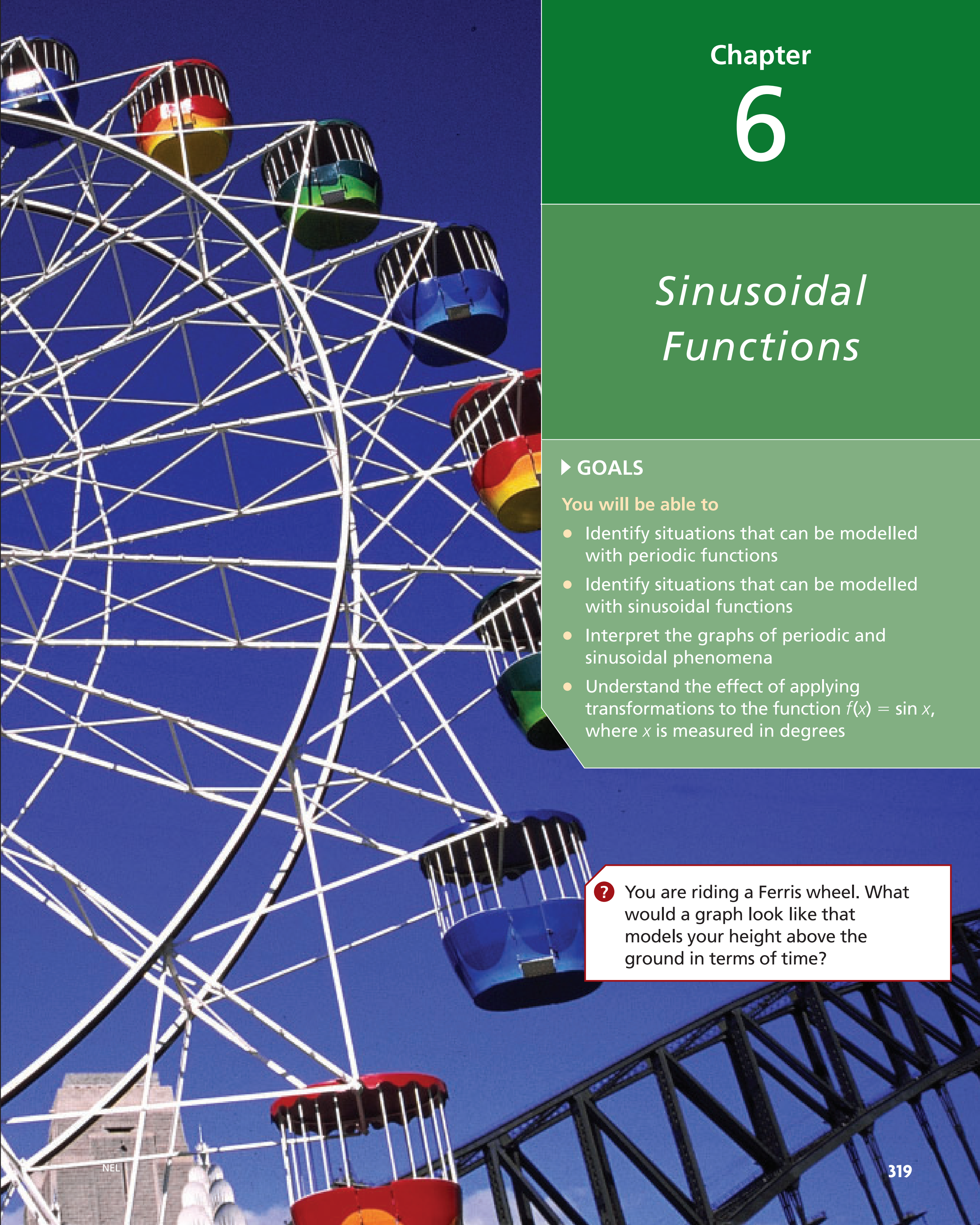


Chapter 6

Sinusoidal Functions

GOALS

You will be able to

- Identify situations that can be modelled with periodic functions
- Identify situations that can be modelled with sinusoidal functions
- Interpret the graphs of periodic and sinusoidal phenomena
- Understand the effect of applying transformations to the function $f(x) = \sin x$, where x is measured in degrees

? You are riding a Ferris wheel. What would a graph look like that models your height above the ground in terms of time?

6 Getting Started

YOU WILL NEED

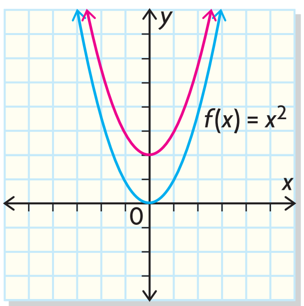
- graph paper
- calculator

WORDS You Need to Know

1. Match each word with the picture or example that best illustrates its definition. The blue curve is the original function $f(x) = x^2$.

a) vertical translation
b) vertical stretch
c) horizontal translation
d) vertical compression
e) minimum value
f) reflection

i)



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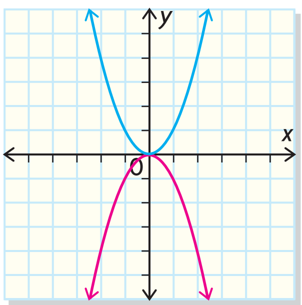

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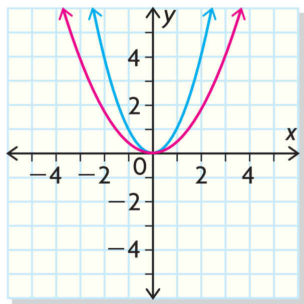

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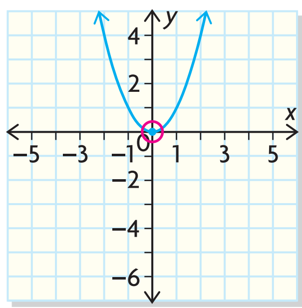

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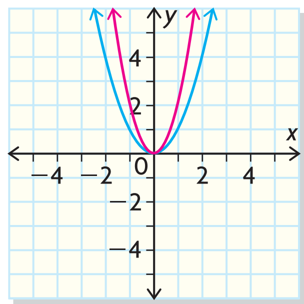

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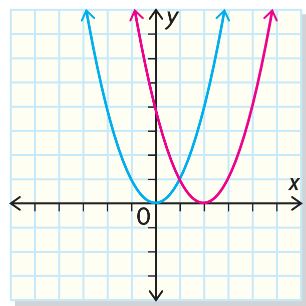


SKILLS AND CONCEPTS You Need

Study Aid

For help, see Essential Skills Appendix, A-14.

