + d$. The nth 1st difference of this relation is

$$t_{n+1} - t_n = a(n+1)^3 + b(n+1)^2 + c(n+1)$$
$$+ d - (an^3 + bn^2 + cn + d)$$
$$= a(n^3 + 3n^2 + 3n + 1) + b(n^2$$
$$+ 2n + 1) + c(n + 1) + d - an^3$$
$$- bn^2 - cn - d$$
$$= an^3 + 3an^2 + 3an + a$$
$$+ bn^2 + 2bn + b + cn + c + d$$
$$- an^3 - bn^2 - cn - d$$
$$= 3an^2 + 2bn + cn$$

This shows that the 1st differences of a cubic relation are quadratic. Since the 2nd differences of a quadratic relation are constant, this means that the 3rd differences of a cubic relation are constant.

13. The first three numbers of the 10th row of Pascal's triangle are 1, 10 (the 10th counting number), and 45 (the 9th triangular number). So,

$$\left(\frac{1}{2} + \frac{1}{2}\right)^{10} = 1\left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1$$
$$+ 45\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \cdots$$
$$= \frac{1}{1024} + \frac{10}{1024} + \frac{45}{1024} + \cdots$$
$$= \frac{1}{1024} + \frac{5}{1024} + \frac{45}{1024} + \cdots$$

The third term above is the number of different ways to flip a coin 10 times

and have 8 heads, multiplied by the probability

of flipping each individual sequence of 8 heads

and 2 tails, which is is $\left(\frac{1}{2}\right)^{10}$. This product is

the probability that 8 heads appear in 10 flips. The cases of the 1st and 2nd terms above can be analyzed similarly.

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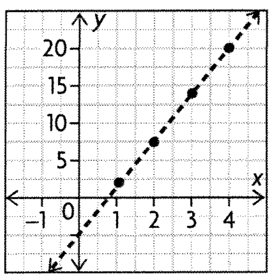
1. a) This is an arithmetic sequence with first term $a = 2$ and where the difference between any pair of consecutive terms is $d = 8 - 2 = 6$.
b) The general term of this sequence is
$t_n = a + (n - 1)d$
$= 2 + 6(n - 1)$
$= 2 + 6n - 6$
$= 6n - 4$
The recursive formula for this sequence is
$t_1 = 2,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 6$, where $n > 1$.
c)


2. To check if a sequence is arithmetic, compute the 1st differences and see if they are constant.
3. a) i) For this arithmetic sequence, $d = 73 - 58 = 15$, and $a = 58$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 58 + 15(n - 1)$
$= 58 + 15n - 15$
$= 15n + 43$
ii) The recursive formula for this sequence is
$t_i = 58,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 15$, where $n > 1$.
b) i) For this arithmetic sequence, $d = (-40) - (-49) = 9$, and $a = -49$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= -49 + 9(n - 1)$
$= -49 + 9n - 9$
$= 9n - 58$
ii) The recursive formula for this sequence is
$t_1 = -49,$
$t_n = t_{n-1} + d$
$= t_{n-1} + 9$, where $n > 1$.
c) i) For this arithmetic sequence, $d = 75 - 81 = -6$, and $a = 81$. The general term for this sequence is
$t_n = a + (n - 1)d$
$= 81 - 6(n - 1)$
$= 81 - 6n + 6$
$= 87 - 6n$
ii) The recursive formula for this sequence is
$t_1 = 81,$
$t_n = t_{n-1} + d$
$= t_{n-1} - 6$, where $n > 1$.
4. To get from t_7 to t_{13}, you must add d six times. So,
$t_{13} = t_7 + 6d$
$219 = 465 + 6d$
$-246 = 6d$
$d = -41$
To get from t_{13} to t_{100}, you must add d 87 times. So,
$t_{100} = t_{13} + 87d$
$= 219 + (-41)87$
$= -3348$
5. Call the height, in mm, of the plant after the nth week t_n. Then it looks like t_n is an arithmetic sequence with common difference $d = 20 - 7 = 13$ and first term $a = 7$. The general term of this sequence is then
$t_n = a + (n - 1)d$
$= 7 + 13(n - 1)$
$= 7 + 13n - 13$
$= 13n - 6$
Find the smallest value of n for which $t_n > 100$.
$13n - 6 > 100$
$13n > 106$
$$n > \frac{106}{13} \doteq 8.1538$$
The smallest such value of n is 9, so the plant will first be taller than 100 mm after the 9th week.
6. To check if a sequence is geometric, compute the ratios of consecutive pairs of terms and see if they are constant.
7. a) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is
$$r = \frac{15}{5} = 3.$$
ii) For this sequence, $r = 3$ and $a = 5$. The general term for this sequence is
$t_n = ar^{n-1}$
$= 5 \times 3^{n-1}$

Find t_6 by substituting in $n = 6$:

$t_6 = 5 \times 3^{6-1}$
$= 5 \times 3^5$
$= 1215$

b) i) This sequence is neither arithmetic nor geometric, since neither the difference nor ratio of consecutive terms is constant.

ii) N/A

c) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $\frac{14.4}{288} = 0.05$.

ii) For this sequence, $r = 0.05$ and $a = 288$. The general term for this sequence is

$t_n = ar^{n-1}$
$= 288 \times 0.05^{n-1}$

Find t_6 by substituting in $n = 6$:

$t_6 = 288 \times 0.05^{6-1}$
$= 288 \times 0.05^5$
$= 0.000\,09$

d) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $50 - 10 = 40$.

ii) For this sequence, $d = 40$ and $a = 10$. The general term for this sequence is

$t_n = a + (n - 1)d$
$= 10 + 40(n - 1)$
$= 10 + 40n - 40$
$= 40n - 30$

Find t_6 by substituting in $n = 6$:

$t_6 = 40(6) - 30$
$= 210$

e) i) This sequence is arithmetic. The common difference d between any pair of consecutive terms is $10 - 19 = -9$.

ii) For this sequence, $d = -9$ and $a = 19$. The general term for this sequence is

$t_n = a + (n - 1)d$
$= 19 + (-9)(n - 1)$
$= 19 - 9n + 9$
$= 28 - 9n$

Find t_6 by substituting in $n = 6$:

