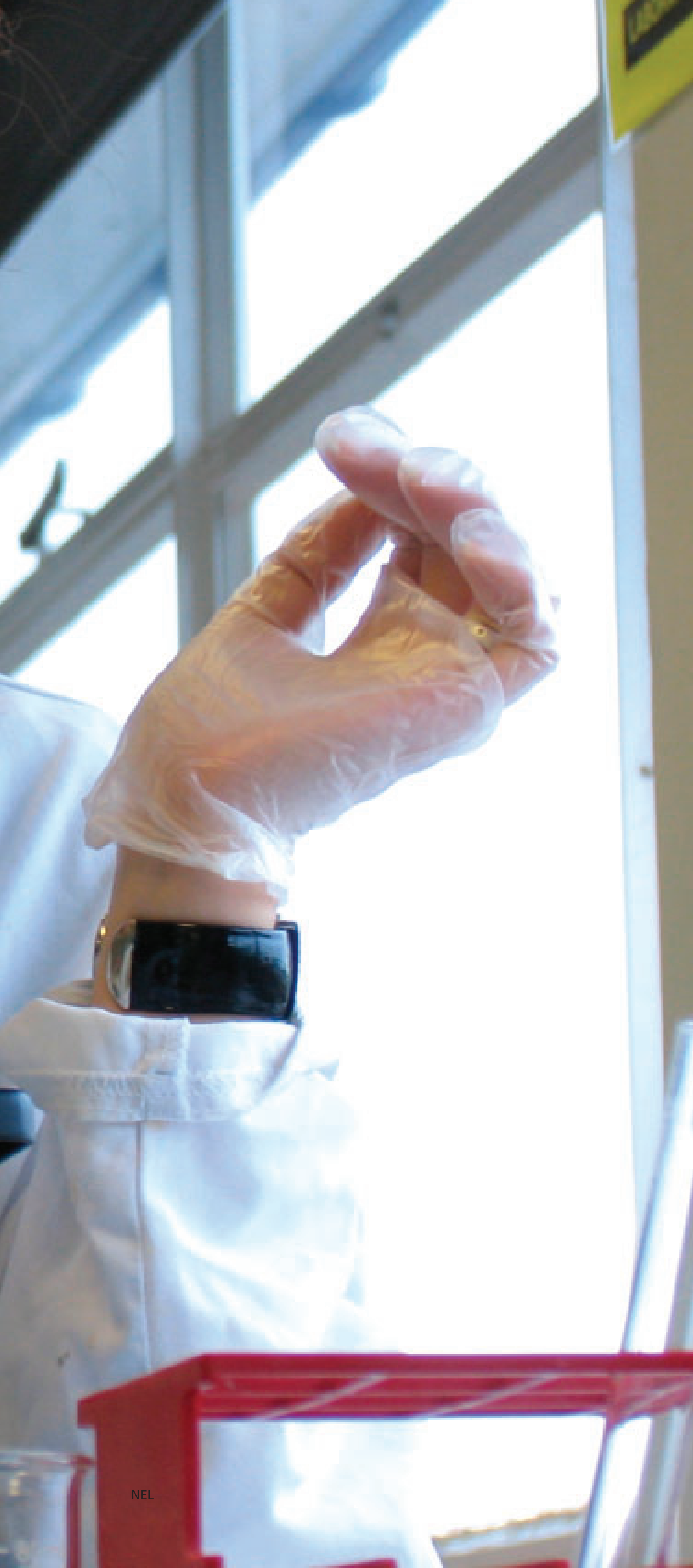


Chapter 2

The Algebra of Quadratic Expressions

GOALS

You will be able to

- Expand and simplify quadratic expressions
- Recognize different types of quadratic expressions and use appropriate strategies to factor them

? Scientists attempt to understand the universe by first determining the basic elements that it contains. When and how have you learned to break down mathematical expressions into their *elements* to understand them better?

2

Getting Started

WORDS You Need to Know

1. Match each word with the expression that best illustrates its definition.

a) variable
b) coefficient
c) like terms
d) unlike terms
e) binomial
f) trinomial
g) expanding
h) factoring

i) $3x + 5$
ii) $2(x - 5) = 2x - 10$
iii) $3x^2 + 7x - 1$
iv) $5x^2$
v) $5x + 15 = 5(x + 3)$
vi) $10y$ and $10y^2$
vii) x
viii) $6x$ and $-3x$

SKILLS AND CONCEPTS You Need

Study Aid
For help, see Essential Skills Appendix, A-9.

Simplifying Algebraic Expressions by Collecting Like Terms

To simplify an algebraic expression, collect like terms by adding or subtracting.

EXAMPLE

Simplify.
$(2x^2 + 3) + (-4x^2 + 8)$
$= 2x^2 + (-4x^2) + 3 + 8$
$= -2x^2 + 11$

2. Simplify each expression.
a) $2x - 5y + 6y - 8x$
b) $7xy - 8x^2 + 6xy - 2x^2 - 12xy + 10x^2$
c) $(4x - 5y) + (6x + 3) - (7x - 2y)$
d) $(2a - 8ab) - (7b + 9a) + (ab - 2a)$

Study Aid
For help, see Essential Skills Appendix, A-3.

Exponent Laws for Multiplication and Division

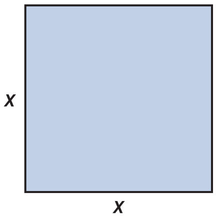
Rule	Written Description	Algebraic Description	Worked Example in Standard Form
Multiplication	To multiply powers with the same base, add the exponents, leaving the base as is.	$b^m \times b^n = b^{m+n}$	$(x^2)(x^4)$ $= x^{2+4}$ $= x^6$
Division	To divide powers with the same base, subtract the exponents, leaving the base as is.	$b^m \div b^n = b^{m-n}$	$x^5 \div x^3$ $= x^{5-3}$ $= x^2$

3. Simplify.

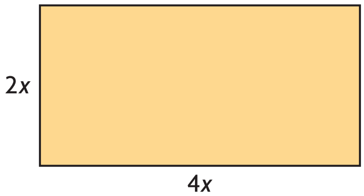
a) $(x^3)(x^2)$ b) $(2x^2)(5x)$ c) $x^4 \div x^2$ d) $4x^5 \div 2x^3$

4. State an expression for the area of each shape.

a)



b)



Expanding Using the Distributive Property

To expand an algebraic expression, use the distributive property to multiply the expression by a constant or a term.

$$a(b + c) = ab + ac$$

Study Aid

For help, see Essential Skills Appendix, A-9.

EXAMPLE

Expand and simplify.

$2(2a + b) - 3(3a - 2b)$

$= 2(2a + b) - 3(3a - 2b)$
$= 4a + 2b - 9a + 6b$
$= 4a - 9a + 2b + 6b$
$= -5a + 8b$

5. Expand and simplify.

a) $3(3x - 8)$
b) $-4(8x^2 - 2x + 1)$
c) $2(7x^2 + 3x + 5) - 2(8x + 1)$
d) $(3d^3 - 6d + 5d^2) + 4(9 - 2d^3 - 4d^2)$
e) $2x^2(3x + 5)$
f) $-5x^2(x^2 - 3x + 4)$

Dividing Out a Common Factor

To factor an algebraic expression, divide out the greatest number or term that will divide into all terms.

EXAMPLE

Factor each expression.

a) $4x^2 - 12x + 4$
$= 4(x^2 - 3x + 1)$

b) $3x^2 + 6x^5 - 9x^4$
$= 3x^2(1 + 2x^3 - 3x^2)$

6. Factor each expression.

a) $2x - 10$
b) $6x^2 + 24x + 30$
c) $25x^2 + 20x - 100$
d) $7x^4 + 12x^3 - 9x^5$

PRACTICE

Study Aid

For help, see Essential Skills Appendix.

Question	Appendix
7	A-3

7. Simplify.

a) $x^2 \times x^3$

b) $-5x^3 \times 6x^4$

c) $x \times 3x^2 \times 5x^4$

d) $2 \times (-x) \times (-x^2)$

8. a) Which expressions are monomials?

b) Which are binomials?

c) Which are trinomials?

d) Which are quadratic?

i) 4

ii) $3x - 2$

iii) $4x^2 + 3x - 1$

iv) $2x(3x + 1)$

v) $7x^3$

vi) $-3x^2(2x - 5)$

9. Name the greatest common factor for each pair.

a) 24, 32

b) 56, 80

c) 108, 90

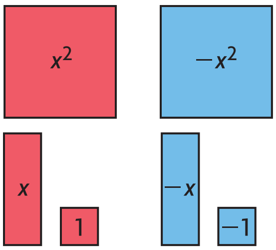
d) $3x$, $2x^2$

e) $3x + 2$, $6x + 4$

f) $25x^2$, $15x$

10. Factor using only prime numbers.

a) 78 **b)** 63 **c)** 3025 **d)** 41



11. Sketch copies of tiles like those shown at the left to represent each expression.

a) $2x + 3$

b) $-4x - 2$

c) $x^2 - 2x + 1$

d) $-x^2 + 1$

e) $3x^2 - 2x - 3$

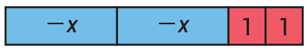
f) $1 + x - x^2$

12. Match each diagram with the correct expression.

a) $2x + 2$ **b)** $-2x + 2$ **c)** $2x - 2$

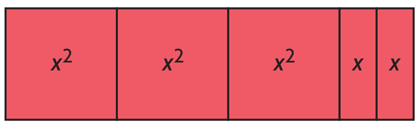
i)

ii)



iii)

13. This rectangle shows $3x^2 + 2x$.



Sketch rectangles to show each expression.

a) $3x^2 - 2x$ **b)** $2x + 4$ **c)** $-2x^2 - x$ **d)** $x^2 + 3x$

14. Decide whether you agree or disagree with each statement. Explain why.

a) You can factor a number by determining the length and width of a rectangle with that area.

b) One way to factor 12 is as $9 \times \frac{4}{3}$.

c) The only way to factor $2x + 6$ is as $2(x + 3)$.

APPLYING What You Know

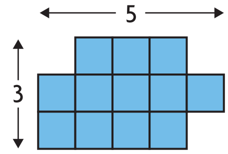
YOU WILL NEED

- graph paper or square tiles

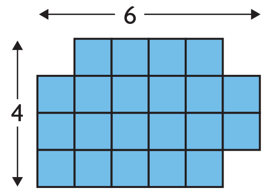
Rearranging Tiles

Using 1×1 square tiles, Fred builds shapes like those shown at the right. Each shape has a length 2 greater than the width.

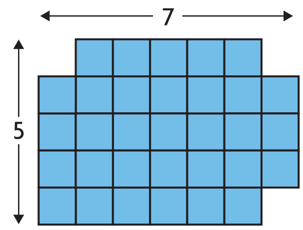
Fred notices that he can rearrange the tiles in each of shapes *R*, *S*, and *T* to form a rectangle.



R



S



T

? Is it always possible to rearrange the tiles to form a rectangle?

A. Rearrange the tiles in shape *R* to form a rectangle. Sketch all possibilities on graph paper. Repeat, using the tiles from *S* and from *T*.

B. Select one of your rectangles from part A for shapes *R*, *S*, and *T* so that the three rectangles chosen form a pattern.

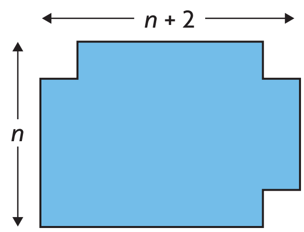
C. How does the length of the rectangle you sketched relate to the original length of each shape?

D. How does the width of the rectangle you sketched relate to the original width of each shape?

E. Use square tiles or graph paper to create new shapes *U* and *V* to extend the pattern of shapes *R*, *S*, *T*, ...

F. Predict how shapes *U* and *V* can be rearranged into rectangles to extend the pattern you created in part B. Check your predictions.

G. Explain how the general shape represents the shapes in the pattern *R*, *S*, *T*, *U*, *V*, ...



general shape

H. What algebraic expression involving n could you use to describe the area of the general shape in part G? Explain.

I. What algebraic expression involving n could you use to describe the length and the width of the rectangle you can make by rearranging the general shape in part G? Explain.