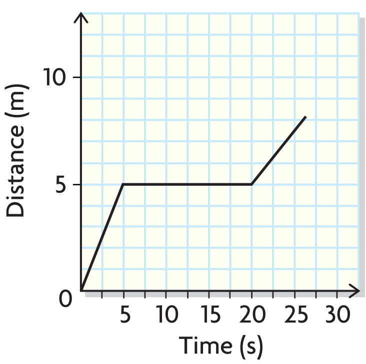
Using a Graphical Model to Predict: Interpolating and Extrapolating

Interpolate: to estimate a value that is between elements of the given data
Extrapolate: to estimate a value that is beyond the range of the given data by following a pattern

EXAMPLE

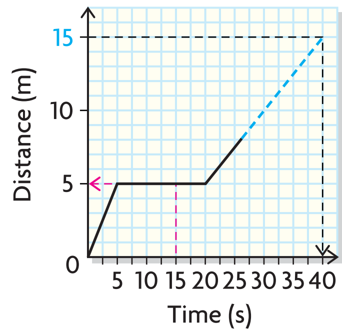
The graph represents a student using a motion sensor with a graphing calculator.

a) What is the distance of the student from the motion sensor at 15 s?
b) At what time is the student 15 m from the origin?

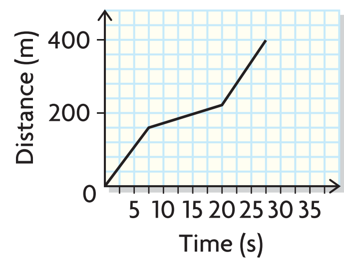


Solution

a) About 5 m
b) About 40 s



2. The graph at the bottom right represents the position of a grade 11 student participating in a 400 m race. Use the graph to determine the following.
   a) What is the distance of the sprinter from the starting line at 10 s?
   b) If the sprinter continues to run at a consistent rate, how long would she take to reach 500 m?
   c) Describe how the sprinter's speed changed during the course of the race.



Identifying Transformations

When a quadratic function is in vertex form, $y = a(x - h)^2 + k$, several properties of the graph of the relation are obvious:

- If $k > 0$, then the graph of $y = x^2$ is translated vertically up by k units. If $k < 0$, then the graph is translated down by k units.
- If $h > 0$, then the graph of $y = x^2$ is translated horizontally h units to the right. If $h < 0$, then the graph of $y = x^2$ is translated horizontally h units to the left.
- If $-1 < a < 1$, then the graph of $y = x^2$ is compressed vertically by a factor of a. The resulting graph is wider than $y = x^2$.
- If $a > 1$ or $a < -1$, then the graph of $y = x^2$ is stretched vertically by a factor of a. The resulting graph is narrower than $y = x^2$.
- If $a > 0$, then the curve opens upward.
- If $a < 0$, then the curve is reflected through the x-axis and opens downward.
- When you use **transformations** to graph $y = a(x - h)^2 + k$ from $y = x^2$, you can apply the transformations in the following order:
  1. translation left or right
  2. vertical stretch or compression and reflection about the x-axis
  3. translation up or down

EXAMPLE

Determine the transformations you would apply to $f(x) = x^2$ to graph $g(x) = -2(x - 1)^2 + 5$.

Solution

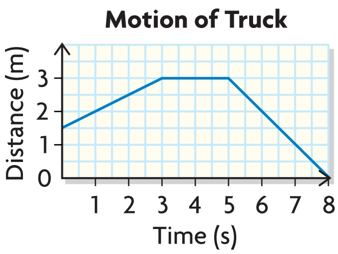
Move to the right 1 unit, reflect in the x-axis and stretch by a factor of 2, and move up 5 units.

3. Determine the transformations you would apply to $y = x^2$ to graph each of the following.

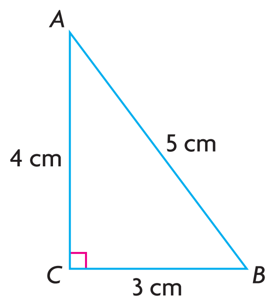
a) $y = 3(x - 5)^2 + 4$ c) $y = 0.5(x + 1)^2 - 3$

b) $y = 2(x - 2)^2 + 1$ d) $y = -\frac{1}{4}(x + 2)^2 - 4$

PRACTICE



4. Earl operates a remote-control truck in front of a motion detector. He moves the truck either toward or away from the detector, never right or left. The graph shows the distance between the truck and the motion detector at regular time intervals.
   a) Determine each value, including units, and explain what they represent in this situation:
      i) the slope of the line between $t = 0$ and $t = 3$
      ii) the distance intercept
      iii) the slope of the line between $t = 3$ and $t = 5$
      iv) the time intercept
   b) State the domain and range of this function.



5. Determine the values of sin A, cos A, and tan A in $\triangle ABC$ at the left.

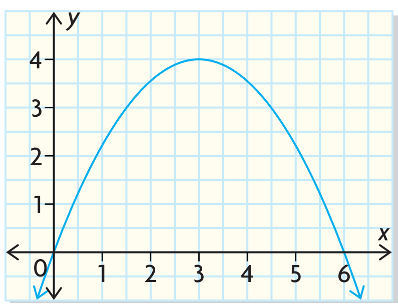
6. Use transformations of the graph of $f(x) = x^2$ to sketch the graphs of each of the following.

a) $y = f(x) + 2$ c) $y = f(x - 1)$

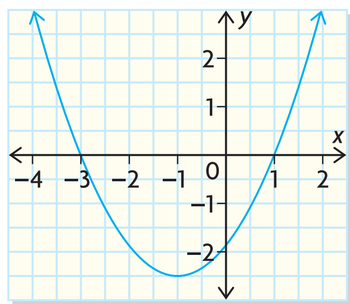
b) $y = 2f(x)$ d) $y = -f(x)$

7. For each function, determine the **maximum** or the **minimum** value, identify the **zeros**, and state the **domain** and **range**.

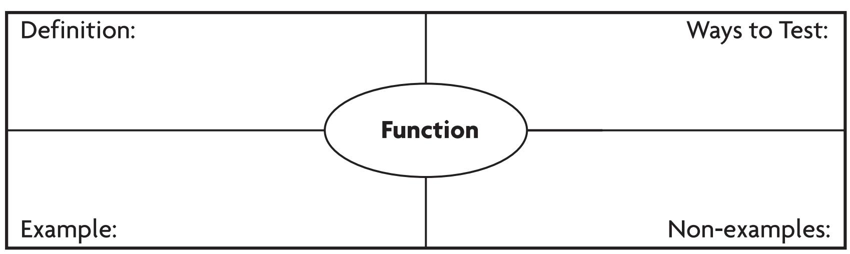
a)



b)



8. Complete a chart like the one below to show what you know about the term *function*.



APPLYING What You Know

YOU WILL NEED

- graph paper

Maximizing Profit

ToyMart sells diecast cars.

- Its current profit per car is 45¢.
- Each week ToyMart sells 2000 cars.
- If ToyMart drops the profit per car by 5¢, it can sell 400 more cars per week.

? How much profit should ToyMart make on each car to maximize its weekly profit?

A. Calculate ToyMart's total profit on current sales of 2000 cars per week.

B. If n is the number of times ToyMart drops the profit by 5¢ per car and $P(n)$ is the weekly profit, explain how the function $P(n) = (2000 + 400n)(45 - 5n)$ models this relationship.