$t_6 = 28 - 9(6)$
$= -26$

f) i) This sequence is geometric. The common ratio r between any pair of consecutive terms is $\frac{384}{512} = 0.75$.

ii) For this sequence, $r = 0.75$ and $a = 512$. The general term for this sequence is

$t_n = ar^{n-1}$
$= 512 \times 0.75^{n-1}$

Find t_6 by substituting in $n = 6$:

$t_6 = 512 \times 0.75^{6-1}$
$= 512 \times 0.75^5$
$= 121.5$

8. a) i) $r = -3$, 7. The recursive formula for this sequence is $t_1 = -3$,

$t_n = rt_{n-1}$
$= -3t_{n-1}$, where $n > 1$.

ii) The general term for this sequence is

$t_n = ar^{n-1}$
$= 7 \times (-3)^{n-1}$

iii) $t_1 = 7 \times (-3)^{1-1}$
$= 7 \times (-3)^0$
$= 7$

$t_2 = 7 \times (-3)^{2-1}$
$= 7 \times (-3)^1$
$= -21$

$t_3 = 7 \times (-3)^{3-1}$
$= 7 \times (-3)^2$
$= 63$

$t_4 = 7 \times (-3)^{4-1}$
$= 7 \times (-3)^3$
$= -189$

$t_5 = 7 \times (-3)^{5-1}$
$= 7 \times (-3)^4$
$= 567$

b) i) $r = \frac{1}{2}$. The recursive formula for this sequence is $t_1 = 12$,

$t_n = rt_{n-1}$
$= \frac{1}{2}t_{n-1}$, where $n > 1$.

ii) The general term for this sequence is

$t_n = ar^{n-1}$
$= 12 \times \left(\frac{1}{2}\right)^{n-1}$

iii) $t_1 = 12 \times \left(\frac{1}{2}\right)^{1-1}$
$= 12 \times \left(\frac{1}{2}\right)^0$
$= 12$

$t_2 = 12 \times \left(\frac{1}{2}\right)^{2-1}$

$= 12 \times \left(\frac{1}{2}\right)^1$
$= 6$
$t_3 = 12 \times \left(\frac{1}{2}\right)^{3-1}$
$= 12 \times \left(\frac{1}{2}\right)^2$
$= 3$
$t_4 = 12 \times \left(\frac{1}{2}\right)^{4-1}$
$= 12 \times \left(\frac{1}{2}\right)^3$
$= \frac{3}{2}$
$t_5 = 12 \times \left(\frac{1}{2}\right)^{5-1}$
$= 12 \times \left(\frac{1}{2}\right)^4$
$= \frac{3}{4}$

c) i) The common ratio of this sequence is $r = \frac{t_3}{t_2} = \frac{144}{36} = 4$. Since r is the ratio between any pair of consecutive terms,
$\frac{t_2}{t_1} = 4$
$\frac{36}{t_1} = 4$
$36 = 4t_1$
$t_1 = 9$
So, the recursive formula for this sequence is
$t_1 = 9,$
$t_n = rt_{n-1}$
$= 4t_{n-1}$, where $n > 1$.

ii) The general term for this sequence is
$t_n = ar^{n-1}$
$= 9 \times 4^{n-1}$

iii) $t_1 = 9 \times 4^{1-1}$
$= 9 \times 4^0$
$= 9$
$t_2 = 9 \times 4^{2-1}$
$= 9 \times 4^1$
$= 36$
$t_3 = 9 \times 4^{3-1}$
$= 9 \times 4^2$
$= 144$
$t_4 = 9 \times 4^{4-1}$
$= 9 \times 4^3$
$= 576$
$t_5 = 9 \times 4^{5-1}$
$= 9 \times 4^4$
$= 2304$

9. a) i) The general term for this sequence is a linear function, so the sequence is arithmetic.

ii) $t_1 = 4(1) + 5$
$= 9$
$t_2 = 4(2) + 5$
$= 13$
$t_3 = 4(3) + 5$
$= 17$
$t_4 = 4(4) + 5$
$= 21$
$t_5 = 4(5) + 5$
$= 25$

b) i) The first few terms of this sequence are
$t_1 = \frac{1}{7(1)-3}$
$= \frac{1}{4}$
$t_2 = \frac{1}{7(2)-3}$
$= \frac{1}{11}$
$t_3 = \frac{1}{7(3)-3}$
$= \frac{1}{18}$
$t_4 = \frac{1}{7(4)-3}$
$= \frac{1}{25}$
$t_5 = \frac{1}{7(5)-3}$
$= \frac{1}{32}$
$t_2 - t_1 = \frac{1}{11} - \frac{1}{4} = -\frac{7}{44}$
$t_3 - t_2 = \frac{1}{18} - \frac{1}{11} = -\frac{7}{198}$
These are not equal, so this sequence is not arithmetic.

$$\frac{t_2}{t_1} = \frac{\left(\frac{1}{11}\right)}{\left(\frac{1}{4}\right)} = \frac{4}{11}$$

$$\frac{t_3}{t_2} = \frac{\left(\frac{1}{18}\right)}{\left(\frac{1}{11}\right)} = \frac{11}{18}$$

These are not equal, so this sequence is not geometric either.

ii) The first five terms are $\frac{1}{4}, \frac{1}{11}, \frac{1}{18}, \frac{1}{25}, \frac{1}{32}$.

c) i) The first few terms of this sequence are

$$\begin{aligned} t_1 &= 1^2 - 1 \\ &= 0 \\ t_2 &= 2^2 - 1 \\ &= 3 \\ t_3 &= 3^2 - 1 \\ &= 8 \\ t_4 &= 4^2 - 1 \\ &= 15 \\ t_5 &= 5^2 - 1 \\ &= 24 \end{aligned}$$

$t_2 - t_1 = 3 - 0 = 0$

$t_3 - t_2 = 8 - 3 = 5$

These are not equal, so this sequence is not arithmetic.