J. How do you know that any shape like the one in part G can be rearranged into a rectangle as long as $n \geq 3$?

2.1 Working with Quadratic Expressions

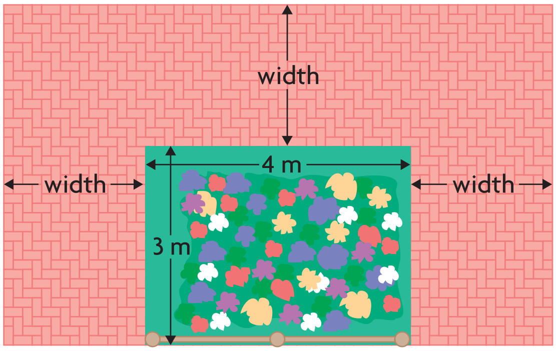
YOU WILL NEED

- algebra tiles

GOAL

Expand and simplify quadratic expressions.

INVESTIGATE the Math



Ali wants to know the area of brick he needs to build a path around his garden.

? What is a simplified expression for the area of the path?

A. Use w to represent the width of the path.

B. Think about the large rectangle that includes the garden and the path. What algebraic expressions represent the length and width of this rectangle?

C. What algebraic expression represents the area of the large rectangle in part B? How do you know?

D. What is the area of the garden without the path?

E. Use your results from parts C and D to write an expression for the area of the path.

F. What simplified expression represents the area of the path?

Reflecting

G. Why was a variable introduced in part A?

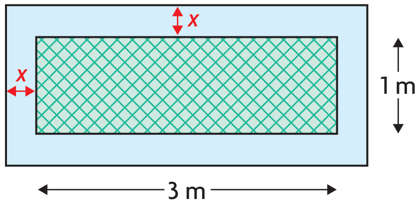
H. Explain how you expanded and simplified the expression you wrote in part E.

I. Is the algebraic expression representing the area of the path linear or quadratic? Explain how you know.

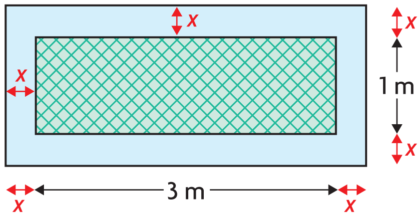
APPLY the Math

EXAMPLE 1 Multiplying two binomials by using algebra tiles

Evelyn is sewing a quilt as shown. If the width of the border is x, state the area of the quilt as a function of x.



Devin's Solution

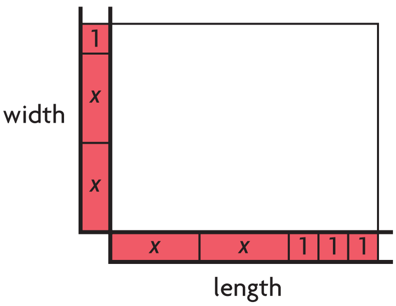


If the width of the border is x, then I know the length and width of the quilt in terms of x.

length: $x + 3 + x = 2x + 3$

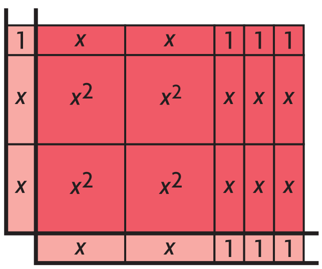
width: $x + 1 + x = 2x + 1$

The dimensions are $2x + 3$ and $2x + 1$.

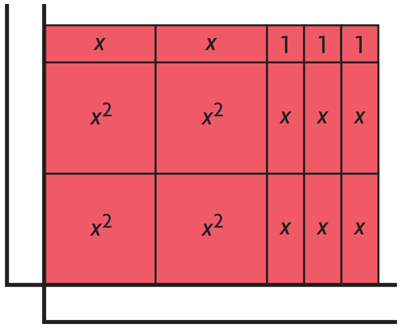


I modelled the area by using algebra tiles.

I represented the length of $2x + 3$ and width of $2x + 1$.



I formed a rectangle with tiles to model the area of the quilt. I used the length and width tiles to make sure I made the rectangle the right size.



I removed the tiles I used to represent the length and width, since they are not part of the area.

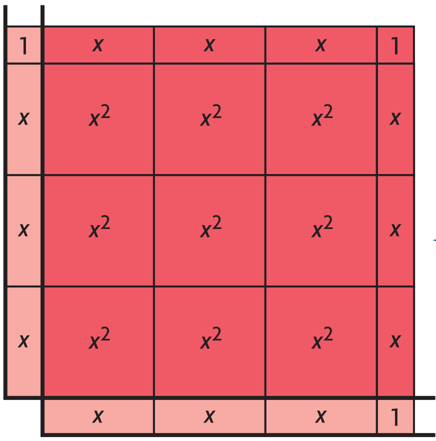
$$A(x) = 4x^2 + 2x + 6x + 3$$
$$= 4x^2 + 8x + 3$$

I counted the tiles in the rectangle to get an expression for its area $A(x)$, then simplified by adding like terms.

EXAMPLE 2 Squaring a binomial by using algebra tiles

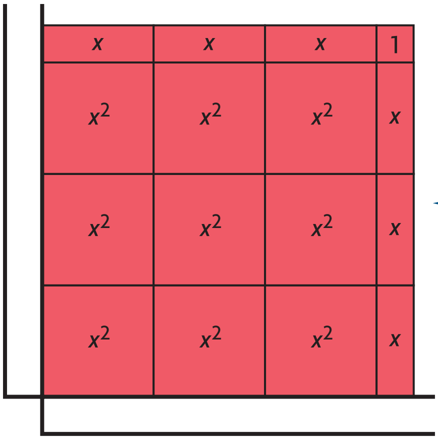
Use algebra tiles to simplify $(3x + 1)^2$.

Lisa's Solution



I used tiles to create a model with a length and width of $3x + 1$.

$(3x + 1)^2$ is the area of a square with side length $3x + 1$.



I removed the tiles that represented the length and width. They were not part of the area of the square.

$$(3x + 1)^2 = 9x^2 + 3x + 3x + 1$$
$$= 9x^2 + 6x + 1$$

To determine the area, I counted the tiles, then simplified the expression by adding like terms.

EXAMPLE 3 Determining the product of a sum and difference of two terms using algebra tiles

Dave claims that

$$(2x + 3)(2x - 3) = (2x)^2 - 3^2$$
$$= 4x^2 - 9$$

a) Confirm the relationship by evaluating each expression when $x = 2$ and $x = 3$.
b) Show that Dave is correct no matter what the value of x.

Tracy's Solution

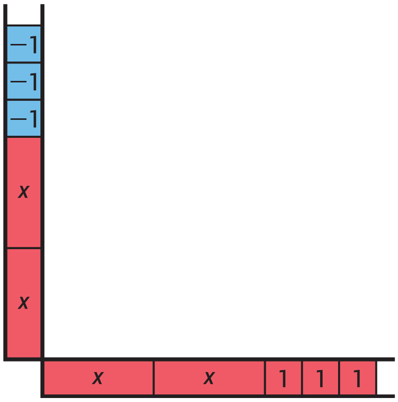
a)

	Left Side	Right Side	
	$(2x + 3)(2x - 3)$	$4x^2 - 9$	
$x = 2$	$(2 \times 2 + 3)(2 \times 2 - 3)$ $= 7 \times 1$ $= 7$	$4 \times 2^2 - 9$ $= 7$	Values are equal.
$x = 3$	$(2 \times 3 + 3)(2 \times 3 - 3)$ $= 9 \times 3$ $= 27$	$4 \times 3^2 - 9$ $= 27$	Values are equal.

I substituted the values 2 and 3 into both sides of the equation.

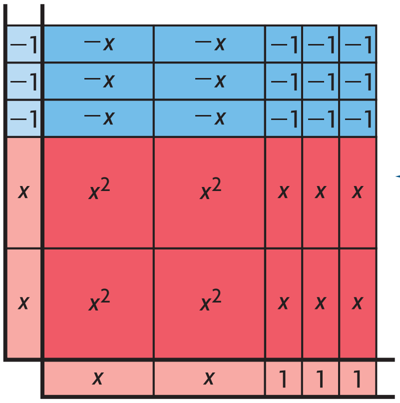
It appears Dave's claim is true, since both expressions give the same result when $x = 2$ and $x = 3$. However, I can't be certain since I only showed that this worked for two cases.

b)

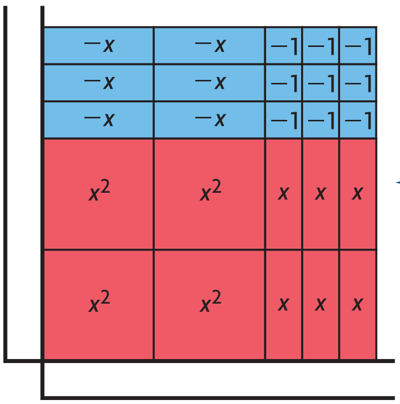


The area of a rectangle with dimensions $2x + 3$ and $2x - 3$ represents the product $(2x + 3)(2x - 3)$.

I used algebra tiles to represent the lengths and widths, red tiles for positives and blue tiles for negatives.



I built the rectangle I needed. I used blue negative tiles for the part that was multiplied by −3.



I removed the tiles representing the length and width.

$$\begin{aligned}(2x + 3)(2x - 3) &= 4x^2 - 6x + 6x - 9\\ &= 4x^2 - 9\end{aligned}$$

To determine the area, I counted the tiles in the square, then simplified. I also noticed that since the 6 blue x tiles are negative and the 6 red x tiles are positive, they cancel out.

You can multiply algebraic expressions without using algebra tiles if you use the distributive property.

EXAMPLE 4 Expanding and simplifying quadratic expressions symbolically

Expand and simplify.

a) $x(2x + 1)$

b) $5(2x - 3)^2$

c) $-2(a - 5)(3a - 2)$

d) $4n(n - 3) + (5n + 1)(3n + 2)$

Link's Solutions

a) $x(2x + 1)$

$= x \times 2x + x \times 1$

$= 2x^2 + x$

I used the distributive property to multiply each term in the brackets by x.

b) $5(2x - 3)^2$

$(2x - 3)^2$ means $(2x - 3)(2x - 3)$.

$= 5(2x - 3)(2x - 3)$

$= 5[(2x)(2x - 3) + (-3)(2x - 3)]$

$= 5[4x^2 - 6x - 6x + 9]$

I used the distributive property to expand the two binomials. I multiplied $2x$ by both $2x$ and -3, then multiplied -3 by both $2x$ and -3. I simplified by collecting like terms.

$= 5[4x^2 - 12x - 9]$

$= 20x^2 - 60x + 45$

I multiplied the product by 5, again using the distributive property.

c) $-2(a - 5)(3a - 2)$

$= -2[a(3a - 2) - 5(3a - 2)]$

$= -2(3a^2 - 2a - 15a + 10)$

I used the distributive property to find the product of the two binomials. I multiplied a by both $3a$ and -2, and -5 by both $3a$ and -2.

$= -2(3a^2 - 17a + 10)$

$= -6a^2 + 34a - 20$

Then I collected like terms and multiplied by -2, using the distributive property again.

d) $4n(n - 3) + (5n + 1)(3n + 2)$

$= (4n \times n - 4n \times 3) + (5n \times 3n + 5n \times 2 + 1 \times 3n + 1 \times 2)$

$= (4n^2 - 12n) + (15n^2 + 10n + 3n + 2)$

$= 4n^2 - 12n + 15n^2 + 13n + 2$

$= 19n^2 + 1n + 2$

$= 19n^2 + n + 2$

I used the distributive property again. I multiplied the first product and then the second one.

Then I collected like terms. Since I was adding $13n$ and $-12n$, only $1n$ was left.

I wrote $1n$ as n.

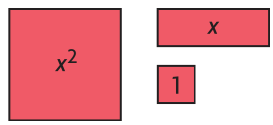
In Summary

Key Idea

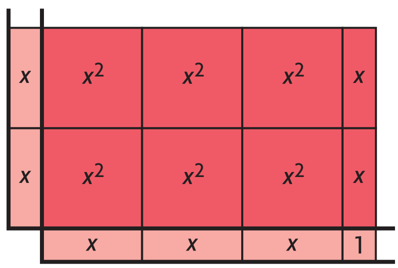
- Quadratic expressions can be expanded by using the distributive property and then simplified by collecting like terms.

Need to Know

- One way to multiply two linear expressions is to use an area model with algebra tiles. If you multiply two expressions, the expressions describe the length and width of a rectangle. The area of the rectangle is the product.
- In these models, x^2 can be represented as the area of a square with side length x.
- x can be represented as the area of a rectangle with side lengths of 1 and x.
- 1 can be represented as a square with side length 1.



- Red is used to represent positive quantities, blue to represent negative quantities.
- For example, to multiply $2x(3x + 1)$, build a rectangle with a width of $2x$ and a length of $3x + 1$ and count the tiles in the area as the product.