C. Complete the table and graph the ordered pairs, using n as the independent variable and $P(n)$ as the dependent variable. What does each ordered pair represent?

n	0	1	2	3	4	5	6	7	8	9
***P*(*n*)**										

D. Calculate $P(10)$. Why should n stop at 9?

E. What is the domain of this function? What is the range?

F. What type of function is this? Explain how you know.

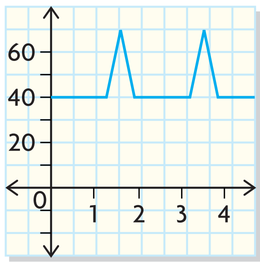
G. What profit is needed on each car to maximize the weekly profit?

6.1 Exploring Periodic Motion

YOU WILL NEED

- Calculator-Based Ranger (CBR)
- graphing calculator
- 20 cm × 25 cm piece of cardboard

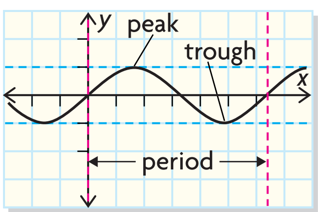
Jordan's Graph



peak
the highest point(s) on the graph

trough
the lowest point(s) on the graph

period
the interval of the independent variable (often time) needed for a repeating action to complete one cycle



cycle
a series of events that are regularly repeated; a complete set of changes, starting from one point and returning to the same point in the same way

GOAL

Explore graphs of motion with repeating patterns.

EXPLORE the Math

Jordan produced different graphs by moving a cardboard paddle in front of a CBR in different ways. Jordan's graph is shown at the left.

? How does the motion of the paddle influence the shape of the graph?

A. Move the paddle to create different linear graphs.

B. Describe how you moved the paddle to control the direction and steepness of the slope of the line.

C. Describe what happens to the graph when you do the following:
- Slowly move the paddle back.
- Hold the paddle steady for a few seconds.
- Move the paddle quickly.
- Move the paddle farther away.

D. Move the paddle in different ways to create three different types of graphs that repeat in regular ways. Describe how each graph was affected by the way you moved the paddle.

E. Move the paddle to reproduce Jordan's graph. Describe how you had to move the paddle to duplicate the peaks, troughs, and period of the cycle in Jordan's graph.

Reflecting

F. How do you move the paddle to control
a) the length of the period of a cycle?
b) the height of the peaks and the depth of the troughs?
c) the steepness of the line segments that form the peaks and troughs?
d) the number of peaks or troughs in a cycle?

G. How can you predict the shape of the graph based on your motions?

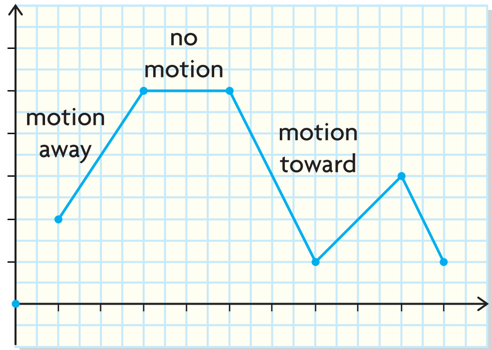
In Summary

Key Idea

- When you repeat a motion in exactly the same way, the graph of that motion is periodic. By looking at the graph, you can figure out details of the motion.

Need to Know

- A linear graph results when there is no motion or the motion is performed at a constant speed in only one direction.
- If the motion is toward the motion detector, the graph falls. If the motion is away, the graph rises.
- The peaks and troughs of the graph describe the maximum and minimum distances, respectively, from the CBR.
- The steepness of the line segments describes the speed with which the object is moving. The steeper the line, the faster the motion.



FURTHER Your Understanding

1. Sketch a graph for each motion pattern made with a paddle in front of a motion sensor.

a)
- Start 30 cm from the sensor.
- Move 60 cm away from your starting point so that you end up 90 cm from the sensor.
- Move back to the starting point.

Repeat five times in about 10 s.

b)
- Start 60 cm from the sensor.
- Move 45 cm from your starting point toward the sensor so that you end up 15 cm from the sensor.
- Move 30 cm from this point away from the sensor so that you end up 45 cm from the sensor.
- Move 30 cm from this point toward the sensor so that you end up 15 cm from the sensor.
- Move 45 cm from this point away from the sensor so that you end up 60 cm from the sensor.

Repeat three times in about 5 s.

2. Draw a graph to represent a different periodic motion of the paddle. Describe how the paddle would have to move to produce this graph.

6.2 Periodic Behaviour

YOU WILL NEED

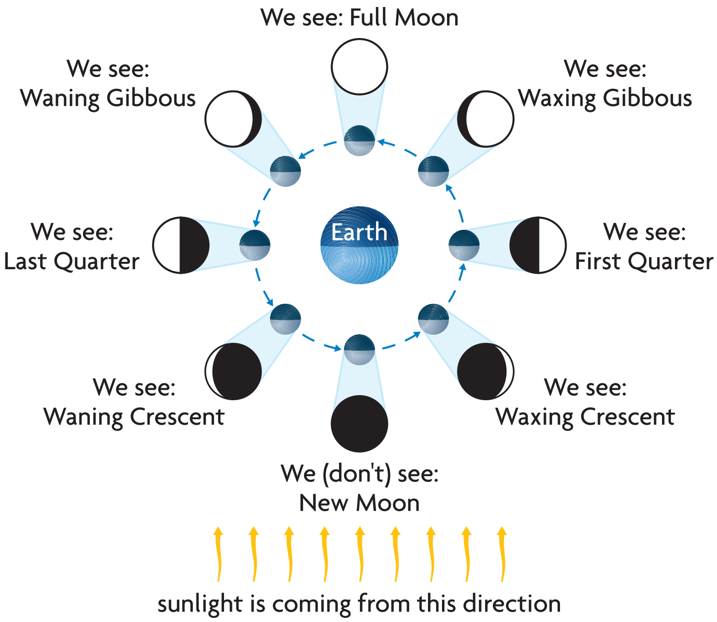
- graph paper

GOAL

Interpret graphs that repeat at regular intervals.

LEARN ABOUT the Math

The Sun always shines on half the Moon. How much of the Moon we see depends on where it is in its orbit around Earth.



The tables show the proportion of the Moon that was visible from Southern Ontario on days 1 to 74 in the year 2006.

Day of Year	1	4	7	10	14	20	24	29	34
Proportion of Moon Visible	0.02	0.22	0.55	0.83	1.00	0.73	0.34	0.00	0.28

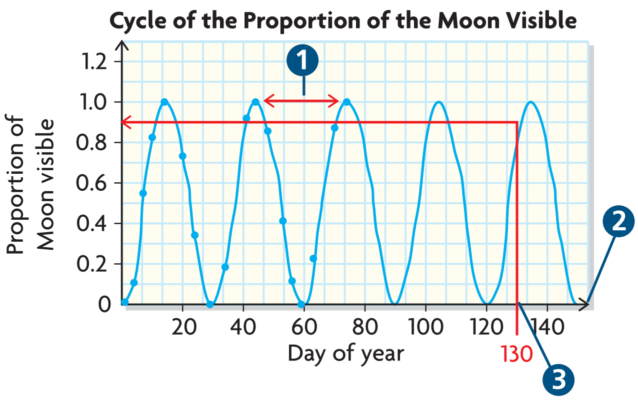
Day of Year	41	44	48	53	56	59	63	70	74
Proportion of Moon Visible	0.92	1.00	0.86	0.41	0.12	0.00	0.23	0.88	1.00

? What proportion of the Moon was visible on day 130?