$\frac{t_3}{t_2} = \frac{8}{3}$

$\frac{t_4}{t_3} = \frac{15}{8}$

These are not equal, so this sequence is not geometric either.

ii) The first five terms are 0, 3, 8, 15, 24.

d) i) The first few terms of this sequence are

$$\begin{aligned} t_1 &= -17 \\ t_2 &= t_1 + 2 - 1 \\ &= -17 + 2 - 1 \\ &= -16 \\ t_3 &= t_2 + 3 - 1 \\ &= -16 + 3 - 1 \\ &= -14 \\ t_4 &= t_5 + 4 - 1 \\ &= -14 + 4 - 1 \\ &= -11 \\ t_5 &= t_4 + 5 - 1 \\ &= -11 + 5 - 1 \\ &= -7 \end{aligned}$$

$t_2 - t_1 = -16 - (-17) = 1$

$t_3 - t_2 = -14 - (-16) = 2$

These are not equal, so this sequence is not arithmetic.

$\frac{t_2}{t_1} = \frac{-16}{-17} = \frac{16}{17}$

$\frac{t_3}{t_2} = \frac{-14}{-16} = \frac{7}{8}$

These are not equal, so this sequence is not geometric either.

ii) The first five terms are $-17, -16, -14, -11, -7$.

10. a) Midnight is 11 hours after 1 p.m., so by midnight, the population will have doubled 11 times from its population at 1 p.m. The number of bacteria present at midnight is then $2^{11}(23\,000) = 47\,104\,000$ bacteria.

b) If t_n represents the population of the bacteria at n hours past noon, then t_n is a geometric sequence with common ratio $r = 2$ and first term $a = 23\,000$. So, the general term of this sequence is $t_n = 23\,000 \times 2^{n-1}$. However, in practice this formula cannot be used to determine the population at times far in the future, because the bacteria will eventually run out of food and space.

11. Call the value, in dollars, of Guy's stamp n years past 2000 t_n. Then, since the value increases by 10% each year, $t_n = (1.1)t_{n-1}$ for $n > 1$. Therefore, this sequence is geometric, with $r = 1.1$ and $a = 820$. Its general term is then

$$\begin{aligned} t_n &= ar^{n-1} \\ &= 820 \times 1.1^{n-1} \end{aligned}$$

The value of the stamp in 2010 is

$$\begin{aligned} t_{10} &= 820 \times 1.1^{10-1} \\ &= 820 \times 1.1^9 \\ &\doteq 1933.52 \end{aligned}$$

The value of Guy's stamp in 2010 will be roughly \$1933.52.

12. The number of horizontally oriented toothpicks in a stack of squares n high is $1 + 2 + \ldots + n + n$ (1 toothpick forms the top of the 1st row, 2 toothpicks form the top of the 2nd row, etc. until n toothpicks form the top of the last row, and n toothpicks form the bottom of the last row.) This is equal to n + the sum of the first n terms of the arithmetic series with common difference $d = 1$ and first term $a = 1$. This sum is

$$S_n = \frac{n[2a + (n-1)d]}{2}$$
$$= \frac{n[2(1) + 1(n-1)}{2}$$
$$= \frac{n^2 + n}{2}$$
So, the number of horizontally oriented toothpicks in this stack is
$$n + S_n = n + \frac{n^2 + n}{2}$$
$$= \frac{n^2 + 3n}{2}$$
The number of vertically oriented toothpicks in the stack is exactly the same as the number of horizontally oriented ones, so the total number of toothpicks in the stack of squares n squares tall is
$$\frac{n^2 + 3n}{2} + \frac{n^2 + 3n}{2} = n^2 + 3n.$$
13. Let's call the numerator of the nth fraction N_n and the denominator of the nth fraction D_n. Then the numerators form the sequence 1, 2, 3, 4, . . . which is an arithmetic sequence with common difference $d = 1$ and first term $a = 1$. The general term for this sequence is
$$N_n = a + (n-1)d$$
$$= 1 + 1(n-1)$$
$$= 1 + n - 1$$
$$= n$$
The denominators form the sequence 2, 5, 8, 11, . . . which is an arithmetic sequence with common difference $d = 5 - 2 = 3$ and first term $a = 2$. The general term for this sequence is
$$D_n = a + (n-1)d$$
$$= 2 + 3(n-1)$$
$$= 2 + 3n - 3$$
$$= 3n - 1$$
The general term of the original sequence is then
$$t_n = \frac{N_n}{D_n}$$
$$= \frac{n}{3n - 1}$$
The 100th term of the sequence is
$$t_{100} = \frac{100}{3(100) - 1}$$
$$= \frac{100}{299}$$
14. a) This is an arithmetic series with common difference $d = 9 - 1 = 8$ and first term $a = 1$. The sum of the first 50 terms is then
$$Sn = \frac{n[2a + (n-1)d]}{2}$$
$$S_{50} = \frac{50[2(1) + 8(50-1)]}{2}$$
$$= 9850$$
b) This is an arithmetic series with common difference $d = 17 - 21 = -4$ and first term $a = 21$. The sum of the first 50 terms is then
$$Sn = \frac{n[2a + (n-1)d]}{2}$$
$$S_{50} = \frac{50[2(21) + (-4)(50-1)]}{2}$$
$$= -3850$$
c) This is an arithmetic series with common difference $d = 52 - 31 = 21$ and first term $a = 31$. The sum of the first 50 terms is then
$$Sn = \frac{n[2a + (n-1)d]}{2}$$
$$S_{50} = \frac{50[2(31) + 21(50-1)]}{2}$$
$$= 27\,275$$
d) This is an arithmetic series with common difference $d = -14 - (-9) = -5$ and first term $a = -9$. The sum of the first 50 terms is then
$$Sn = \frac{n[2a + (n-1)d]}{2}$$
$$S_{50} = \frac{50[2(-9) + (-5)(50-1)]}{2}$$
$$= -6575$$
e) This is an arithmetic series with common difference $d = 18.9 - 17.5 = 1.4$ and first term $a = 17.5$. The sum of the first 50 terms is then
$$Sn = \frac{n[2a + (n-1)d]}{2}$$
$$S_{50} = \frac{50[2(17.5) + 1.4(50-1)]}{2}$$
$$= 2590$$
f) This is an arithmetic series with common difference $d = -31 - (-39) = 8$ and first term $a = -39$. The sum of the first 50 terms is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{50} = \frac{50[2(-39) + 8(50 - 1)]}{2}$$

$$= 7850$$

15. a) For this arithmetic series, $n = 25$, $a = 24$, and $d = 11$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(24) + 11(25 - 1)]}{2}$$