- For the product of a monomial and a binomial, the distributive property states that

$$a(b + c) = ab + ac$$

- For the product of a binomial and a binomial, apply the distributive property twice:

$$(a + b)(c + d)$$
$$= a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

- Like terms have the same variables with the same exponents.
- Three special multiplication patterns are

$$(a + b)^2 = (a + b)(a + b) = (a^2 + ab + ba + b^2) = a^2 + 2ab + b^2$$

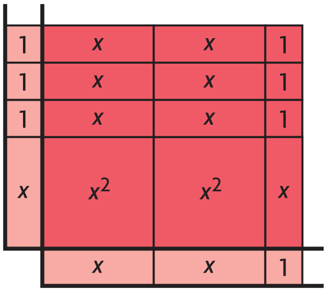
$$(a - b)^2 = (a - b)(a - b) = (a^2 - ab - ba + b^2) = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = (a^2 - ab + ba - b^2) = a^2 - b^2$$

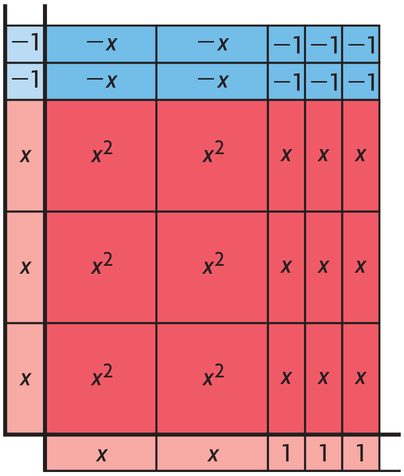
CHECK Your Understanding

1. For each diagram, state the terms representing the length and the width of the rectangle. Then determine the product represented by the area.

a)



b)



2. Expand and simplify.

a) $(x + 7)(x - 3)$

b) $(a + 6)(a + 6)$

c) $(2x - 5)^2$

d) $(m - 9)(m + 9)$

3. Expand and simplify

a) $3(x - 6)(x + 5)$

b) $3a(a - 5) - (2a + 1)(a - 7)$

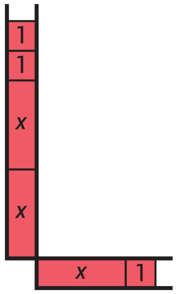
c) $-2n(2n + 1) + (n + 2)^2$

d) $3(2x + 1)^2 - 2(3x - 1)^2$

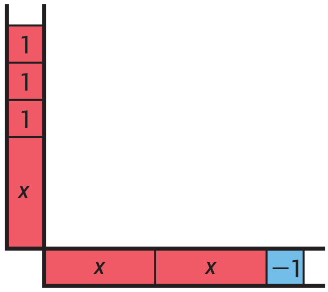
PRACTISING

4. For each diagram, describe the terms representing the length and the width of the rectangle and the product represented by the area.

a)



b)



5. Expand and simplify.

a) $(3x - 2)(4x + 5)$

b) $5(3x + 2)^2$

c) $2(x - 3)^2 - (4x + 1)(4x - 1)$

d) $-a(2a + 3) + 2(a + 4)(3a + 4)$

e) $2n(n + 3) - 3n(5 - 2n) + 7n(1 - 4n)$

f) $(2x + 5)^2 + (2x - 5)^2 - (2x - 5)(2x + 5)$

6. a) Evaluate $3 - 2x$ for $x = -4$.

b) Evaluate $x + 2$ for $x = -4$.

c) Expand and simplify $(3 - 2x)(x + 2)$.

d) Evaluate your answer from part (c) for $x = -4$.

e) How are your answers to parts (a) and (b) related to the answer in part (d)?

7. Expand and simplify.

K **a)** $2x(x - 5)$

b) $(a + 7)(a - 9)$

c) $(3x + 7)(6x - 5)$

d) $2m(3m + 1) + (m - 4)(5m + 3)$

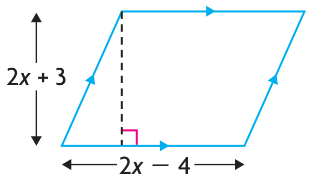
8. How do you know that the product will be quadratic when you expand an expression such as $(2x + 3)(3x - 2)$?

9. a) Sketch two different rectangles with an area of $4x^2 + 8x$.

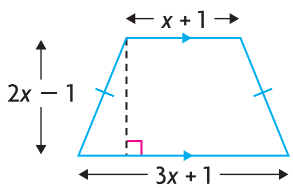
b) List the dimensions of each rectangle.

10. Write an expression for the area of each shape. Expand and simplify.

A **a)**



b)



Communication | *Tip*

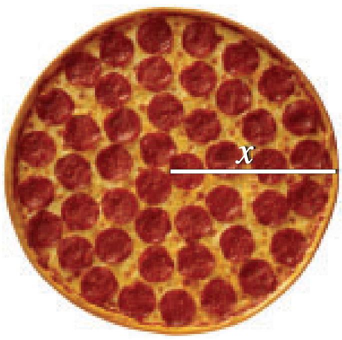
The formula for the area of
- a parallelogram is $A = bh$
- a trapezoid is $A = h(b_1 + b_2) \div 2$

11. A circular pizza has a radius of x cm.

T **a)** Write an expression for the area of the pizza.

b) Write an expression for the area of a pizza with a radius that is 5 cm greater.

c) How much greater is the second area? Write the difference as a simplified expression.



Communication | *Tip*

The formula for the area of a circle is $A = \pi r^2$.

12. When you add $(6x^2 - 8x)$ and $(-15x^2 - 18)$ and collect like terms, you end up with three terms.

a) Give an example to show that the sum of two quadratic binomials could have only two terms.

b) Give an example to show that the sum of two quadratic binomials could have only one term.

13. **a)** Use an example to show that you can multiply two binomials and end up with three terms after you simplify.

C **b)** Use other binomials to show that you can end up with two terms after you multiply and simplify.

Extending

14. Expand and simplify.

a) $(2x - y)(3x + y)$

b) $(3a - 5b)^2$

c) $(5m - 7n)(5m + 7n)$

d) $-2(x + 3y)(2x - y)$

15. A Pythagorean triple is a triple of natural numbers a, b, c that satisfies the Pythagorean theorem $a^2 + b^2 = c^2$, or equivalently, $a^2 = c^2 - b^2$. For example, 3, 4, 5 is a Pythagorean triple, since $3^2 + 4^2 = 5^2$, or $3^2 = 5^2 - 4^2$.

a) Show that $(n + 1)^2 - n^2 = 2n + 1$.

b) Determine n if $5^2 = 2n + 1$.

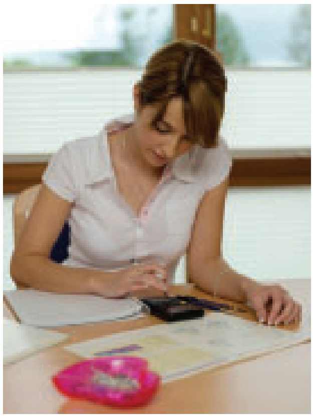
c) Use your answer from part (b) to determine a Pythagorean triple.

d) Determine three more Pythagorean triples.

2.2 Factoring Polynomials: Common Factoring

YOU WILL NEED

- graph paper



GOAL

Factor polynomials by dividing out the greatest common factor.

LEARN ABOUT the Math

Elvira squared the numbers 3, 4, and 5 and then added 1 to get a sum of 51.

$$\begin{aligned} 3^2 + 4^2 + 5^2 + 1 &= 9 + 16 + 25 + 1 \\ &= 51 \end{aligned}$$

She repeated this process with the numbers 8, 9, and 10 and got a sum of 246.

$$\begin{aligned} 8^2 + 9^2 + 10^2 + 1 &= 64 + 81 + 100 + 1 \\ &= 246 \end{aligned}$$

Both of her answers are divisible by 3.

$$\frac{51}{3} = 17 \quad \text{and} \quad \frac{246}{3} = 82$$

? Is one more than the sum of the squares of three consecutive integers always divisible by 3?

EXAMPLE 1 Selecting a strategy to determine the greatest common factor

Ariel's Solution: Using Algebra

$n^2 + (n + 1)^2 + (n + 2)^2 + 1$

I let the three consecutive integers be n, $n + 1$, and $n + 2$, where n represents the first integer. I then wrote an expression for the sum of their squares and added 1.

$$\begin{aligned} &= n^2 + (n + 1)(n + 1) + (n + 2)(n + 2) + 1 \\ &= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4) + 1 \\ &= 3n^2 + 6n + 6 \end{aligned}$$

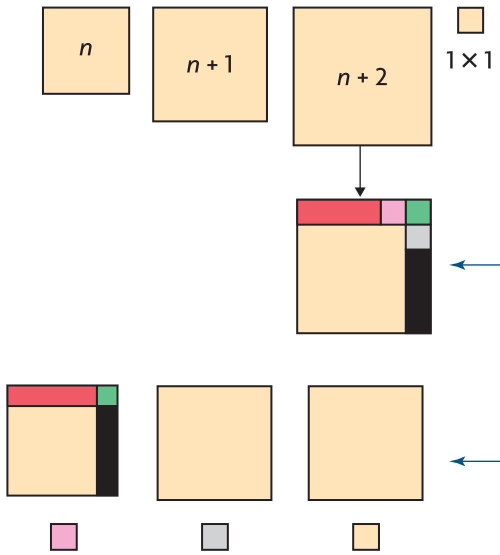
I simplified the resulting expression by squaring and collecting like terms.

$= 3(n^2 + 2n + 2)$

I factored the polynomial by finding the greatest common factor (GCF) of its terms: 3. I divided it out.

Three is a factor of the expression. This shows that one more than the sum of the squares of three consecutive integers is always divisible by 3.

David's Solution: Using Area Models



From construction paper, I made three squares, each 1 unit longer than the next.

These represent the squares of any three consecutive integers. I also made a 1 × 1 square.

I made straight cuts in the largest square to create five pieces: three equal squares and two rectangles.

I moved three of those pieces to the smallest square to make it as big as the middle square.

Altogether, I had three groups, each with one large square and one small square. The number represented by each group is $(n + 1)^2 + 1$.

Since there are three equal groups, the total value is a multiple of 3. This shows that one more than the sum of the squares of three consecutive integers is always divisible by 3.

Reflecting

A. Why did Ariel introduce a variable to solve this problem?

B. Why did Ariel want to write the expression with a factor of 3?

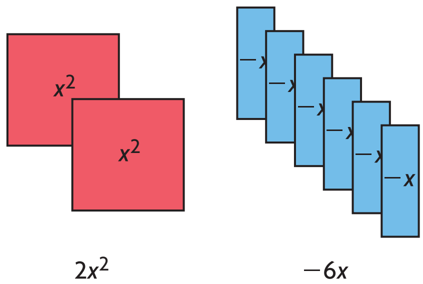
C. Ariel used the expression $3(n^2 + 2n + 2)$ to show that the sum was divisible by 3. David used $3[(n + 1)^2 + 1]$. Are the two expressions equivalent? Explain.

D. What are the advantages and disadvantages of each method of showing divisibility?

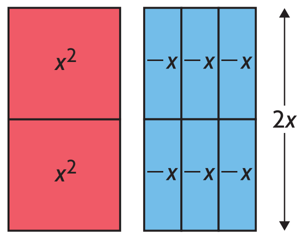
EXAMPLE 2 Selecting a strategy to represent the greatest common factor

Factor $2x^2 - 6x$.

Chloe's Solution: Using Algebra Tiles



I used algebra tiles to represent each term, red tiles for positives and blue tiles for negatives.



To get the greatest common factor of $2x^2$ and $-6x$, I rearranged the two terms separately into two rectangles with width $2x$ for each term. This is the greatest width I could use with the x^2 tiles.

The greatest common factor of the terms of $2x^2 - 6x$ is $2x$.

$2x^2 - 6x = 2x(x - 3)$

To find the other factor, I looked at my model to determine the length. It was $x - 3$.

Luis's Solution: Using Symbols

$2x^2 - 6x = 2(x^2) + 2(-3)x$

The greatest common factor of the coefficients 2 and 6 is 2.

$= 2x(x) + 2x(-3)$

The greatest common factor of the variable parts x^2 and x is x.

The greatest common factor of the terms of $2x^2 - 6x$ is $2x$.

$2x^2 - 6x = 2x(x - 3)$

I used the distributive property to write the product of the factors.

Sometimes algebraic expressions have a binomial as a common factor.

EXAMPLE 3 Using reasoning to factor

Factor $3x(x + 1) - 2(x + 1)$.