EXAMPLE 1 Representing data as a periodic graph to make estimates

Create a graph using the Moon data, and use it to estimate what proportion of the Moon was visible on day 130.

David's Solution



I drew a scatter plot with Day of year as the independent variable and Proportion of Moon visible as the dependent variable. Then I interpreted the pattern.

1. This graph has a repeating pattern. Its period is 30 days. I can tell because the proportion returns to 0 at 30 days, and the next part of the curve looks the same as the previous part.
2. I used the repeating pattern to extend the graph to 150 days.
3. I used the graph to estimate the proportion of the Moon visible on day 130.

At day 130, about 0.9 of the Moon is visible.

Reflecting

A. Why does it make sense to call the graph of the Moon data a periodic function?

B. How does the table help you predict the period of the graph?

C. How does knowing the period of a graph help you predict future events?

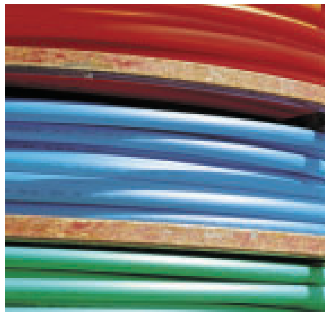
periodic function
a function whose values are repeated at equal intervals of the independent variable

APPLY the Math

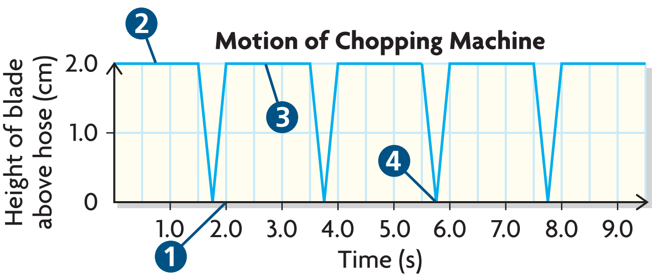
If you are given a graph of a periodic function, it can be used to help you understand how the process or situation repeats.

EXAMPLE 2 Using reasoning to interpret a periodic graph

Tanya's mother works at a factory that makes rubber hoses. A chopping machine cuts each hose to 5.0 m lengths. How can Tanya interpret the graph that shows the process?



Tanya's Solution



The period of this function is 2.0 s.
The maximum height of the blade is 2.0 cm.
The minimum height is 0 cm.
The blade stops for 1.5 s intervals.
The blade takes 0.5 s to go down and up.

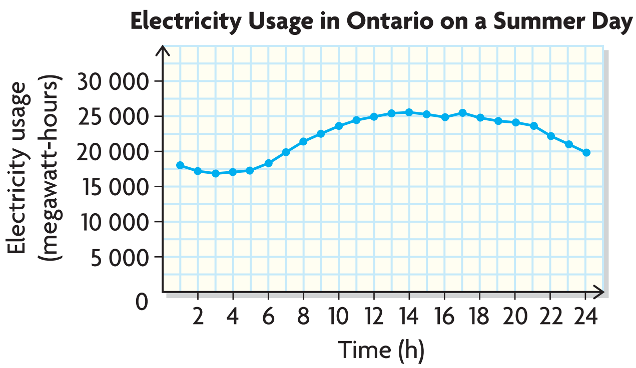
I studied the graph and identified patterns. Then I related the patterns on the graph to the process of the chopping machine.

1. The cutting blade cuts a new section of hose every 2.0 s since the graph has a pattern that repeats every 2.0 s.
2. The height is always 2.0 cm or less, so the blade can't be higher than 2.0 cm. When the height is 0 cm, the blade is hitting the cutting surface.
3. Flat sections like the one from 2.0 to 3.5 mean that the blade stops for these intervals. The machine is probably pulling the next 5.0 m section of hose through before it's cut.
4. Parts of the graph, like from $t = 5.5$ to $t = 5.75$, show that the blade takes 0.25 s to go down. The part from $t = 5.75$ to $t = 6.0$ shows that the blade takes 0.25 s to go up.

Some graphs may appear to be periodic. They are periodic only if the pattern repeats in exactly the same way over the same interval.

EXAMPLE 3 Connecting the features of a graph to the situation

The graph shows the demand for electricity in Ontario on a day in August 2002. Discuss and interpret the graph, and suggest possible reasons for its shape.



Enrique's Solution

The peak of the graph occurs at 14:00 h, or 2 p.m., and the trough of the graph occurs at 3:00 h, or 3 a.m.

Demand for electricity is highest at 2 p.m. This could be because many factories and industries are working to capacity at this time. The demand is lowest at 3 a.m. because most people are sleeping and not using electricity.

The graph increases between 3 a.m. and 2 p.m. and again between 4 p.m. and 5 p.m.
The graph decreases between 2 p.m. and 4 p.m. and again between 5 p.m. and 3 a.m.

As people get up to go to work, they use more electricity. Between 4 p.m. and 5 p.m., people arrive home from work so electricity usage rises. Between 2 p.m. and 4 p.m., people are travelling between home and work. Between 5 p.m. and 3 a.m., people use less electricity as they go to sleep.

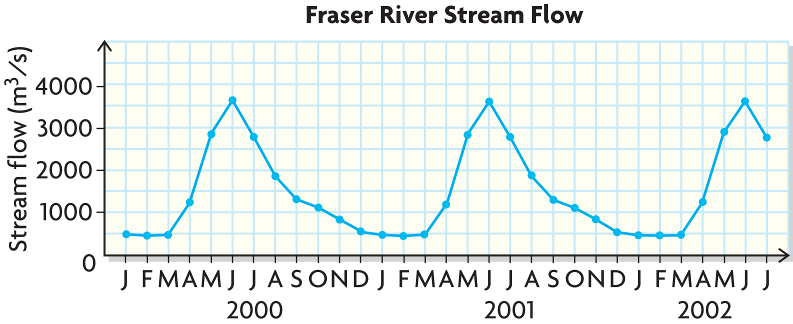
I don't think the graph will be periodic. It will have this same pattern over 24 h, but I think the peaks and troughs will be different each day.

The graph suggests that the pattern of electricity usage from day to day might repeat. But I can't be sure it repeats in exactly the same way each day. On hotter days, more people will use their air conditioners, making the demand higher than it shows for this day.

In Summary

Key Idea

- A function that produces a graph that has a regular repeating pattern over a constant interval is called a *periodic function*. It describes something that happens in a cycle, repeating in the same way over and over.