$$= 3900$$

b) To get from t_1 to t_{25}, you must add d 24 times. This means that

$$t_{25} = t_1 + 24d$$
$$374 = 91 + 24d$$
$$283 = 24d$$
$$d = \frac{283}{24}$$

For this arithmetic series, $n = 25$, $a = 91$, and $d = \frac{283}{24}$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25\left[2(91) + \frac{283}{24}(25 - 1)\right]}{2}$$

$$= 5812.5$$

c) Find d from t_2 and t_1:
$d = t_2 - t_1 = 57 - 84 = -27$. Then, for this series, $n = 25$, $a = 84$, and $d = -27$. The sum is then

$$S_{25} = \frac{n[2a + (n - 1)d]}{2}$$

$$= \frac{25[2(84) + (-27)(25 - 1)]}{2}$$

$$= -6000$$

d) Since the terms of this series decrease by 11, $d = -11$. To get from t_1 to t_3, you must add d twice. So,

$$t_3 = t_1 + 2d$$
$$42 = t_1 + 2(-11)$$
$$42 = t_1 - 22$$
$$t_1 = 64$$

For this arithmetic series, $n = 25$, $a = 64$, and $d = -11$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(64) + (-11)(25 - 1)]}{2}$$

$$= -1700$$

e) Since the terms of this series decrease by 4, $d = -4$. To get from t_1 to t_{12}, you must add d 11 times. So,

$$t_{12} = t_1 + 11d$$
$$19 = t_1 + 11(-4)$$
$$19 = t_1 + 11(-4)$$
$$t_1 = 63$$

For this arithmetic series, $n = 25$, $a = 63$, and $d = -4$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(63) + (-4)(25 - 1)]}{2}$$

$$= 375$$

f) To get from t_5 to t_{15}, you must add d 10 times. So,

$$t_{15} = t_5 + 10d$$
$$12 = 142 + 10d$$
$$-130 = 10d$$
$$d = -13$$

To get from t_1 to t_5, you must add d 4 times. So,

$$t_5 = t_1 + 4d$$
$$142 = t_1 + 4(-13)$$
$$142 = t_1 - 52$$
$$t_1 = 194$$

For this arithmetic series, $n = 25$, $a = 194$, and $d = -13$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{25} = \frac{25[2(194) + (-13)(25 - 1)]}{2}$$

$$= 950$$

16. a) The first and last terms of this series, so compute the sum once the number of terms n is known. The common difference of the sequence of terms of this series is $13 - 1 = 12$. So, the general term for this sequence is

$$t_n = a + (n - 1)d$$
$$= 1 + 12(n - 1)$$
$$= 1 + 12n - 12$$
$$= 12n - 11$$

Use this to solve for the number of terms n: n is the value so that $t_n = 145$.

$$t_n = 145$$
$$12n - 11 = 145$$
$$12n = 156$$
$$n = 13$$

The sum of this series is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{13} = \frac{13(1 + 145)}{2}$$
$$= 949$$

b) The first and last terms of this series, so compute the sum once the number of terms n is known. The common difference of the sequence of terms of this series is $42 - 9 = 33$. So, the general term for this sequence is

$$t_n = a + (n - 1)d$$
$$= 9 + 33(n - 1)$$
$$= 9 + 33n - 33$$
$$= 33n - 24$$

Use this to solve for the number of terms n: n is the value so that $t_n = 4068$.

$$t_n = 4068$$
$$33n - 24 = 4068$$
$$33n = 4092$$
$$33n = 124$$

The sum of this series is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{124} = \frac{124(9 + 4068)}{2}$$
$$= 252\,774$$

c) The first and last terms of this series, so compute the sum once the number of terms n is known. The common difference of the sequence of terms of this series is $118 - 123 = -5$. So, the general term for this sequence is

$$t_n = a + (n - 1)d$$
$$= 123 + (-5)(n - 1)$$
$$= 123 - 5n + 5$$
$$= 128 - 5n$$

Use this to solve for the number of terms n: n is the value so that $t_n = -122$.

$$t_n = -122$$
$$128 - 5n = -122$$
$$-5n = -250$$
$$n = 50$$

The sum of this series is then

$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{50} = \frac{50(123 + (-122))}{2}$$
$$= 25$$

17. The distance that the spacecraft descended during the first second is 64 metres. The distance that it travels in the next second is 7 metres less, or $64 - 7 = 57$ metres. Continuing in this way, the total distance that the spacecraft descends in 10 seconds, in metres, is the sum of the first 10 terms of the arithmetic series $64 + 57 + 50 + \ldots$ For this series, $n = 10$, $a = 64$, and $d = 57 - 64 = -7$. The sum is then

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$
$$S_{10} = \frac{10[2(64) + (-7)(10 - 1)]}{2}$$
$$= 325$$

The spacecraft descended a total of 325 metres.

18. a) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{33}{11} = 3$ and first term $a = 11$. The general term is then

$$t_n = ar^{n-1}$$
$$= 11 \times 3^{n-1}$$

Substitute $n = 6$:

$$t_6 = 11 \times 3^{6-1}$$
$$= 11 \times 3^5$$
$$= 2673$$

Since $r = 3$ and $a = 11$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{11(3^6 - 1)}{3 - 1}$$
$$= 4004$$

b) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{1.111\,11}{0.111\,111} = 10$ and first term $a = 0.111\,111$. The general term is then

$$t^n = ar^{n-1}$$
$$= 0.111\,111 \times 10^{n-1}$$

Substitute $n = 6$:

$t_6 = 0.111\ 111 \times 10^{6-1}$
$= 0.111\ 111 \times 10^5$
$= 11\ 111.1$
Since $r = 10$ and $a = 0.111\ 111$, calculate S_6:
$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{0.111\ 111(10^6 - 1)}{10 - 1}$$
$\doteq 12\ 345.654$

c) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{-12}{6} = -2$ and first term $a = 6$. The general term is then
$t_n = ar^{n-1}$
$= 6 \times (-2)^{n-1}$
Substitute $n = 6$:
$t_6 = 6 \times (-2)^{6-1}$
$= 6 \times (-2)^5$
$= -192$
Since $r = -2$ and $a = 6$, calculate S_6:
$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{6((-2)^6 - 1)}{(-2) - 1}$$
$= -126$

d) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{21\ 870}{32\ 805} = \frac{2}{3}$ and first term $a = 32\ 805$. The general term is then
$t_n = ar^{n-1}$
$$= 32\ 805 \times \left(\frac{2}{3}\right)^{n-1}$$
Substitute $n = 6$:
$$t_6 = 32\ 805 \times \left(\frac{2}{3}\right)^{6-1}$$
$$= 32\ 805 \times \left(\frac{2}{3}\right)^5$$
$= 4320$
Since $r = \frac{2}{3}$ and $a = 32\ 805$, calculate S_6:
$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{32\ 805\left(\left(\frac{2}{3}\right)^6 - 1\right)}{\frac{2}{3} - 1}$$
$= 89\ 775$

e) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{-25.5}{17} = -1.5$ and first term $a = 17$. The general term is then
$t_n = ar^{n-1}$
$= 17 \times 1.5^{n-1}$
Substitute $n = 6$:
$t_6 = 17 \times 1.5^{6-1}$
$= 17 \times 1.5^5$
$= -129.093\ 75$
Since $r = -1.5$ and $a = 17$, calculate S_6:
$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_6 = \frac{17((-1.5)^6 - 1)}{(-1.5) - 1}$$
$= -70.656\ 25$

f) The sequence of the terms of this series is a geometric sequence with common ratio $r = \frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}$ and first term $a = \frac{1}{2}$. The general term is then
$t_n = ar^{n-1}$
$$= \frac{1}{2} \times \left(\frac{3}{5}\right)^{n-1}$$
Substitute $n = 6$:
$$t_6 = \frac{1}{2} \times \left(\frac{3}{5}\right)^{6-1}$$
$$= \frac{1}{2} \times \left(\frac{3}{5}\right)^5$$
$$= \frac{243}{6250}$$
Since $r = \frac{3}{5}$ and $a = \frac{1}{2}$, calculate S_6:

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{\frac{1}{2}\left(\left(\frac{3}{5}\right)^6 - 1\right)}{\frac{3}{5} - 1}$$

$$= \frac{\left(-\frac{7448}{15\,625}\right)}{\left(-\frac{2}{5}\right)}$$

$$= \frac{3724}{3125}$$

19. a) For this series, $n = 8$, $a = -6$, and $r = 4$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 = \frac{(-6)(4^8 - 1)}{4 - 1}$$

$$= -131\,070$$

b) To get from t_1 to t_9, you must multiply by r 8 times. So,

$$t_9 = t_1 r^8$$

$$2112 = 42r^8$$

$$\frac{2112}{42} = r^8$$

$$\frac{352}{7} = r^8$$

$$r = \sqrt[8]{\frac{352}{7}} \doteq 1.631\,851\,285$$

Then, for this series, $n = 8$, $a = 42$, and $r \doteq 1.631\,851\,285$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 \doteq \frac{42(1.631\,851\,285^8 - 1)}{1.631\,851\,285 - 1}$$

$$\doteq 3276.087\,34$$

c) For this series, $r = \frac{t_2}{t_1} = \frac{80}{320} = 0.25$. Then, for this series, $n = 8$, $a = 320$, and $r = 0.25$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 = \frac{320(0.25^8 - 1)}{0.25 - 1}$$

$$= 426.660$$

d) Since the terms increase by a factor of 5, $r = 5$. To get from t_1 to t_3, you multiply by r twice. So,

$$t_3 = t_1 r^2$$

$$35 = 25t_1$$

$$t_1 = 1.4$$

Then, for this series, $n = 8$, $a = 1.4$, and $r = 5$. The sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 = \frac{1.4(5^8 - 1)}{5 - 1}$$

$$= 136\,718.4$$

20. The number of orders filled in the first year is the sum of the first 12 terms of the geometric series $15 + 30 + 60 + \ldots$ with first term $a = 15$ and common ratio $r = 2$. This sum is

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{15(2^{12} - 1)}{2 - 1}$$

$$= 61\,425$$

The company filled 61 425 orders in the first year.

21. The general term of the arithmetic sequence is $t_n = a + (n - 1)d$

Since the 1st, 5th, and 13th terms of this sequence form the first three terms of a geometric sequence with common ratio 2,

$$\frac{t_5}{t_1} = 2$$

$$t_5 = 2t_1$$

$$a + (5 - 1)d = 2[a + (1 - 1)d]$$

$$a + 4d = 2a$$

$$4d = a$$

Since the 21st term of the arithmetic sequence is 72,

$$t_{21} = 72$$

$$a + (21 - 1)d = 72$$

$$a + 20d = 72$$

Since $a = 4d$,

$$4d + 20d = 72$$

$$24d = 72$$

$$d = 3$$

So, $a = 4d = 12$.

The geometric series then has $n = 10$, $a = 12$, and $r = 2$. Its sum is then

$$Sn = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{12(2^{10} - 1)}{2 - 1}$$

$$= 12\,276$$

22. a) The common ratio of the terms of this series is $r = \frac{14}{7} = 2$, and the first term a is 7.
The general term for the geometric sequence is then

$$t_n = ar^{n-1}$$
$$= 7 \times 2^{n-1}$$

Check how many terms are in the series by setting $t_n = 3584$ and solving for n.

$$t_n = 3584$$
$$7 \times 2^{n-1} = 3584$$
$$2^{n-1} = 512$$
$$n - 1 = 9$$
$$n = 10$$

For this series, $n = 10$, $a = 7$, and $r = 2$.

$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_{11} = \frac{7(2^{10} - 1))}{2 - 1}$$
$$= 7161$$

b) The common ratio of the terms of this series is $r = \frac{-6}{-3} = 2$, and the first term a is -3. The general term for the geometric sequence is then

$$t_n = ar^{n-1}$$
$$= -3 \times 2^{n-1}$$

Check how many terms are in the series by setting $t_n = -768$ and solving for n.

$$t_n = -768$$
$$-3 \times 2^{n-1} = -768$$
$$2^{n-1} = 256$$
$$n - 1 = 8$$
$$n = 9$$

For this series, $n = 9$, $a = -3$, and $r = 2$.

$$Sn = \frac{a(r^n - 1))}{r - 1}$$
$$S_9 = \frac{-3(2^9 - 1)}{2 - 1}$$
$$= -1533$$

c) The common ratio of the terms of this series is $r = \frac{5}{2}$, and the first term a is 1. The general term for the geometric sequence is then

$$t_n = ar^{n-1}$$
$$= 1 \times \left(\frac{5}{2}\right)^{n-1}$$

Check how many terms are in the series by setting $t_n = \frac{15\,625}{64}$ and solving for n.