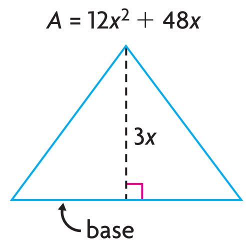
Ida's Solution

$$3x(x + 1) - 2(x + 1) = (3x - 2)(x + 1)$$

Both $3x$ and -2 are being multiplied by $x + 1$, so $x + 1$ is a common factor. I wrote an equivalent expression using the distributive property.

EXAMPLE 4 Solving a problem by factoring

A triangle has an area of $12x^2 + 48x$ and a height of $3x$. What is the length of its base?



Aaron's Solution

$$A = \frac{1}{2}bh$$

The formula for the area of a triangle is $A = \frac{1}{2}bh$.

$$12x^2 + 48x = \frac{1}{2}b \times 3x$$

I knew A and h; I had to solve for b.

$$3x(4x + 16) = \frac{1}{2}b \times 3x$$

I factored $3x$ out of the terms on the left side.

$$4x + 16 = \frac{1}{2}b$$

I divided both sides of the equation by $3x$. If $x = 0$, I can't carry out this operation, but if $x = 0$, then the triangle doesn't exist.

$$2(4x + 16) = \left(\frac{1}{\cancel{2}_1}b\right)\cancel{2}^1$$

I multiplied both sides of the equation by 2.

$$8x + 32 = b$$

The length of the base is $8x + 32$, where $x \neq 0$.

In Summary

Key Ideas

- Factoring algebraic expressions is the opposite of expanding. Expanding involves multiplying, while factoring involves looking for the expressions to multiply. For example:

expanding →

$$2x(3x - 5) = 6x^2 - 10x$$

← factoring

- One way to factor a polynomial is to look for the greatest common factor of its terms as one of its factors. For example, $6x^2 + 2x - 4$ can be factored as $2(3x^2 + x - 2)$, since 2 is the greatest common factor of each term.

Need to Know

- It is possible to factor a polynomial by dividing by a common factor that is not the greatest common factor. This will result in another polynomial that still has a common factor. For example:

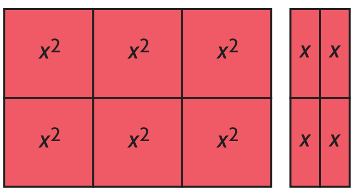
$$\begin{aligned} 4x + 8 &= 2(2x + 4) \\ &= 2(2)(x + 2) \\ &= 4(x + 2) \end{aligned}$$

- A polynomial is factored fully when only 1 or -1 is a common factor of every term.
- A common factor can have any number of terms. For example, a common factor of $8x^2 + 6x$ is $2x$, a monomial. But a common factor of $(2x - 1)^2 - 4(2x - 1)$ is $(2x - 1)$, a binomial.

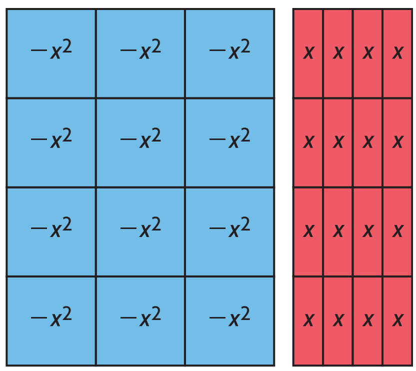
CHECK Your Understanding

1. In each diagram, two terms of a polynomial have been rearranged to show their common factor. For each, identify the terms of the polynomial and the common factor.

a)



b)



2. Name the common factor of the terms of the polynomial.

a) $3x^2 - 9x + 12$ **b)** $5x^2 + 3x$

3. Factor, using the greatest common factor.

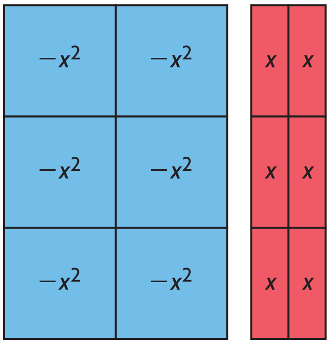
a) $4x^2 - 6x + 2$ **c)** $5a(a + 7) + 2(a + 7)$

b) $5x^2 - 20x$ **d)** $4m(3m - 2) - (3m - 2)$

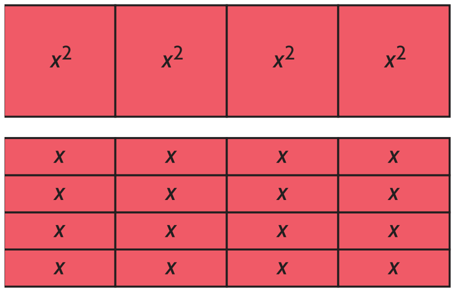
PRACTISING

4. In each diagram, two terms of a polynomial have been rearranged to show their common factor. For each, identify the terms of the polynomial and the common factor.

a)



b)



5. Name the greatest common factor of each polynomial.

a) $6x^2 + 12x - 18$ **c)** $16x^2 - 8x + 10$

b) $4x^2 + 14x$ **d)** $-15x^2 - 10$

6. Factor.

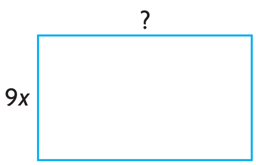
K **a)** $27x^2 - 9x$ **d)** $-2a^2 - 4a + 6$

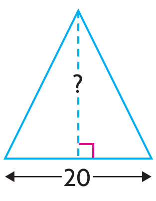
b) $-8m^2 + 20m$ **e)** $3x(x + 7) - 2(x + 7)$

c) $10x^2 - 5x + 25$ **f)** $x(3x - 2) + (3x - 2)(x + 1)$

7. The area, A, of each figure is given. Determine the unknown measurement.

a) $A = 18x^2 - 9x$ **b)** $A = 10m^2 - 20m + 20$





8. The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.
A A cylinder has a height of 10 units and a radius of r units. Determine a factored expression for its total surface area.

9. Show that $(3a - 2)$ is a common factor of $2a(3a - 2) + 7(2 - 3a)$.

10. a) Give three examples of a quadratic binomial with greatest common factor $6x$, and then factor each one.
b) Give three examples of a quadratic trinomial with greatest common factor 7, and then factor each one.

11. Colin says that the greatest common factor of $-8x^2 + 4x - 6$ is 2, but Colleen says that it is -2. Explain why both answers could be considered correct.

12. **T** For what values of k is it possible to divide out a common factor from $6x^2 + kx - 12$, but not from $6x^2 + kx + 4$? Explain.

13. It seems that the sum of the squares of two consecutive even or odd integers is always even. For example:

$$4^2 + 6^2 = 16 + 36$$
$$= 52$$
$$7^2 + 9^2 = 49 + 81$$
$$= 130$$

Let n represent the first integer.
a) What expression represents the second integer?
b) What expression represents the sum of the two squares?
c) Use algebra to show that the result is always even.

14. **C** How does knowing how to determine the greatest common factor of two numbers help you factor polynomials?

Extending

15. Factor.
a) $5x^2y - 10xy^2$
b) $10a^2b^3 + 20a^2b - 15a^2b^2$
c) $3x(x + y) - y(x + y)$
d) $5y(x - 2) - 7(2 - x)$

16. A factor might be common to only some terms of a polynomial, but grouping these terms sometimes allows the polynomial to be factored. For example:

$$ax - ay + bx - by$$
$$= a(x - y) + b(x - y)$$
$$= (x - y)(a + b)$$

Now you see the common factor $(x - y)$.

Factor these expressions by grouping.
a) $9xa + 3xb + 6a + 2b$
b) $10x^2 - 5x - 6xy + 3y$
c) $(x + y)^2 + x + y$
d) $1 + xy + x + y$

2.3 Factoring Quadratic Expressions: $x^2 + bx + c$

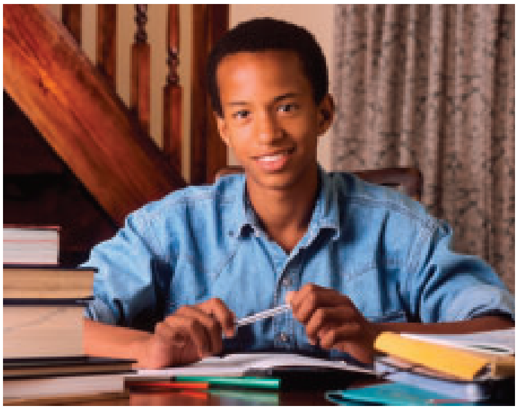
GOAL

Factor quadratic expressions of the form $ax^2 + bx + c$, where $a = 1$.

YOU WILL NEED

- algebra tiles

INVESTIGATE the Math



Seth claims that for any whole number n, the function $f(n) = n^2 + 8n + 15$ always produces a number that has factors other than 1 and itself.

He tested several examples:

$f(1) = 24 = 4 \times 6$, so $f(1)$ is not prime.

$f(2) = 35 = 5 \times 7$, so $f(2)$ is not prime.

$f(3) = 48 = 6 \times 8$, so $f(3)$ is not prime.

$f(4) = 63 = 7 \times 9$, so $f(4)$ is not prime.

? Is Seth's claim true?

A. Identify a pattern relating the value of n to the two factors of $f(n)$.

B. Use this pattern to factor $f(n)$; verify Seth's claim by expanding.

C. Arrange an x^2 tile, 8 x tiles, and 15 unit tiles to form a rectangle.

D. What are the dimensions of the rectangle?

E. Is Seth's claim true? Explain.

Reflecting

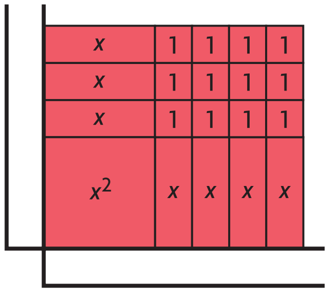
F. What are the advantages and disadvantages of the two methods of factoring (using patterns in parts A and B, and using algebra tiles in parts C and D)?

APPLY the Math

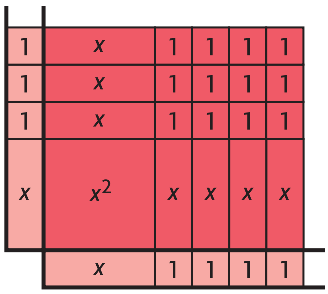
EXAMPLE 1 Factoring using algebra tiles

Factor $x^2 + 7x + 12$.

Andy's Solution



I arranged 1 x^2 tile, 7 x tiles, and 12 unit tiles in a rectangle.



I placed x tiles and unit tiles to make it easier to see the length and width.

The dimensions of the rectangle are $x + 3$ and $x + 4$.

The sum $3 + 4$ (7) determines the number of x tiles, and the product 3×4 (12) determines the number of unit tiles.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

EXAMPLE 2 Factoring by using the sum-and-product method

Factor $x^2 + 4x - 5$.

Yusef's Solution

$x^2 + 4x - 5 = (x\ ?)(x\ ?)$

The two factors for the quadratic could be binomials that start with *x*.

I needed two numbers whose sum is 4 and whose product is −5.

$= (x + ?)(x - ?)$

I started with the product. Since −5 is negative, one of the numbers must be negative.

Since the sum is positive, the positive number must be farther from zero than the negative one.

$= (x + 5)(x - 1)$

The numbers are 5 and −1.

Check:

I checked by multiplying.

$$(x + 5)(x - 1) = x^2 - 1x + 5x - 5$$
$$= x^2 + 4x - 5$$

EXAMPLE 3 Factoring quadratic expressions

Factor each expression.

a) $x^2 + 7x + 12$ **b)** $a^2 - 10a + 21$ **c)** $x^2 + 4x + 5$

Ryan's Solution

a) $x^2 + 7x + 12$

$= (x + 4)(x + 3)$

I needed two numbers whose sum is 7 and whose product is 12.

The product is positive, so either both numbers are negative or both are positive.

Since the sum is positive, both numbers must be positive.

The numbers are 4 and 3.

b) $a^2 - 10a + 21$

$= (a - 3)(a - 7)$

I needed two numbers whose sum is -10 and whose product is 21.

The product is positive, so either both numbers are negative or both are positive.

Since the sum is negative, both numbers are negative.

The numbers are -3 and -7.

c) $x^2 + 4x + 5$

Cannot be factored

I needed two numbers whose sum is 4 and whose product is 5.

The product is positive, so either both numbers are negative or both are positive.

Since the sum is positive, both numbers are positive.

There are no such numbers, because the only factors of 5 are 1 and 5. But their sum is 6, not 4.

EXAMPLE 4 Factoring quadratic expressions by first using a common factor

Factor $4x^2 + 16x - 48$.

Larry's Solution

$4x^2 + 16x - 48$

First, I divided out the greatest common factor, 4, since all terms are divisible by 4.