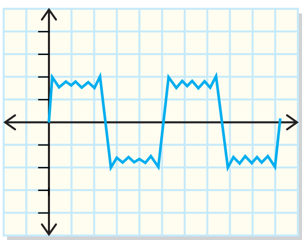
Need to Know

- A function that produces a graph that does not have a regular repeating pattern over a constant interval is called a *nonperiodic function*.
- Extending the graph of a periodic function by using the repeating pattern allows you to make reasonable predictions by extrapolating.
- The graph of a periodic function allows you to figure out information about the repeating pattern it represents.

CHECK Your Understanding

1. Which graphs represent periodic functions? Justify your decision.

a)



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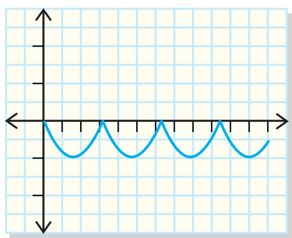
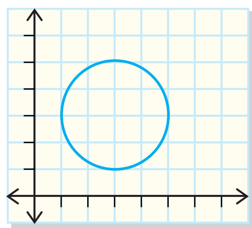


c)



2. According to the table on page 326, there was a new Moon on day 59. When will the next two new Moons occur?

3. Use the data from the table on page 326. In the first 74 days of 2006, how many times will 50% of the Moon be visible in the clear night sky?

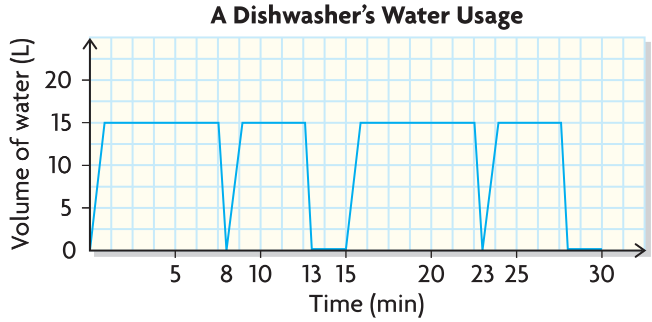
PRACTISING

4. A coach measures the velocity of air as a gymnast inhales and exhales after working out.
K

a) Plot the data from the table and draw a curve.
b) State the period of this function.
c) Explain why some velocities are positive and others are negative.
d) Extend the graph for the period between 9 s and 19 s.
e) Predict when the velocity of air will be 0 m/s for the period between 9 s and 19 s.

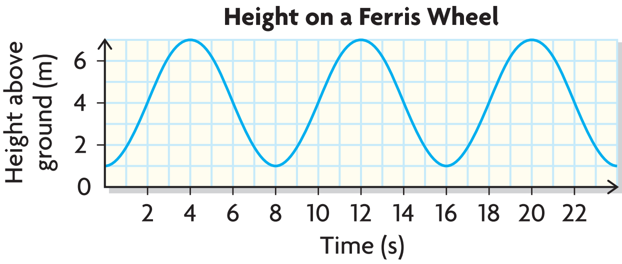
Time (s)	Velocity of Air (L/s)
1.0	1.75
1.5	1.24
2.0	0
2.5	−1.24
3.0	−1.75
3.5	−1.24
4.0	0
4.5	1.24
5.0	1.75
5.5	1.24
6.0	0
6.5	−1.24
7.0	−1.75
7.5	−1.24
8.0	0
8.5	1.24
9.0	1.75

5. The graph shows the amount of water used by an automatic dishwasher as a function of time.



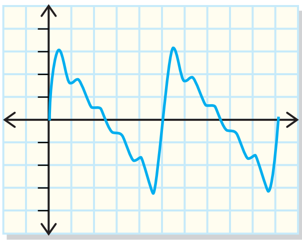
a) Why does the operation of the dishwasher model a periodic function?
b) What is the period? What does one complete cycle mean?
c) Extend the graph for one more complete cycle.
d) How much water is used if the dishwasher runs through eight complete cycles?
e) For part (d), state the domain and range of the function.

6. This is a graph of Nali's height above the ground in terms of time while riding a Ferris wheel.
A

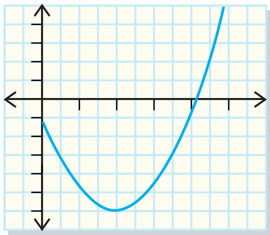


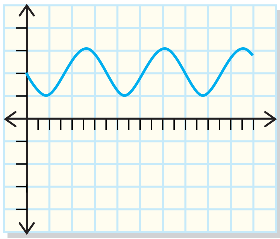
a) What is the period of this function?
b) What does the period represent?
c) What is the diameter of the Ferris wheel? How do you know?
d) Approximately how high above the ground is Nali at 10 s?
e) At what times is Nali at the top of the wheel?
f) When is Nali 4 m above the ground?

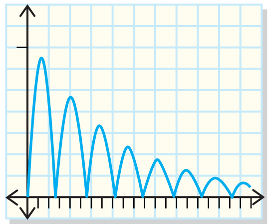
7. Which graphs represent periodic functions? Explain how you know.

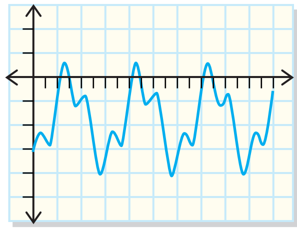
a)


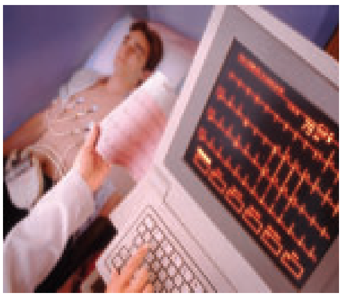
b)


c)


d)


e)


f)




8. Which of the following situations would produce periodic graphs? Justify your decision, and draw a sketch of what you think the graph might look like.

a) An electrocardiograph monitors the electrical currents generated by the heart.
- independent variable: time
- dependent variable: voltage

b) Some forms of bacteria double in number every 20 min.
- independent variable: time
- dependent variable: number of bacteria

c) When you purchase a load of gravel, you pay for the gravel by the tonne plus a delivery fee.
- independent variable: mass
- dependent variable: cost

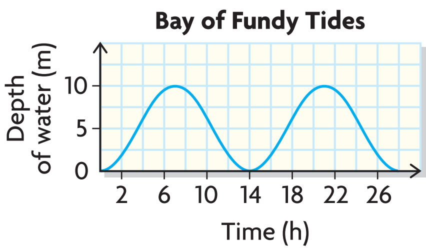
d) Alex dribbles a basketball at a steady pace and height.
- independent variable: time
- dependent variable: height

e) A tow truck is parked by the side of a dark road with its rotating amber light on. You are viewing it from a spot across the street.
- independent variable: time
- dependent variable: light intensity

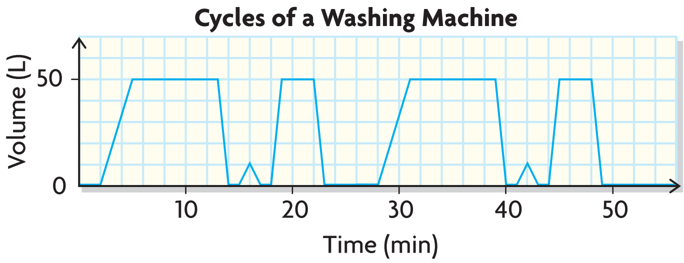
f) You throw a basketball to a friend, but she is so far away that the ball bounces on the ground four times.
- independent variable: distance
- dependent variable: height

9. The Bay of Fundy, which is between New Brunswick and Nova Scotia, has the highest tides in the world. There can be no water on the beach at low tide, while at high tide the water covers the beach.

a) Why can you use periodic functions to model the tides?
b) What is the change in depth of the water from low to high tide?