$$t_n = \frac{15\,625}{64}$$
$$1 \times \left(\frac{5}{2}\right)^{n-1} = \frac{15\,625}{64}$$
$$\left(\frac{5}{2}\right)^{n-1} = \frac{15\,625}{64}$$
$$n - 1 = 6$$
$$n = 7$$

For this series, $n = 7$, $a = 1$, and $r = \frac{5}{2}$.

$$Sn = \frac{a(r^n - 1))}{r - 1}$$
$$S_7 = \frac{1\left(\left(\frac{5}{2}\right)^7 - 1\right)}{\frac{5}{2} - 1}$$
$$= \frac{25\,999}{64}$$

d) The common ratio of the terms of this series is $r = -\frac{1}{2}$, and the first term a is 96 000. The general term for the geometric sequence is then

$$t_n = ar^{n-1}$$
$$= 96\,000 \times -\frac{1}{2}^{n-1}$$

Check how many terms are in the series by setting $t_n = 375$ and solving for n.

$$t_n = 375$$
$$96\,000 \times -\frac{1}{2}^{n-1} = 375$$
$$-\frac{1}{2}^{n-1} = \frac{2}{375}$$
$$n - 1 = 8$$
$$n = 9$$

For this series, $n = 9$, $a = 96\,000$, and $r = -\frac{1}{2}$.

$$Sn = \frac{a(r^n - 1))}{r - 1}$$
$$S_9 = \frac{96\,000\left(\left(-\frac{1}{2}\right)^9 - 1\right)}{-\frac{1}{2} - 1}$$
$$= 64\,125$$

e) The common ratio of the terms of this series is $r = 1.06$, and the first term a is 1000. The general term for the geometric sequence is then

$t_n = ar^{n-1}$
$= 1000 \times 1.06^{n-1}$

For this series, $n = 13$, $a = 1000$, and $r = 1.06$.

$$Sn = \frac{a(r^n - 1))}{r - 1}$$

$$S_7 = \frac{1000(1.06^{13} - 1)}{1.06 - 1}$$
$$= 18\,882.14$$

23. a) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so

$$(a + 6)^4$$
$$= 1(a)^4 + 4(a)^3(6)^1 + 6(a)^2(6)^2 + 4(a)^1(6)^3 + 1(6)^4$$
$$= a^4 + 24a^3 + 216a^2 + 864a + 1296$$

b) The 5th row of Pascal's triangle is 1, 5, 10, 10, 5, 1, so

$$(b - 3)^5$$
$$= 1(b)^5 + 5(b)^4(-3)^1 + 10(b)^3(-3)^2 + 10(b)^2(-3)^3 + 5(b)^1(-3)^4 + 1(-3)^5$$
$$= b^5 - 15b^4 + 90b^3 - 270b^2 + 405b - 243$$

c) The 3rd row of Pascal's triangle is 1, 3, 3, 1, so

$$(2c + 5)^3$$
$$= 1(2c)^3 + 3(2c)^2(5)^1 + 3(2c)^1(5)^2 + 1(5)^3$$
$$= 8c^3 + 60c^2 + 150c + 125$$

d) The 6th row of Pascal's triangle is 1, 6, 15, 20, 15, 6, 1, so

$$(4 - 3d)^6$$
$$= 1(4)^6 + 6(4)^5(-3d)^1 + 15(4)^4(-3d)^2 + 20(4)^3(-3d)^3 + 15(4)^2(-3d)^4 + 6(4)^1(-3d)^5 + (-3d)^6$$
$$= 4096 - 18\,432d + 34\,560d^2 - 34\,560d^3 + 19\,440d^4 - 5832d^5 + 729d^6$$

e) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so

$$(5e - 2f)^4$$
$$= 1(5e^4) + 4(5e)^3(-2f)^1 + 6(5e)^2(-2f)^2 + 4(5e)^1(-2f)^3 + 1(-2f)^4$$
$$= 625e^4 - 1000e^3f + 600e^2f^2 - 160ef^3 + 16f^4$$

f) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so

$$\left(3f^2 - \frac{2}{f}\right)^4$$
$$= 1(3f^2)^4 + 4(3f^2)^3\left(-\frac{2}{f}\right)^1 + 6(3f^2)^2\left(-\frac{2}{f}\right)^2 + 4(3f^2)^1\left(-\frac{2}{f}\right)^3 + 1\left(-\frac{2}{f}\right)^4$$
$$= 81f^8 - 216f^5 + 216f^2 - 96f^{-1} + 16f^{-4}$$

Chapter Self-Test, p. 470

1. a) i) $t_1 = 5 \times 3^{1+1}$
$= 5 \times 3^2$
$= 45$

$t_2 = 5 \times 3^{2+1}$
$= 5 \times 3^3$
$= 135$

$t_3 = 5 \times 3^{3+1}$
$= 5 \times 3^4$
$= 405$

$t_4 = 5 \times 3^{4+1}$
$= 5 \times 3^5$
$= 1215$

$t_5 = 5 \times 3^{5+1}$
$= 5 \times 3^6$
$= 3645$

ii) This sequence is geometric since its general term is a discrete exponential function.

b) i) $t_1 = \dfrac{3(1) + 2}{2(1) + 1} = \dfrac{5}{3}$

$t_2 = \dfrac{3(2) + 2}{2(2) + 1} = \dfrac{8}{5}$

$t_3 = \dfrac{3(3) + 2}{2(3) + 1} = \dfrac{11}{7}$

$t_4 = \dfrac{3(4) + 2}{2(4) + 1} = \dfrac{14}{9}$

$t_5 = \dfrac{3(5) + 2}{2(5) + 1} = \dfrac{17}{11}$

ii) $t_2 - t_1 = \dfrac{8}{5} - \dfrac{5}{3} = \dfrac{-1}{24}$

$t_3 - t_2 = \dfrac{11}{7} - \dfrac{8}{5} = \dfrac{-1}{56}$

These are not equal, so this sequence is not arithmetic.