$= 4(x^2 + 4x - 12)$

$= 4(x - 2)(x + 6)$

To factor the trinomial, I needed two numbers whose sum is 4 and whose product is -12.

The numbers are -2 and 6.

In Summary

Key Idea

- If they can be factored, quadratic expressions of the form $x^2 + bx + c$ can be factored into two binomials $(x + r)(x + s)$, where $r + s = b$ and $r \times s = c$, and r and s are integers.

Need to Know

- To factor $x^2 + bx + c$ as $(x + r)(x + s)$, the signs in the trinomial can help you determine the signs of the numbers you are looking for:

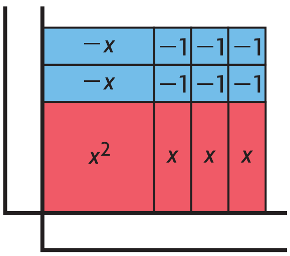
Trinomial	Factors
$x^2 + bx + c$	$(x + r)(x + s)$
$x^2 - bx + c$	$(x - r)(x - s)$
$x^2 - bx - c$	$(x - r)(x + s)$, where $r > s$
$x^2 + bx - c$	$(x + r)(x - s)$, where $r > s$

- To factor $x^2 + bx + c$ by using algebra tiles, form a rectangle from the tiles. The factors are given by the dimensions of the rectangle.
- It is easier to factor any polynomial expression if you factor out the greatest common factor first.

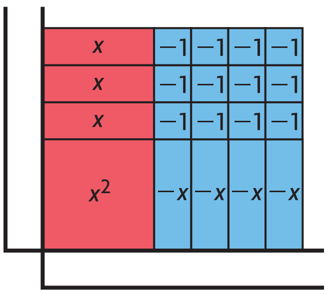
CHECK Your Understanding

1. The tiles in each diagram represent a polynomial. Identify the polynomial and its factors.

a)



b)



2. One factor is provided and one is missing. What is the missing factor?

a) $x^2 + 11x + 30 = (x + 6)(?)$

b) $x^2 - 3x - 28 = (x - 7)(?)$

c) $x^2 + 3x - 40 = (?)(x - 5)$

d) $x^2 - 8x - 20 = (?)(x - 10)$

3. Factor.

a) $x^2 + 9x + 20$

b) $a^2 - 11a + 30$

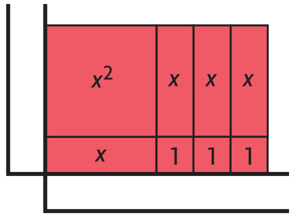
c) $m^2 + m - 6$

d) $2n^2 - 4n - 70$

PRACTISING

4. The tiles in each diagram represent a polynomial. Identify the polynomial and its factors.

a)



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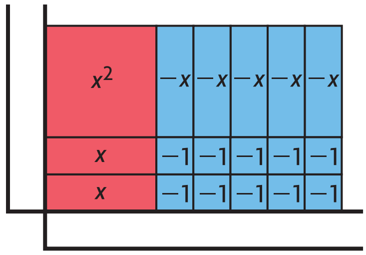


5. One factor is provided and one is missing. What is the missing factor?

a) $x^2 + 10x + 24 = (x + 4)(?)$
b) $x^2 - 13x + 42 = (x - 6)(?)$
c) $x^2 - 3x - 40 = (?)(x - 8)$
d) $x^2 + 6x - 27 = (?)(x + 9)$

6. Factor.

a) $x^2 - 7x + 10$
b) $y^2 + 6y - 55$
c) $x^2 - 3x - 10$
d) $w^2 - 3w - 18$
e) $x^2 - 14x + 33$
f) $n^2 - n - 90$

7. Write three different quadratic trinomials that have $(x + 5)$ as a factor.

8. Tony factored $x^2 - 7x + 10$ as $(x - 2)(x - 5)$. Fred factored it as $(2 - x)(5 - x)$. Why are both answers correct?

9. Factor.

K **a)** $a^2 + 7a + 10$
b) $-3x^2 - 27x - 54$
c) $z^2 - 10z + 25$
d) $x^2 + 11x - 60$
e) $x^2 + 3x - 10$
f) $y^2 + 13y + 42$

10. Explain why the function $f(n) = n^2 - 2n - 3$ results in a prime number when $n = 4$, but not when n is any integer greater than 4.

11. How can writing $x^2 - 16$ as $x^2 + 0x - 16$ help you factor it?

12. Choose a pair of integers for b and c that will make each statement true.

T **a)** $x^2 + bx + c$ can be factored, but $x^2 + cx + b$ cannot.
b) Both $x^2 + bx + c$ and $x^2 + cx + b$ can be factored.
c) Neither $x^2 + bx + c$ nor $x^2 + cx + b$ can be factored.

13. For what values of k can the polynomial be factored? Explain.

C **a)** $x^2 + kx + 4$
b) $x^2 + 4x + k$
c) $x^2 + kx + k$

Extending

14. Factor.

a) $x^2 + 3xy - 10y^2$
b) $a^2 + 4ab + 3b^2$
c) $-5m^2 + 15mn - 10n^2$
d) $(x + y)^2 - 5(x + y) + 6$

15. Factor $x + 7 + \frac{12}{x}$.

2 Mid-Chapter Review

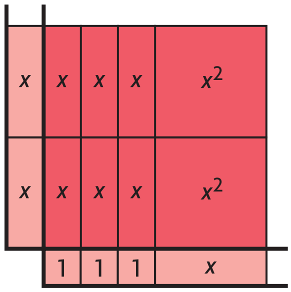
FREQUENTLY ASKED Questions

Q: How do you simplify quadratic expressions?

A: You can simplify by expanding and collecting like terms. For example, $2x(3 + x) + 4x + 2$ can be simplified by first expanding using the distributive property:

$$2x(3 + x) = 2x \times 3 + 2x \times x$$
$$= 6x + 2x^2$$

Another way to expand is to determine the area of a rectangle with length and width based on the factors. For example, $2x(3 + x)$ is the area of the rectangle shown at the right.



After expanding, collect like terms.

$$2x(3 + x) + 4x + 2 = 6x + 2x^2 + 4x + 2$$
$$= 2x^2 + 10x + 2$$

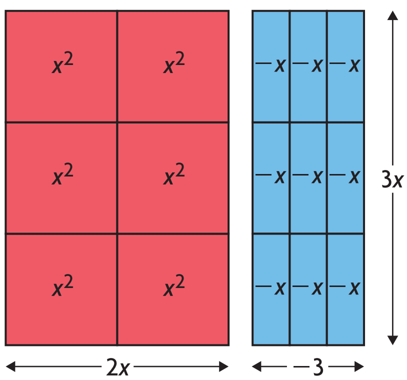
Study *Aid*

- See Lesson 2.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 to 4.

Q: How do you use the greatest common factor of the terms of a polynomial to factor it?

A1: You can represent the terms with algebra tiles and rearrange them into rectangles with the same and greatest possible width. That width is the greatest common factor.

For example, the greatest common factor of the terms of $6x^2 - 9x$ is $3x$, since each term can be rearranged into rectangles with width $3x$, which is the largest possible width.



Once you divide out the common factor, the remaining terms represent the other dimension of each rectangle.

$$6x^2 - 9x = 3x(2x - 3)$$

Study *Aid*

- See Lesson 2.2, Examples 1 to 4.
- Try Mid-Chapter Review Questions 5, 6, and 7.

A2: You can determine the greatest common factor of the coefficients and of the variables, and then multiply them together.

For example, for $6x^2 - 9x$, the GCF of 6 and 9 is 3. The GCF of x^2 and x is x. GCF of $6x^2 - 9x$ is $3x$.

Then you can divide out the common factor:

$$6x^2 - 9x = 3x(2x - 3)$$

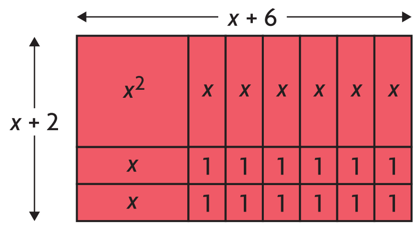
Study *Aid*

- See Lesson 2.3, Examples 1 to 4.
- Try Mid-Chapter Review Questions 8 to 11.

Q: How can you factor quadratic expressions of the form $x^2 + bx + cx$?

A1: You can form a rectangle using algebra tiles. The length and width are the factors.

For example, to factor $x^2 + 8x + 12$:



$$x^2 + 8x + 12 = (x + 6)(x + 2)$$

A2: Look for two numbers with a sum of b and a product of c and use them to factor.

For example, to factor $x^2 + 3x - 18$:

- Numbers whose product is -18 are 6 and -3, 9 and -2, 1 and -18, -6 and 3, -9 and 2, and -1 and 18.
- The only pair that adds to $+3$ is 6 and -3.

The factors are $(x + 6)(x - 3)$.

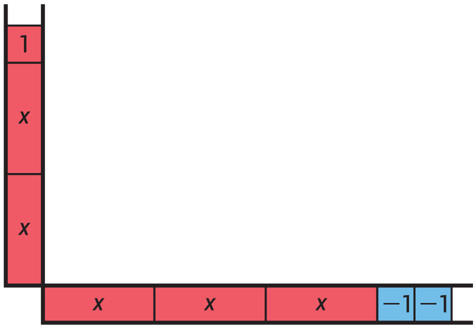
PRACTICE Questions

Lesson 2.1

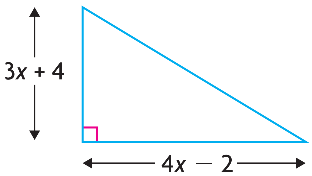
1. Expand and simplify.

a) $2x(x - 6) - 3(2x - 5)$
b) $(3n - 2)^2 + (3n + 2)^2$
c) $3x(2x - 1) - 4x(3x + 2) - (-x^2 + 4x)$
d) $-2(3a + b)(3a - b)$

2. The diagram below represents a polynomial multiplication. Which two polynomials are being multiplied and what is the product?



3. Write a simplified expression to represent the area of the triangle shown.



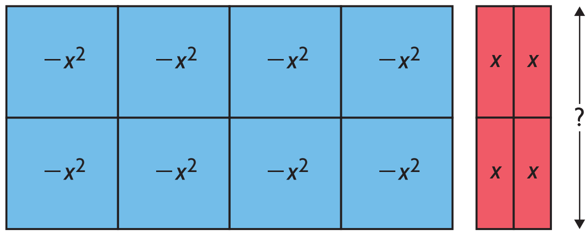
4. A rectangle has dimensions $2x + 1$ and $3x - 2$, where $x > 0$. Determine the increase in its area if each dimension is increased by 1.

Lesson 2.2

5. Factor.

a) $-8x^2 + 4x$
b) $3x^2 - 6x + 9$
c) $5m^2 - 10m - 5$
d) $3x(2x - 1) + 5(2x - 1)$

6. The tiles shown represent the terms of a polynomial. Identify the polynomial and the common factor of its terms.

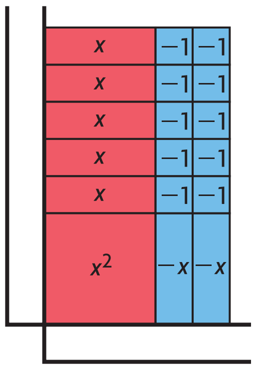


7. Consider the binomials $2x + 4$ and $3x + 6$. The greatest common factor of the first pair of terms is 2 and of the second pair is 3.

a) Determine the product of the polynomials.
b) Is the greatest common factor of the terms of their product equal to the product of 2 and 3?
c) Why might you have expected the answer you got in part (b)?

Lesson 2.3

8. The tiles shown represent a polynomial. Identify the polynomial and its factors.



9. Factor.

a) $x^2 + 2x - 15$
b) $n^2 - 8n + 12$
c) $x^2 - 12x + 35$
d) $2a^2 - 2a - 24$

10. How do you know that $(x - 4)$ can't be a factor of $x^2 - 18x + 6$?

11. If $x^2 + bx + c$ can be factored, then can $x^2 - bx + c$ be factored? Explain.

2.4 Factoring Quadratic Expressions: $ax^2 + bx + c$

GOAL

Factor quadratic expressions of the form $ax^2 + bx + c$, where $a \neq 1$.