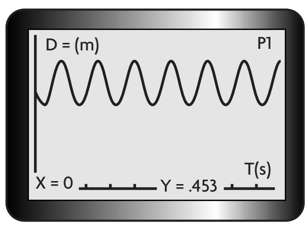


10. The graph shows the number of litres of water that a washing machine uses over several wash cycles.



a) Explain what each part of the graph represents in terms of the cycles of the washing machine.
b) What is the period of one complete cycle?
c) What is the maximum volume of water used for each part of the cycle?
d) What is the total volume of water used for one complete cycle?

11. Describe the motion of the paddle in front of a CBR that would have produced the graph shown.

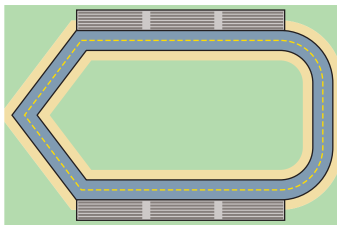


12. Denis holds a cardboard paddle 60 cm from a CBR for 3 s, and then within 0.5 s moves the paddle so that it is 30 cm from the detector. He holds the paddle there for 2 s, and then within 0.5 s moves the paddle back to the 60 cm location. Denis repeats this process three times. Sketch the graph. Include a scale.
T

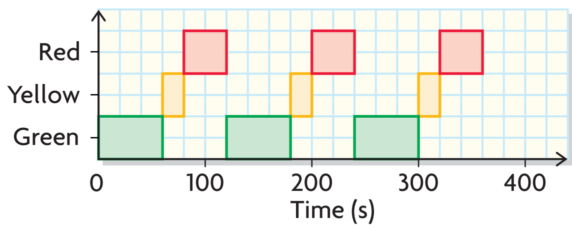
13. Write a definition of a periodic function. Include an example, and use your definition to explain why it is periodic.
C

Extending

14. Katrina is racing her car around the track shown. Assuming that she is at least two laps into the race, draw a graph showing the relationship between her speed and time as she completes two additional laps. Is this a periodic function, or is it an approximate periodic function? Justify your answer.



15. A traffic light at either end of a bridge under construction reduces traffic to a single lane and changes colour over time as shown.



a) Describe one complete cycle.
b) What is the period?
c) A 20 s advanced green arrow is added to the beginning of the cycle. What is the period now? Draw two full cycles of the graph.

6.3 Investigating the Sine Function

GOAL

Examine a specific type of periodic function called a sinusoidal function.

YOU WILL NEED

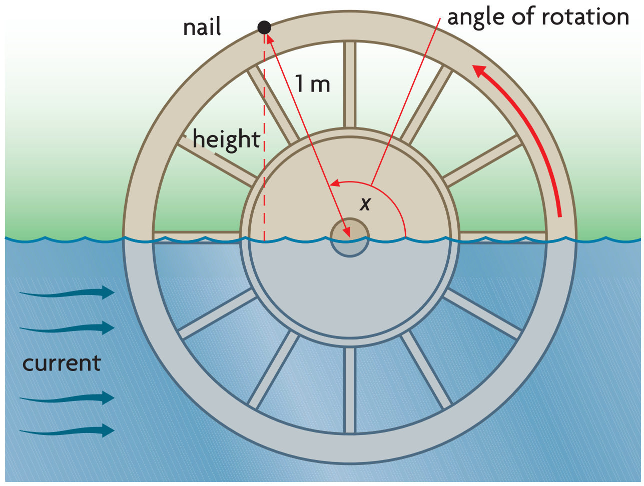
- cardboard
- ruler
- thumbtack
- protractor
- metre stick
- graphing calculator

INVESTIGATE the Math

Steve uses a generator powered by a water wheel to produce his own electricity.

- Half the water wheel is below the surface of the river.
- The wheel has a radius of 1 m.
- The wheel has a nail on its circumference.

As the current flows, the wheel rotates in a counterclockwise direction to power the generator. The height of the nail, relative to the water level, as the wheel rotates is graphed in terms of the angle of rotation, x.

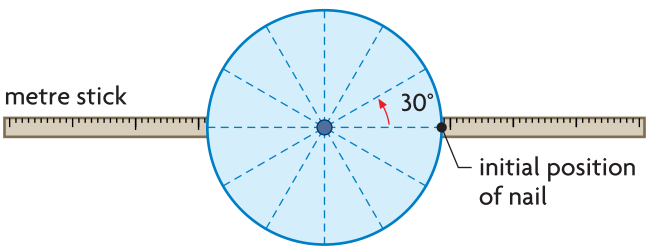


? How can the resulting graph be described?

A. Construct a scale model of the water wheel by drawing a circle with a radius of 10 cm on cardboard to represent the water wheel's 1 m radius.

B. Locate the centre of the circle. Use a protractor to divide your cardboard wheel into 30° increments through the centre. Draw a dot to represent the nail on the circumference of the circle at one of the lines you drew to divide the wheel.

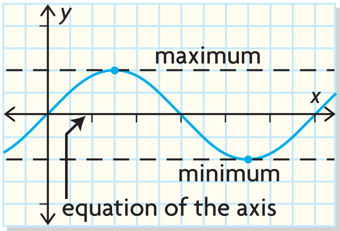
C. Cut out the wheel and attach it to a metre stick by pushing a thumbtack through the centre of the wheel at the 50 cm mark.



equation of the axis

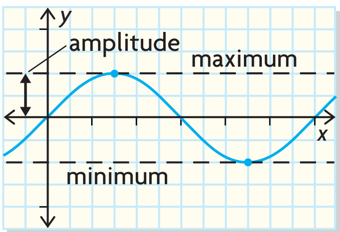
the equation of the horizontal line halfway between the maximum and the minimum is determined by

$$y = \frac{(\text{maximum value} + \text{minimum value})}{2}$$



amplitude

the distance from the function's equation of the axis to either the maximum or the minimum value

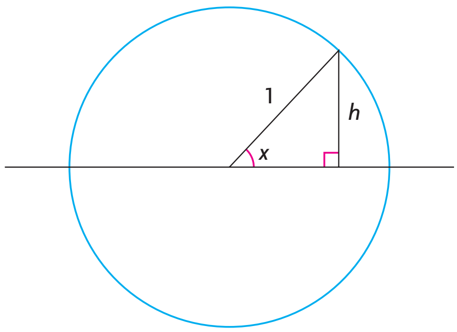


D. Copy the table below. Rotate the wheel 30°, and measure the height of the "nail" (the perpendicular distance from the middle of the metre stick to the "nail"). Convert your measurement to metres and record in the table. Continue to rotate the wheel in 30° increments, and record the measurements in metres. If the "nail" goes below the surface of the centre of the metre stick, record the height as a negative value. Continue until the "nail" has rotated 720°. Use your data to graph the height versus angle of rotation.