$$\frac{t_2}{t_1} = \frac{\left(\frac{8}{5}\right)}{\left(\frac{5}{3}\right)} = \frac{24}{25}$$

$$\frac{t_3}{t_2} = \frac{\left(\frac{11}{7}\right)}{\left(\frac{8}{5}\right)} = \frac{55}{56}$$

These are not equal, so this sequence is not geometric either.

c) i) $t_1 = 5(1)$
$= 5$
$t_2 = 5(2)$
$= 10$
$t_3 = 5(3)$
$= 15$
$t_4 = 5(4)$
$= 20$
$t_5 = 5(5)$
$= 25$

ii) This sequence is arithmetic since its general term is a linear function.

d) i) $t_1 = 5$
$t_2 = 7(t_1)$
$= 7(5)$
$= 35$
$t_3 = 7(t_2)$
$= 7(35)$
$= 245$
$t_4 = 7(t_3)$
$= 7(245)$
$= 1715$
$t_5 = 7(t_4)$
$= 7(1715)$
$= 12\,005$

ii) This sequence is geometric since it is defined by the recursive formula $t_n = rt_{n-1}$ where $n > 1$, for the value $r = 7$.

e) i) $t_1 = 19$
$t_2 = 1 - t_1$
$= 1 - 19$
$= -18$
$t_3 = 1 - t_2$
$= 1 - (-18)$
$= 19$
$t_4 = 1 - t_3$
$= 1 - 19$
$= -18$
$t_5 = 1 - t_4$
$= 1 - (-18)$
$= 19$

ii) $t_2 - t_1 = -18 - 19 = -37$
$t_3 - t_2 = 19 - (-18) = 37$

These are not equal, so this sequence is not arithmetic.

$\frac{t_2}{t_1} = -\frac{18}{19}$

$\frac{t_3}{t_2} = -\frac{19}{18}$

These are not equal, so this sequence is not geometric either.

f) i) $t_1 = 7$
$t_2 = 13$
$t_3 = 2t_2 - t_1$
$= 2(13) - 7$
$= 19$
$t_4 = 2t_3 - t_2$
$= 2(19) - 13$
$= 25$
$t_5 = 2t_4 - t_3$
$= 2(25) - 19$
$= 31$

ii) These terms 7, 13, 19, 25, 31, …, seem to be an arithmetic sequence. Since this sequence has first term $a = 7$ and common difference $d = 13 - 7 = 6$, its general term appears to be
$t_n = a + (n - 1)d$
$= 7 + 6(n - 1)$
$= 7 + 6n - 6$
$= 6n + 1$

Check this. This general term is certainly accurate for $n = 1$ and $n = 2$:
$t_1 = 7$
$= 6(1) + 1$
$t_2 = 13$
$= 6(2) + 1$

Assuming that $t_n - 1 = 6(n - 1) + 1$ and $t_n - 2 = 6(n - 2) + 1$, then
$t_n = 2t_{n-1} - t_{n-2}$
$= 2(6(n - 1) + 1) - (6(n - 2) + 1)$
$= 2(6n - 6 + 1) - (6n - 12 + 1)$
$= 2(6n - 5) - (6n - 11)$
$= 12n - 10 - 6n + 11$
$= 6n + 1$

Therefore, this general term is accurate and this sequence is an arithmetic sequence.

2. a) i) This is a geometric sequence with $a = -9$ and $r = -11$. Its general term is then

$t_n = ar^{n-1}$
$= (-9) \times (-11)^{n-1}$
ii) The recursive formula for this sequence is
$t_1 = -9,$
$t_n = rt_{n-1}$
$= -11t_{n-1}$ where $n > 1$.
b) i) The common difference of this arithmetic sequence is $d = t_3 - t_2 = -456 - 123 = -579$. Use this to find t_1:
$d = t_2 - t_1$
$-579 = 123 - t_1$
$-702 = -t_1$
$t_1 = 702$
The general term for this arithmetic sequence is then
$t_n = a + (n - 1)d$
$= 702 + (-579)(n - 1)$
$= 702 - 579n + 579$
$= 1281 - 579n$
ii) The recursive formula for this sequence is
$t_1 = 702,$
$t_n = t_{n-1} + d$
$= t_{n-1} - 579$ where $n > 1$
3. a) This is an arithmetic sequence with common difference $d = 25 - 18 = 7$. So, the general term for this sequence is
$t_n = a + (n - 1)d$
$= 18 + 7(n - 1)$
$= 18 + 7n - 7$
$= 7n + 11$
Use this to solve for the number of terms n: n is the value so that $t_n = 193$.
$t_n = 193$
$7n + 11 = 145$
$7n = 182$
$n = 26$
b) This is a geometric sequence with common ratio $r = -5$. So, the general term for this sequence is
$t_n = ar^{n-1}$
$= 2 \times (-5)^{n-1}$
Use this to solve for the number of terms n: n is the value so that $t_n = -156\,250$.
$t_n = -156\,250$
$2 \times (-5)^{n-1} = -156\,250$
$(-5)^{n-1} = -78\,125$
$n - 1 = 7$
$n = 8$
4. a) The 4th row of Pascal's triangle is 1, 4, 6, 4, 1, so
$(x - 5)^4$
$= 1(x)^4 + 4(x)^3(-5)^1 + 6(x)^2(-5)^2 + 4(x)^1(-5)^3 + 1(-5)^4$
$= x^4 - 20x^3 + 150x^2 - 500x + 625$
b) The 3rd row of Pascal's triangle is 1, 3, 3, 1, so
$(2x + 3y)^3$
$= 1(2x)^3 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(3y)^3$
$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$
5. a) $d = 33 - 19 = 14$ This appears to be an arithmetic sequence with common difference, since $33 - 19 = 14$ and $47 - 33 = 14$. So, the general term for this sequence is
$t_n = a + (n - 1)d$
$= 19 + 14(n - 1)$
$= 19 + 14n - 14$
$= 14n + 5$
Use this to solve for the number of terms n: n is the value so that $t_n = 439$.
$t_n = 439$
$14n + 5 = 439$
$14n = 434$
$n = 31$
The sum of this series is then
$$Sn = \frac{n(t_1 + t_n)}{2}$$
$$S_{31} = \frac{31(19 + 439)}{2}$$
$= 7099$
b) This series is a geometric sequence with common ratio $r = \frac{12\,000}{10\,000} = 1.2$. This is then a geometric sequence with $n = 10$, $a = 10\,000$, $r = 1.2$. Its sum is then
$$Sn = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{10\,000(1.2^{10} - 1)}{1.2 - 1}$$
$= 259\,586.821\,1$
6. Compute the first few terms of this sequence:
$t_1 = 4$
$t_2 = 5$
$$t_3 = \frac{t_2 + 1}{t_1}$$
$$= \frac{5 + 1}{4}$$
$= 1.5$