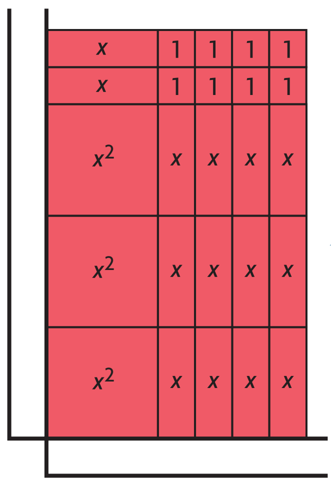
Martina is asked to factor the expression $3x^2 + 14x + 8$. She is unsure of what to do because the first term of the expression has the coefficient 3, and she hasn't worked with these kinds of polynomials before.

? How do you factor $3x^2 + 14x + 8$?

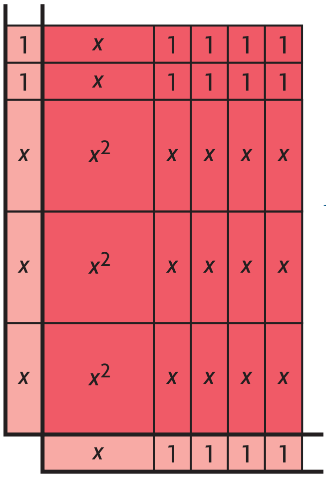
LEARN ABOUT the Math

EXAMPLE 1 Selecting a strategy to factor a trinomial where $a \neq 1$

Llewelyn's Solution: Using Algebra Tiles



I used 3 x^2 tiles, 14 x tiles, and 8 unit tiles to create a rectangle area model of the trinomial.



I placed the tiles along the length and width of the rectangle to read off the factors. The length was $3x + 2$ and the width was $x + 4$.

$$3x^2 + 14x + 8 = (3x + 2)(x + 4)$$

Albert's Solution: Using Guess and Check

$$3x^2 + 14x + 8 = (x + ?)(3x + ?)$$

To get $3x^2$, I had to multiply x by $3x$, so I set up the equation to show the factors.

$$(x + 8)(3x + 1) = 3x^2 + 25x + 8$$

WRONG FACTORS

$$(x + 1)(3x + 8) = 3x^2 + 11x + 8$$

WRONG FACTORS

$$(x + 4)(3x + 2) = 3x^2 + 14x + 8$$

WORKED!

$$3x^2 + 14x + 8 = (x + 4)(3x + 2)$$

I had to multiply two numbers together to get 8.
The ways to get a product of 8 are 8×1, 4×2, $-8 \times (-1)$, and $-4 \times (-2)$. Since the middle term was positive, I tried the positive values only.
I used guess and check to see which pair of numbers worked.

decomposing

breaking a number or expression into parts that make it up

Murray's Solution: Using Decomposition

$$3x^2 + 14x + 8 = 3x^2 + 12x + 2x + 8$$

I rewrote the original expression by **decomposing** $14x$ into $12x + 2x$.

$$\begin{aligned} 3x^2 &+ 12x + 2x + 8 \\ &= (3x^2 + 12x) + (2x + 8) \\ &= 3x(x + 4) + 2(x + 4) \\ &= (x + 4)(3x + 2) \end{aligned}$$

I divided out the greatest common factors from the first two terms and then from the last two terms.

I factored out the binomial common factor.

Reflecting

A. Why did Llewlyn try to form a rectangle with the algebra tiles?

B. What was Murray's goal when he "decomposed" the coefficient of the x-term?

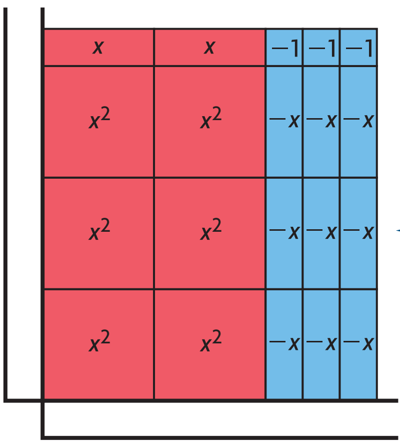
C. How was Albert's factoring method similar to the sum-and-product method?

D. What are the advantages and disadvantages of each of the three factoring methods?

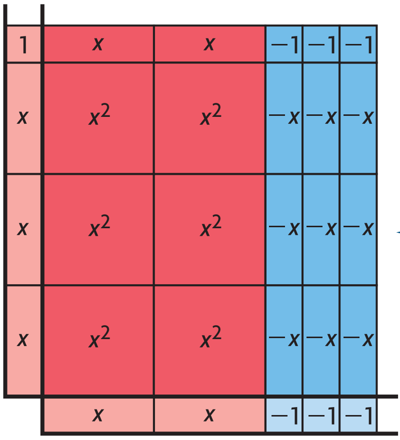
EXAMPLE 2 Factoring a trinomial by using algebra tiles

Use algebra tiles to factor $6x^2 - 7x - 3$.

Ariel's Solution



I used 6 x^2 tiles, 9 $-x$ tiles, 2 x tiles, and 3 negative unit tiles to build a rectangle area model of the trinomial.



I placed tiles along the length and width of the rectangle to read off the factors.

$6x^2 - 7x - 3 = (3x + 1)(2x - 3)$

EXAMPLE 3 Factoring a trinomial by decomposition

Factor $4x^2 - 3x - 1$.

Florian's Solution

$$\begin{aligned} &4x^2 - 3x - 1 \\ &= 4x^2 - 4x + x - 1 \end{aligned}$$

I found two numbers whose sum is −3 and whose product is $(4)(-1) = -4$.

The numbers are −4 and 1. I used them to "decompose" the middle term.

$$\begin{aligned} &= 4x(x - 1) + 1(x - 1) \\ &= (x - 1)(4x + 1) \end{aligned}$$

I factored the first two terms and then the last two terms. Next, I divided out the common factor $x - 1$.

Check:

I checked by multiplying.

$$\begin{aligned} (x - 1)(4x + 1) &= 4x^2 + 1x - 4x - 1 \\ &= 4x^2 - 3x - 1 \end{aligned}$$

EXAMPLE 4 Factoring a trinomial by guess and check

Factor $3x^2 - 5x + 2$.

Nadia's Solution

$3x^2 - 5x + 2$

$(3x - 1)(x - 2) = 3x^2 - 7x + 2$

WRONG FACTORS

$(3x - 2)(x - 1) = 3x^2 - 5x + 2$

WORKED

$$3x^2 - 5x + 2 = (3x - 2)(x - 1)$$

The factors of $3x^2$ are $3x$ and x.

The factors of 2 are 2 and 1 or -2 and -1.

I tried the negative values, since the middle term was negative.

I used guess and check to place the values in each set of brackets.

In Summary

Key Idea

- If the quadratic expression $ax^2 + bx + c$, where $a \neq 1$ can be factored, then the factors are of the form $(px + r)(qx + s)$, where $pq = a$, $rs = c$, and $ps + rq = b$.

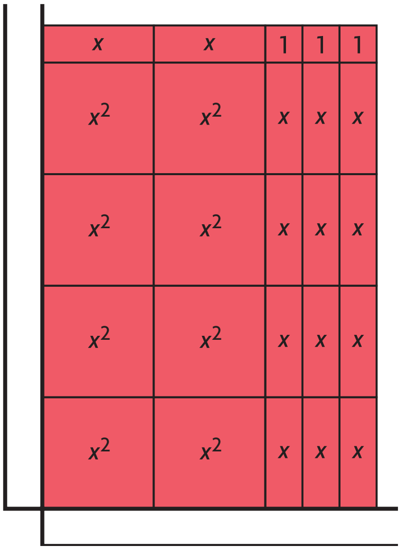
Need to Know

- If the quadratic expression $ax^2 + bx + c$, where $a \neq 1$ can be factored, then the factors can be found by a variety of strategies, such as
  - forming a rectangle with algebra tiles
  - decomposition
  - guess and check
- A trinomial of the form $ax^2 + bx + c$ can be factored if two integers can be found whose product is ac and whose sum is b.
- The decomposition method involves decomposing b into a sum of two numbers whose product is ac.

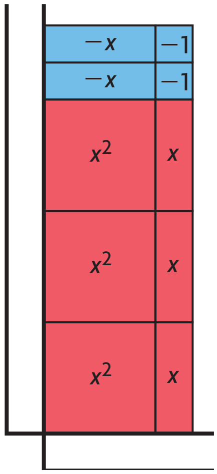
CHECK *Your Understanding*

1. Each diagram represents a polynomial. Identify the polynomial and its factors.

a)



b)



2. State the missing factor.

a) $6a^2 + 5a - 4 = (3a + 4)(?)$

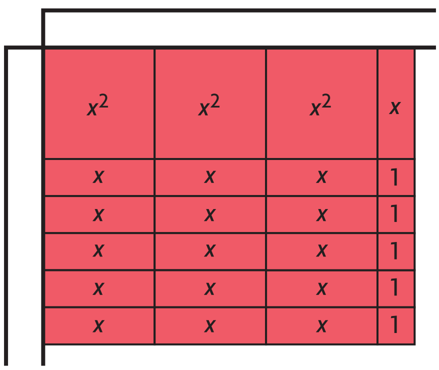
b) $5x^2 - 22x + 8 = (?)(x - 4)$

c) $3x^2 + 7x + 2 = (3x + 1)(?)$

d) $4n^2 + 8n - 60 = (?)(n + 5)$

PRACTISING

3. The diagram below represents a polynomial. Identify the polynomial and its factors.



4. Factor. You may first need to determine a common factor.
   a) $2x^2 - 7x - 4$
   b) $3x^2 + 18x + 15$
   c) $5x^2 + 17x + 6$
   d) $2x^2 + 10x + 8$
   e) $3x^2 + 12x - 63$
   f) $2x^2 - 15x + 7$

5. Factor.
   a) $8x^2 + 10x + 3$
   b) $6m^2 - 3m - 3$
   c) $2a^2 - 11a + 12$
   d) $15x^2 - 4x - 4$
   e) $6n^2 + 26n - 20$
   f) $16x^2 + 4x - 6$

6. Write three different quadratic trinomials that have $(2x - 5)$ as a factor.

7. For each expression, name an integer, k, such that the quadratic trinomial can be factored.
   a) $kx^2 + 4x + 1$
   b) $4x^2 + kx - 10$
   c) $8x^2 - 14x + k$

8. Can the guess and check method for factoring trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, be applied when $a = 1$? Explain.

9. Factor.
   K
   a) $6x^2 - x - 12$
   b) $8k^2 + 43k + 15$
   c) $30r^2 - 85r - 70$
   d) $12n^3 - 75n^2 + 108n$
   e) $3k^2 - 6k - 24$
   f) $24y^2 - 10y - 25$

10. Factor.
   a) $x^2 + 5x + 6$
   b) $x^2 - 36$
   c) $5a^2 - 13a - 6$
   d) $a^2 - a - 12$
   e) $4x^2 + 16x - 48$
   f) $6x^2 + 7x - 3$

11. Is there an integer, n, such that $6n^2 + 10n + 4$ is divisible by 50?
   T
   Explain.

12. How does knowing that factoring is the opposite of expanding help
   C
   you factor a polynomial such as $-4x^2 + 38x - 48$?

Extending

13. Factor.
   a) $6x^2 + 11xy + 3y^2$
   b) $5a^2 - 7ab - 6b^2$
   c) $8x^2 - 14xy + 3y^2$
   d) $12a^2 + 52a - 40$

14. Can $ax^2 + bx + c$ be factored if a, b, and c are odd? Explain.

2.5 Factoring Quadratic Expressions: Special Cases

GOAL

Factor perfect-square trinomials and differences of squares.

LEARN ABOUT the Math

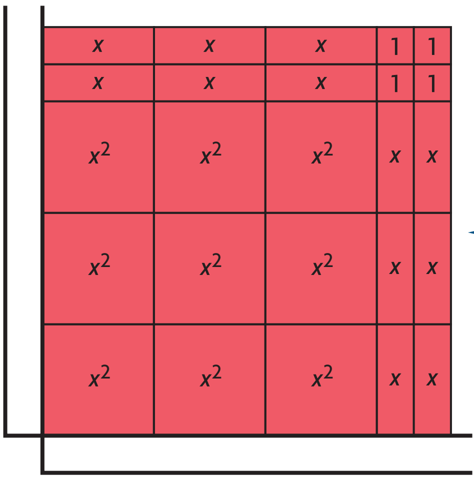
The area of a square is given by $A(x) = 9x^2 + 12x + 4$, where x is a natural number.

? What are the dimensions of the square?