Rotation, x (°)	0	30	60	90	120	• • •	690	720
Height of Nail, h (m)	0							

E. What are the period, the equation of the axis, and the amplitude of this function?

F. Describe how the sine ratio can be used to relate the height, h, of the nail to the angle of rotation, x.



sine function

a sine function is the graph of $f(x) = \sin x$, where x is an angle measured in degrees; it is a periodic function

G. Use a graphing calculator to graph the sine function. How does this compare to your water wheel graph?

H. Describe the shape of the sine function, and determine its domain and range.

Tech Support

For help on graphing the sine function, see Technical Appendix, B-13.

Reflecting

I. If you needed to sketch the graph $f(x) = \sin x$, what five key points would you use?

J. Which one of these equations best models this water wheel?

$x = \sin h$ $\quad$ $h = \sin x$ $\quad$ $h = \cos x$

$h = 90\,x$ $\quad$ $x = h^2$ $\quad$ $h = x^2$

sinusoidal function

a type of periodic function created by transformations of $f(x) = \sin x$

K. Why is the term sinusoidal function appropriate for this type of periodic function?

APPLY the Math

EXAMPLE 1 Identifying sinusoidal functions

Which situations can be modelled by a sinusoidal function? Justify your decision.

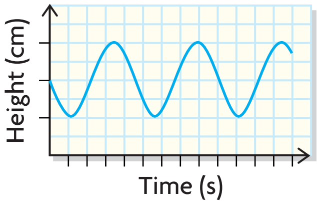
a) the height above the ground of a person on a Ferris wheel

b) the depth of a programmable radio-controlled toy submarine

c) the motion of a flagpole in a gust of wind

Tina's Solution

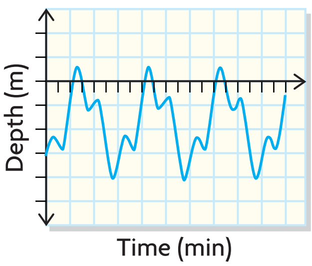
a) the height above the ground of a person on a Ferris wheel



The graph repeats over the same interval, so it's periodic. The graph has the same shape as $f(x) = \sin x$.

The graph is periodic and sinusoidal.

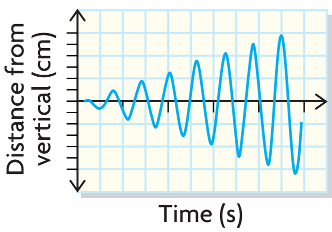
b) the depth of a programmable radio-controlled toy submarine



The graph repeats over the same interval, so it's periodic. But the graph doesn't have the same shape as the graph of $f(x) = \sin x$.

The graph is periodic but not sinusoidal.

c) the motion of a flagpole in a gust of wind



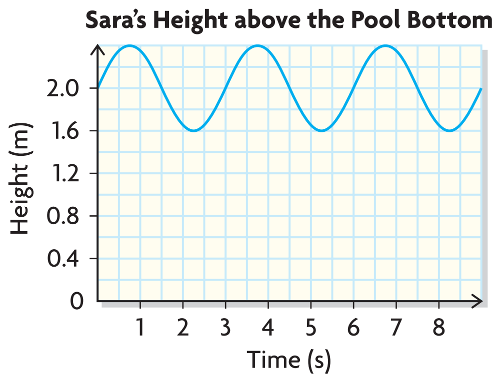
The graph does not repeat over the same interval, so it's not periodic. Also, the graph doesn't have the same shape as the graph of $f(x) = \sin x$.

The graph is neither periodic nor sinusoidal.

If you are given a graph of a sinusoidal function, you can determine information about the situation that it models.

EXAMPLE 2 Reasoning and interpreting the graph

Sarah is sitting in an inner tube in a wave pool. The depth of the water below her in terms of time can be represented by the graph shown. Discuss and interpret the graph.



Jared's Solution

$7 - 4 = 3$

The period is 3 s.

The period is the change in *x* corresponding to one cycle. I found the change in time between two maximum values. This represents the time it takes for Sarah to go from the peak of one wave to the peak of the next.

$$y = \frac{2.4 + 1.6}{2}$$

$$= 2.0$$

The equation of the axis is $y = 2$.

I found the halfway point between the maximum and the minimum value of the graph. The value $y = 2$ represents Sarah's height above the bottom of the pool in still water.

$2.4 - 2.0 = 0.4$

The amplitude is 0.4 m.

The amplitude of a sinusoidal function is the vertical distance from its axis, which is $y = 2$, to the maximum value, which is 2.4 m. The amplitude represents the distance the wave rises and falls from the water level in still water.

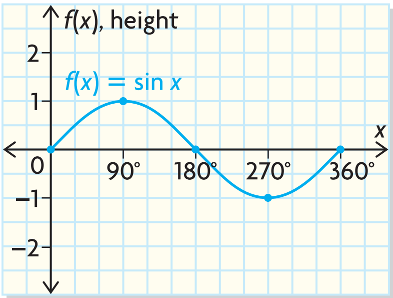
In Summary

Key Idea

- The height of a point above its starting position is a function of the angle of rotation, x, measured in degrees. The height, $f(x)$, can be modelled by the periodic function $f(x) = \sin x$, the sine function.

Need to Know

- The sine function has the following properties:
  - It has an amplitude of 1.
  - It has a period of 360°.
  - Its axis is defined by $y = 0$.
  - The domain is $\{x \in \mathbf{R}\}$, and the range is $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$.
- The sine function passes through five key points: (0°, 0), (90°, 1), (180°, 0), (270°, −1), and (360°, 0).

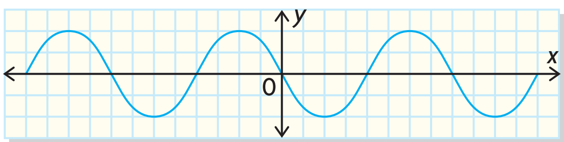


- Graphs that are periodic and have the same shape and characteristics as the sine function are called sinusoidal functions.