$$t_4 = \frac{t_3 + 1}{t_2} = \frac{1.5 + 1}{5} = 0.5$$

$$t_5 = \frac{t_4 + 1}{t_3} = \frac{0.5 + 1}{1.5} = 1$$

$$t_6 = \frac{t_5 + 1}{t_4} = \frac{1 + 1}{0.5} = 4$$

$$t_7 = \frac{t_6 + 1}{t_5} = \frac{4 + 1}{1} = 5$$

Since each term of this sequence depends only on the two previous terms, the five terms 4, 5, 1.5, 0.5, 1 will repeat forever. This means that $t_{123} = t_{118} = t_{113} = \ldots = t_8 = t_3$. Since $t_3 = 1.5$, this means that $t_{123} = 1.5$.

7. Since your grandparents put aside \$75 more each year than in the previous year, the total amount of money, in dollars, that they set aside over 21 years is the sum of the first 21 terms of the arithmetic series $100 + 175 + 250 + \ldots$ with common difference $d = 75$. This series then has $n = 21$, $a = 100$, and $d = 75$. Its sum is

$$Sn = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{21} = \frac{21[2(100) + 75(21 - 1)]}{2} = 17\,850.$$

They will have put aside \$17 850 over 21 years.

8. a) To find the next term in the sequence, add the two previous terms. The recursive formula for this sequence is $t_1 = 1$ and $t_2 = 7$, $t_n = t_{n-1} + t_{n-2}$, where $n > 2$.

$t_7 = t_6 + t_5 = 38 + 23 = 61$

$t_8 = t_7 + t_6 = 61 + 38 = 99$

$t_9 = t_8 + t_7 = 99 + 61 = 160$

b) To find the next term in the sequence, add 1 to the exponent of p, switch the sign between the terms, and increase the coefficient of q by 1. The general term is $t_n = p^{n+1} + (-1)^{n+1}(n + 1)q$.

$$t_5 = p^{5+1} + (-1)^{5+1}(5 + 1)q = p^6 + 6q$$

$$t_6 = p^{6+1} + (-1)^{6+1}(6 + 1)q = p^7 - 7q$$

$$t_7 = p^{7+1} + (-1)^{7+1}(7 + 1)q = p^8 - 8q$$

c) To find the next term, subtract 6 from the numerator of the previous term and add 3 to the denominator of the previous term. The general term is $t_n = \frac{25 - 6(n - 1)}{3n}$.

$$t_6 = \frac{25 - 6(6 - 1)}{3(6)} = -\frac{5}{18}$$

$$t_7 = \frac{25 - 6(7 - 1)}{3(7)} = -\frac{11}{21}$$

$$t_8 = \frac{25 - 6(8 - 1)}{3(8)} = -\frac{17}{24}$$

Chapter 8: Discrete Functions: Financial Applications

Note: Answers are given to the same number of decimal points as the numbers in each question.

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1. a) Each term in the sequence is equal to the previous term plus 4. So the next two terms are 23 and 27:
$t_4 = 19 + 4 = 23$
$t_5 = 23 + 4 = 27$
The general term, t_n, is $3 + 4n$.
For example,
$t_1 = 3 + (4 \times 1) = 7$
$t_2 = 3 + (4 \times 2) = 11$
The recursive formula is $t_1 = 7$, $t_n = t_{n-1} + 4$, where $n > 1$.
b) Each term in the sequence is equal to the previous term minus 27. So the next two terms are -50 and -77:
$t_4 = -23 - 27 = -50$
$t_5 = -50 - 27 = -77$
The general term, t_n, is $85 - 27n$.
For example,
$t_1 = 85 - (27 \times 1) = 58$
$t_2 = 85 - (27 \times 2) = 31$
The recursive formula is $t_1 = 58$, $t_n = t_{n-1} - 27$, where $n > 1$.
c) Each term in the sequence is equal to the previous term multiplied by 4. So the next two terms are 1280 and 5120:
$t_4 = 320 \times 4 = 1280$
$t_5 = 1280 \times 4 = 5120$
The general term, t_n, is $5 \times 4^{n-1}$.
For example,
$t_1 = 5 \times 4^0 = 5$
$t_2 = 5 \times 4^1 = 20$
The recursive formula is $t_1 = 5$, $t_n = 4t_{n-1}$, where $n > 1$.
d) Each term in the sequence is equal to the previous term multiplied by $-\frac{1}{2}$. So the next two terms in the series are -125 and $62\frac{1}{2}$:
$t_4 = 250 \times -\frac{1}{2} = -125$
$t_5 = -125 \times -\frac{1}{2} = 62\frac{1}{2}$
The general term, t_n, is $1000 \times \left(-\frac{1}{2}\right)^{n-1}$. For example,
$t_1 = 1000 \times \left(-\frac{1}{2}\right)^0$
$t_2 = 1000 \times \left(-\frac{1}{2}\right)^1$
The recursive formula is $t_1 = 1000$,
$t_n = \left(-\frac{1}{2}\right)t_{n-1}$, where $n > 1$.

2. a) In an arithmetic sequence, there is a common difference, d, between terms. If $t_4 = 46$ then $t_5 = 46 + d$ and $t_6 = (46 + d) + d$.
Substituting $t_6 = 248$,
$248 = 46 + 2d$
$2d = 202$
$d = 101$
So the fifth term, t_5, is $46 + 101 = 147$.
b) As determined in **a)**, $d = 101$.
c) Start with t_4 and subtract the common difference to find the first term, t_1:
$t_4 = 46$
$t_3 = 46 - d$
$t_2 = (46 - d) - d = 46 - 2d$
$t_1 = (46 - 2d) - d = 46 - 3d = -257$.
d) The general term, t_n, is $-358 + 101n$. So the 100th term, t_{100}, is $-358 + (100 \times 101) = 9742$.