EXAMPLE 1 Factoring a perfect-square trinomial

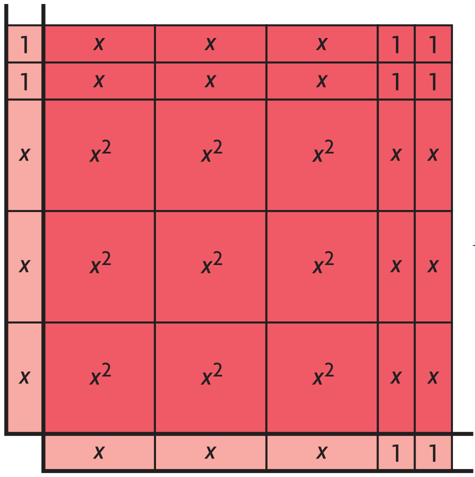
Factor $9x^2 + 12x + 4$.

Fred's Solution: Using Algebra Tiles



I used 9 x^2 tiles, 12 x tiles, and 4 unit tiles to create an area model of the trinomial.

The only arrangement of tiles that seemed to work was a square.



I lined up tiles along two edges to determine the length (or width) of the square.

Each edge is $(3x + 2)$ long.

$9x^2 + 12x + 4 = (3x + 2)^2$

Haroun's Solution: Using a Pattern

$A(x) = 9x^2 + 12x + 4$

$9 = 3^2$

$4 = 2^2$

9 and 4 are both perfect squares.

Maybe $9x^2 + 12x + 4 = (3x + 2)^2$.

I tested to see if the polynomial was a perfect square.

$$\begin{aligned}(3x + 2)^2 &= (3x + 2)(3x + 2)\\ &= 9x^2 + 6x + 6x + 4\\ &= 9x^2 + 12x + 4\end{aligned}$$

WORKED!

This worked, since I got the correct coefficient of *x*.

$9x^2 + 12x + 4 = (3x + 2)^2$

Reflecting

A. Why is $9x^2 + 6x + 4$ not a perfect square?

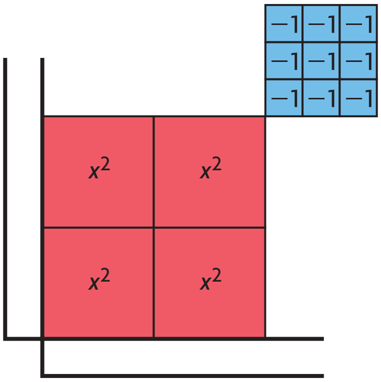
B. How can you recognize a trinomial that might be a perfect square? Explain.

APPLY the Math

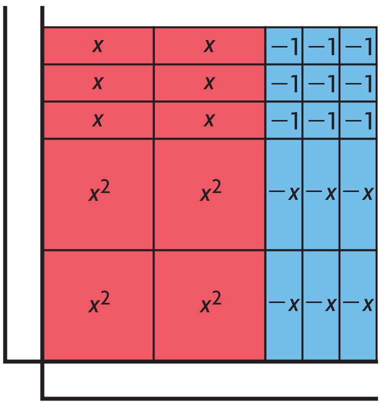
EXAMPLE 2 | Factoring a difference of squares by forming a square

Factor $4x^2 - 9$.

Lisa's Solution

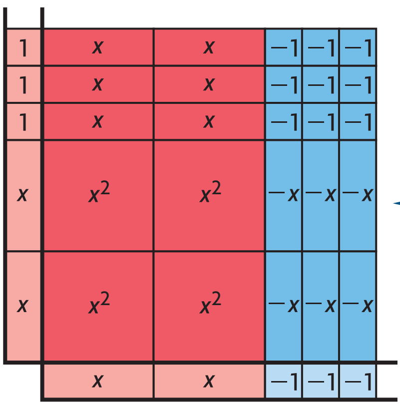


I started with 4 x^2 tiles and 9 negative unit tiles. I made a square out of each term. When using algebra tiles to factor, the ones tiles are always diagonal from the x^2 tiles, so I put them that way.



The rectangle was not complete, so I added 6 horizontal $-x$ tiles and 6 vertical x tiles. This is adding zero to the polynomial, because $6x + (-6x) = 0x$.

The rectangle turned out to be a square.



The resulting dimensions, $2x - 3$ and $2x + 3$, are the factors of $4x^2 - 9$.

$$4x^2 - 9 = (2x - 3)(2x + 3)$$

EXAMPLE 3 Factoring a perfect-square trinomial by decomposition

Factor $16x^2 + 24x + 9$.

Tolbert's Solution

$16 \times 9 = 144$

$24 = 12 + 12$ ← I needed to decompose 24, the coefficient of x, as the sum of two numbers whose product is 144.

and $12 \times 12 = 144$

$16x^2 + 12x + 12x + 9$ ← I wrote the x-term in decomposed form.

$= 4x(4x + 3) + 3(4x + 3)$ ← I divided out the GCF from the first two terms and the GCF from the last two terms.

$= (4x + 3)(4x + 3)$ ← Then I divided out the binomial common factor.

$= (4x + 3)^2$

Check: ← I checked by multiplying.

$(4x + 3)^2 = (4x + 3)(4x + 3)$

$= 16x^2 + 12x + 12x + 9$

$= 16x^2 + 24x + 9$

$16x^2 + 24x + 9 = (4x + 3)^2$

In Summary

Key Ideas

- A polynomial of the form $a^2x^2 \pm 2abx + b^2$ is a perfect-square trinomial and can be factored as $(ax \pm b)^2$.
- A polynomial of the form $a^2x^2 - b^2$ is a difference of squares and can be factored as $(ax - b)(ax + b)$.

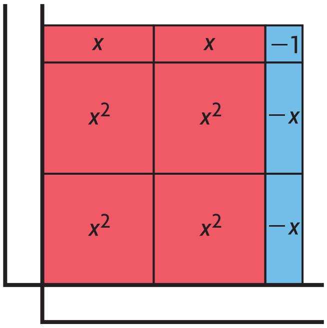
Need to Know

- A perfect-square trinomial and a difference of squares can be factored by
  - forming a square using algebra tiles
  - decomposition
  - guess and check or sum and product

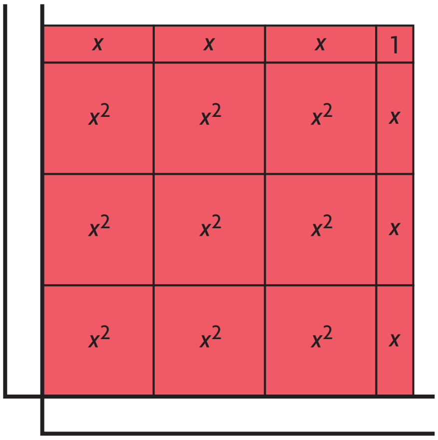
CHECK Your Understanding

1. Each diagram represents a polynomial. Identify the polynomial and its factors.

a)



b)



2. State the missing factor.

a) $x^2 - 25 = (x + 5)(?)$
b) $n^2 + 8n + 16 = (?)(n + 4)$
c) $25a^2 - 36 = (?)(5a - 6)$
d) $28x^2 - 7 = (?)(2x - 1)(2x + 1)$
e) $4m^2 - 12m + 9 = (?)^2$
f) $18x^2 + 12x + 2 = 2(?)^2$

PRACTISING

3. Factor.

a) $x^2 - 36$
b) $x^2 + 10x + 25$
c) $x^2 - 64$
d) $x^2 - 24x + 144$
e) $x^2 - 100$
f) $x^2 + 4x + 4$

4. Factor, if possible.

a) $49a^2 + 42a + 9$
b) $x^2 - 121$
c) $-8x^2 + 24x - 18$
d) $20a^2 - 180$
e) $16 - 36x^2$
f) $(x + 1)^2 + 4(x + 1) + 4$

5. **A** Some shortcuts in mental arithmetic are based on factoring. For example, $21^2 - 19^2$ can be easily calculated mentally with the difference-of-squares method.

$$\begin{aligned} 21^2 - 19^2 &= (21 - 19)(21 + 19) \\ &= 2(40) = 80 \end{aligned}$$

Calculate mentally.

a) $52^2 - 48^2$

b) $34^2 - 24^2$

6. **K** Explain how you know that $5x^2 + 20x + 9$ cannot be factored in the indicated way.

a) as $(ax + b)^2$

b) as $(ax + b)(ax - b)$

7. Factor $x^4 - 13x^2 + 36$.

8. **T** Determine all integers, m and n, such that $m^2 - n^2 = 24$.

9. Fred claims that the difference between the squares of any two consecutive odd numbers is 4 times their median. For example,

$$\begin{aligned} 9^2 - 7^2 &= 81 - 49 \\ &= 32 = 4(8) \\ 15^2 - 13^2 &= 225 - 169 \\ &= 56 = 4(14) \end{aligned}$$

Use variables to explain why Fred is correct.

10. **C** Explain how you would recognize a polynomial as a perfect-square trinomial or as a difference of squares. Then explain how you would factor each one.

Extending

11. Factor.

a) $100x^2 - 9y^2$

b) $4x^2 + 4xy + y^2$

c) $(2x - y)^2 - 9$

d) $90x^2 - 120xy + 40y^2$

12. The method of grouping can sometimes be applied to factor polynomials that contain a perfect-square trinomial. For example:

$x^2 + 6x + 9 - y^2$ ← The first three terms form a perfect-square trinomial, which is factored.

$= (x + 3)^2 - y^2$

$= (x + 3 - y)(x + 3 + y)$ ← This produces a difference of squares, which is factored to complete the factoring.

Factor.

a) $4x^2 - 20xy + 25y^2 - 4z^2$

b) $81 - x^2 + 14x - 49$

Curious Math

Factoring Using Number Patterns

Many people are more comfortable working with numbers than algebraic expressions. Number patterns can be used to help factor algebraic expressions.

For example, to factor $4x^2 - 1$, you can substitute numbers for x in a systematic way and try to identify a pattern.

$$
\begin{aligned}
\text{Let}\quad & x = 1 \rightarrow 4(1)^2 - 1 = 3 && \rightarrow 1 \times 3 \\
& x = 2 \rightarrow 4(2)^2 - 1 = 15 && \rightarrow 3 \times 5 \\
& x = 3 \rightarrow 4(3)^2 - 1 = 35 && \rightarrow 5 \times 7 \\
& x = 4 \rightarrow 4(4)^2 - 1 = 63 && \rightarrow 7 \times 9 \\
& x = 5 \rightarrow 4(5)^2 - 1 = 99 && \rightarrow 9 \times 11
\end{aligned}
$$

One possible pattern you might observe is that the factors are two apart from each other. Also, the factors are 1 greater and 1 less than double the number that was substituted. So, $(2x + 1)(2x - 1)$ are the factors.

Can you see any other patterns that might help you factor this expression?

Here is another example. To factor $9x^2 + 6x + 1$:

$$
\begin{aligned}
\text{Let}\quad & x = 1 \rightarrow 9(1)^2 + 6(1) + 1 = 9 + 6 + 1 = 16 && \rightarrow 4 \times 4 \\
& x = 2 \rightarrow 9(2)^2 + 6(2) + 1 = 36 + 12 + 1 = 49 && \rightarrow 7 \times 7 \\
& x = 3 \rightarrow 9(3)^2 + 6(3) + 1 = 81 + 18 + 1 = 100 && \rightarrow 10 \times 10 \\
& x = 4 \rightarrow 9(4)^2 + 6(4) + 1 = 144 + 24 + 1 = 169 && \rightarrow 13 \times 13 \\
& x = 5 \rightarrow 9(5)^2 + 6(5) + 1 = 225 + 30 + 1 = 256 && \rightarrow 16 \times 16
\end{aligned}
$$

Here the pattern shows the factors are identical, so this must be a perfect square. Each factor is 1 greater than 3 times the number substituted. So, $(3x + 1)(3x + 1)$ are the factors.

1. Use number patterns to factor each expression.

a) $x^2 + 3x + 2$

b) $2x^2 + x - 2$

c) $4x^2 + 4x + 1$

d) $4x^2 + 10x + 6$

e) $9x^2 - 1$

f) $x^2 + 10x + 25$

2 Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

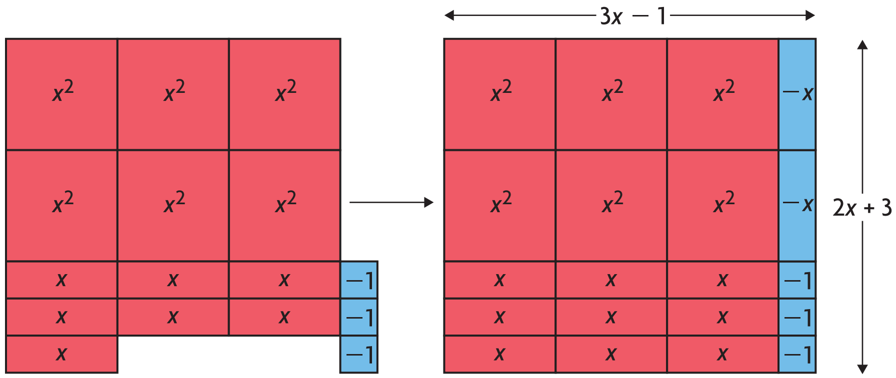
- See Lesson 2.4, Examples 1 to 4.
- Try Chapter Review Questions 11, 12, and 13.

Q: How can you factor quadratic expressions of the form $ax^2 + bx + c$, where $a \neq 1$?

A: Some polynomials of this form can be factored, but others cannot. Try to factor the expression by using one of the methods below. If none of them works, the polynomial cannot be factored.

Method 1:

Arrange algebra tiles to form a rectangle, and read off the length and width as the factors. For example, for $6x^2 + 7x - 3$, the rectangle wasn't filled, so add $+2x + (-2x)$ to make it work.



$6x^2 + 7x - 3 = (3x - 1)(2x + 3)$

Method 2:

Use guess and check.

If there are factors, the factored expression looks like (__ x + __)(__ x − __), since the constant is -3.

The coefficients of x must multiply to 6, so they could be 6 and 1, -6 and -1, 3 and 2, or -3 and -2.

The constants must multiply to -3. They could be 1 and -3, or -3 and 1.

Try different combinations until one works.

$$6x^2 + 7x - 3 = (3x - 1)(2x + 3)$$

Method 3:

Use decomposition. Decompose $+7$ as the sum of two numbers that multiply to $-3 \times 6 = -18$. Use -2 and 9.

$$\begin{aligned} 6x^2 + 7x - 3 &= 6x^2 - 2x + 9x - 3 \\ &= 2x(3x - 1) + 3(3x - 1) \\ &= (3x - 1)(2x + 3) \end{aligned}$$

Q: How do you recognize a perfect-square trinomial and how do you factor it?

A: If the coefficient of x^2 is a perfect square and the constant is a perfect square, test to see if the middle coefficient is twice the product of the two square roots.

For example, $16x^2 + 40x + 25$ is a perfect square, since $16 = 4^2$, $25 = 5^2$, and $40 = 2 \times 4 \times 5$. So, $16x^2 + 40x + 25 = (4x + 5)^2$.

Study *Aid*

- See Lesson 2.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 14 to 19.

Q: How do you recognize a difference of squares and how do you factor it?

A: If a polynomial is made up of two perfect-square terms that are subtracted, it can be factored.

For example, $36x^2 - 49$ is a difference of squares, since $36x^2 = (6x)^2$ and $49 = 7^2$.

So, $36x^2 - 49 = (6x + 7)(6x - 7)$.

PRACTICE Questions

Lesson 2.1

1. Expand and simplify.

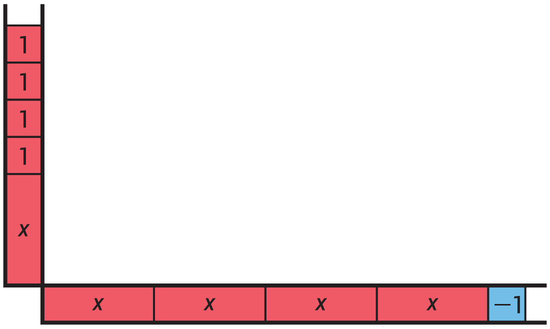
a) $3x(2x - 3) + 9(x - 1) - x(-x - 11)$

b) $-9(4a - 5)(4a + 5)$

c) $2(x^2 - 5) - 7x(8x - 9)$

d) $-5(2n - 5)^2$

2. Which two polynomials are being multiplied and what is the product?



Lesson 2.2

3. Determine the missing factor.

a) $7x^2 - 14x = 7x(?)$

b) $3a^2 + 15a - 9 = (?)(a^2 + 5a - 3)$

c) $10b^4 - 20b^2 = 10b^2(?)$

d) $4x(x - 3) + 5(x - 3) = (?)(x - 3)$

4. Factor.

a) $10x^2 - 5x$

b) $12n^2 - 24n + 48$

c) $-2x^2 - 6x + 8$

d) $3a(5 - 7a) - 2(7a - 5)$

5. A rectangle has an area of $6x^2 - 8$.

a) Determine the dimensions of the rectangle.

b) Is there more than one possibility? Explain.

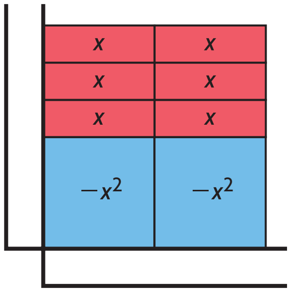
6. a) Give three examples of polynomials that have a greatest common factor of $7x$.

b) Factor each polynomial from part (a).

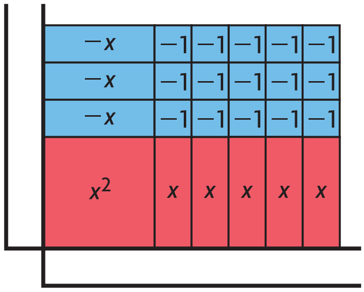
Lesson 2.3

7. What are the factors of each polynomial being modelled?

a)



b)



8. Determine the missing factor.

a) $x^2 + 9x + 14 = (x + 2)(?)$

b) $a^2 + 3a - 28 = (?)(a + 7)$

c) $b^2 - b - 20 = (b - 5)(?)$

d) $-8x + x^2 + 15 = (?)(x - 3)$

9. Factor.

a) $x^2 + 7x + 10$

b) $x^2 - 12x + 27$

c) $x^2 + x - 42$

d) $x^2 - x - 90$

10. Determine consecutive integers b and c, and also m and n, such that

$$x^2 + bx + c = (x + m)(x + n)$$

Lesson 2.4

11. How would you decompose the x-term to factor each polynomial?

a) $6x^2 + x - 1$

b) $12x^2 + 9x - 30$

c) $7x^2 - 50x - 48$

d) $30x^2 - 9x - 3$

12. Determine the missing factor.

a) $2x^2 + 7x + 5 = (x + 1)(?)$
b) $3a^2 + 10a - 8 = (?)(3a - 2)$
c) $4b^2 - 4b - 15 = (2b - 5)(?)$
d) $20 + 27x + 9x^2 = (?)(3x + 5)$

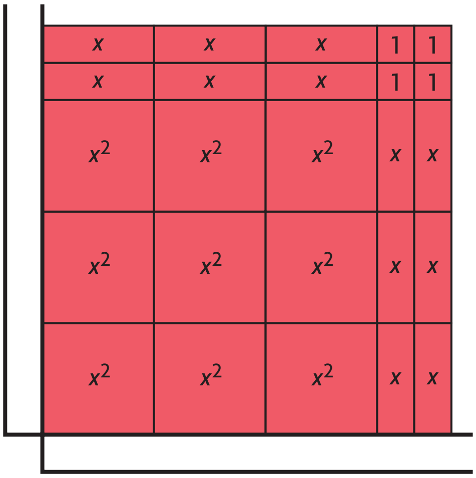
13. Factor.

a) $6x^2 - 19x + 10$
b) $10a^2 - 11a - 6$
c) $20x^2 + 9x - 18$
d) $6n^2 + 13n + 7$

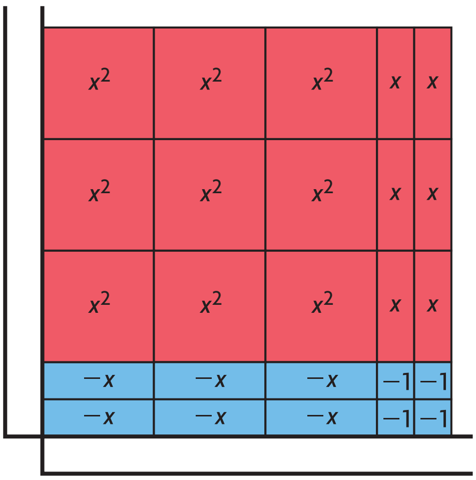
Lesson 2.5

14. Each diagram represents a polynomial. Identify the polynomial and its factors.

a)



b)



15. Determine the missing factor.

a) $x^2 - 25 = (x + 5)(?)$
b) $9a^2 + 6a + 1 = (?)(3a + 1)$
c) $4b^2 - 20b + 25 = (2b - 5)(?)$
d) $9x^2 - 64 = (?)(3x + 8)$

16. Factor.

a) $4x^2 - 9$
b) $16a^2 - 24a + 9$
c) $x^8 - 256$
d) $(x - 2)^2 + 6(x - 2) + 9$

17. The polynomial $x^2 - 1$ can be factored. Can the polynomial $x^2 + 1$ be factored? Explain.

18. Factor each expression. Remember to divide out all common factors first.

a) $x^2 + 2x - 15$
b) $5m^2 + 15m - 20$
c) $2x^2 - 18$
d) $18x^2 + 15x - 3$
e) $36x^2 + 48x + 16$
f) $15c^3 + 25c^2$

19. How is factoring a polynomial related to expanding a polynomial? Use an example in your explanation.

Chapter Self-Test

1. Expand and simplify.

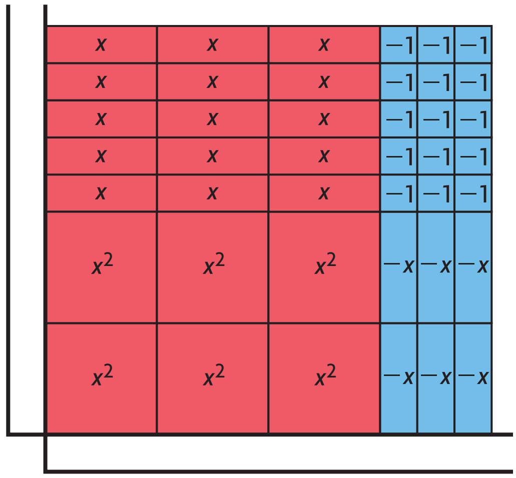
a) $-2x(3x - 4) - x(x + 6)$

b) $-3(5n - 4)^2 - 5(5n + 4)^2$

c) $-8(x^2 - 5x + 7) + 5(2x - 5)(3x - 7)$

d) $-3(5a - 4)(5a + 4) - 3a(a - 7)$

2. What two binomials are being multiplied and what is the product?



3. A rectangle has a width of $2x - 3$ and a length of $3x + 1$.

a) Write its area as a simplified polynomial.

b) Write expressions for the dimensions if the width is doubled and the length is increased by 2.

c) Write the new area as a simplified polynomial.

4. Use pictures and words to show how to factor $-2x^2 + 8x$.

5. Factor.

a) $x^2 + x - 12$

b) $a^2 + 16a + 63$

c) $-5x^2 + 75x - 280$

d) $y^2 + 3y - 54$

6. Factor.

a) $2x^2 - 9x - 5$

b) $12n^2 - 67n + 16$

c) $6x^2 - 15x + 6$

d) $8a^2 - 14a - 15$

7. What dimensions can a rectangle with an area of $12x^2 - 3x - 15$ have?

8. State all the integers, m, such that $x^2 + mx - 13$ can be factored.

9. Factor.

a) $121x^2 - 25$

b) $36a^2 - 60a + 25$

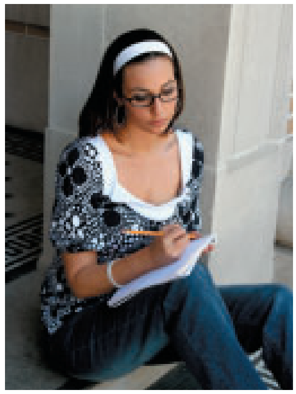
c) $x^4 - 81$

d) $(3 - n)^2 - 12(3 - n) + 36$

10. Determine all integers, m and n, such that $m^2 - n^2 = 45$.

2 Chapter Task

The Algebra Challenge



Part 1:

Make each statement true by replacing each $\square$ with one of these digits:

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Use each of the 18 digits once.

$$(3x + 2)^2 = 9x^2 + \square\,\square\, x + 4$$
$$\square x^2 - \square = (2x + \square)(2x - 3)$$
$$\square x^2 + 14x - 2\square = \square(x^2 + 2x - 3)$$
$$(\square x + \square)(\square x + \square) = 40x^2 + 93x + \square 4$$
$$\square(\square x^2 - \square) = \square x^2 - \square$$

Part 2:

Make up a puzzle of your own, like the one you just solved, for a partner to solve.

Task Checklist

- ✔ Did you use each of the 18 digits listed once?
- ✔ Did you verify that each statement is correct?
- ✔ How do you know the puzzle you created works?