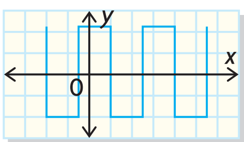
CHECK Your Understanding

1. Which graphs are sinusoidal functions? Justify your decision.

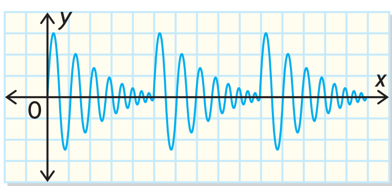
a)



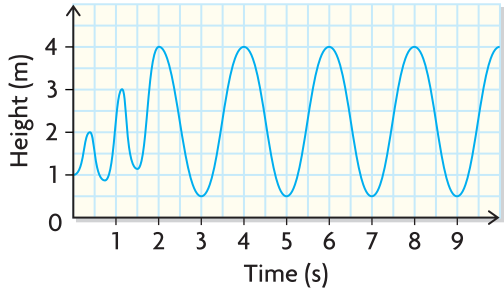
b)



c)



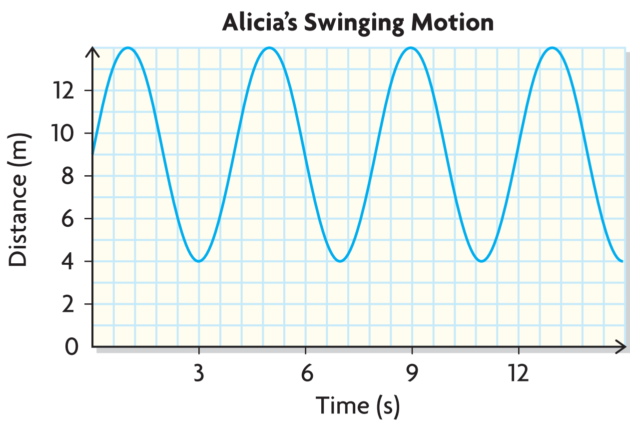
2. Nolan is jumping on a trampoline. The graph shows how high his feet are above the ground.



a) How long does it take for Nolan's jumping to become periodic? What is happening during these first few seconds?
b) What is the period of the curve? What does *period* mean in this context?
c) Write an equation for the axis of the periodic portion of the curve.
d) What is the amplitude of the sinusoidal portion of the curve? What does *amplitude* mean in this context?

PRACTISING

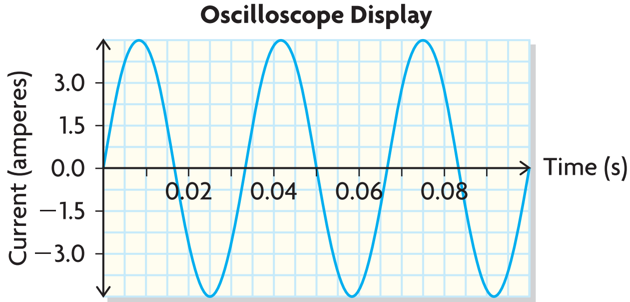
3. **K** Alicia was swinging back and forth in front of a motion detector. Her distance from the motion detector in terms of time can be modelled by the graph shown.



a) What is the equation of the axis, and what does it represent in this situation?
b) What is the amplitude of this function?

c) What is the period of the function, and what does it represent in this situation?
d) How close did Alicia get to the motion detector?
e) If the motion detector was activated as soon as Alicia started to swing, how would the graph change? (You may draw a diagram or a sketch.) Would the resulting graph be sinusoidal? Why or why not?
f) At $t = 8$ s, would it be safe to run between Alicia and the motion detector? Explain your reasoning.

4. An oscilloscope hooked up to an AC (alternating current) circuit shows a sine curve on its display:

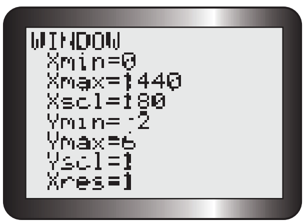


a) What is the period of the function?
b) What is the equation of the axis of the function?
c) What is the amplitude of the function?
d) State the units of measure for each of your answers above.

5. Using a graphing calculator in DEGREE mode, graph each function. Use the WINDOW settings shown. After you have the graph, state the period, the equation of the axis, and the amplitude for each function.

Tech **Support**

For help on changing the window settings and graphing functions, see Technical Appendix, B-4.



a) $f(x) = 2\sin x + 3$
b) $f(x) = 3\sin x + 1$
c) $f(x) = \sin(0.5x) + 2$
d) $f(x) = \sin(2x) - 1$
e) $f(x) = 2\sin(0.25x)$
f) $f(x) = 3\sin(0.5x) + 2$

6. Sketch the sinusoidal graphs that satisfy the given properties.

	Period (s)	Amplitude (m)	Equation of the Axis	Number of Cycles
a)	4	3	$y = 5$	2
b)	20	6	$y = 4$	3
c)	80	5	$y = -2$	2

7. Tides are the result of the gravitational attraction between the Sun, the Moon, and Earth. There are two high and low tides each day. The two high tides may not be the same height. The same is true of the low tides. In addition, tides differ in height on a daily basis. However, every 14.8 days, the oceans and other bodies of water that are affected by the Sun–Moon–Earth system go through a new tidal cycle.

a) Are tides a periodic phenomenon? Why or why not?

b) Are tides a sinusoidal phenomenon? Why or why not?

8. A The table shows the hours of daylight for the city of Regina, at a latitude of 50°. The amount of daylight is calculated as the time between sunrise and sunset.

Regina, latitude of 50°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	8.5	10.1	11.8	13.7	16.4	17.1	15.6	14.6	12.7	10.8	9.1	8.1

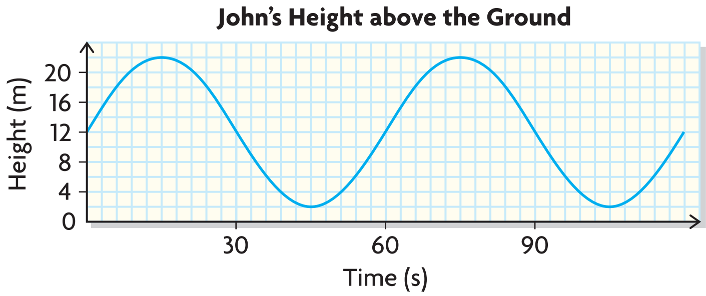
a) Graph the data, and draw a curve through the points.

b) The hours of daylight in terms of day of the year can be modelled by a sinusoidal function. What is the period of this function?

c) Approximate the equation of the axis. What does it represent in this situation?

d) Approximate the amplitude. What does it represent in this situation?

9. The graph shows John's height above the ground as a function of time
T as he rides a Ferris wheel.



a) What is the diameter of the Ferris wheel?
b) What is John's initial height above the ground?
c) At what height did John board the Ferris wheel?
d) How high above the ground is the axle on the wheel?

10. a) Sketch the graph of $f(x) = \sin x$, where $-360° \leq x \leq 360°$.
C **b)** State the period, equation of the axis, and amplitude of $f(x) = \sin x$.

Extending

11. a) Create a table of values for the function defined by $f(\theta) = \cos \theta$, where $0° \leq \theta \leq 360°$.

Rotation (θ)	0°	30°	60°	90°	120°	• • •	330°	360°
cos θ	1							

b) Plot these points, and draw a curve.
c) Is this a sinusoidal function? Explain.
d) Determine the period, the equation of the axis, and the amplitude of the function.
e) How does this periodic function compare with the function $f(\theta) = \sin \theta$?

12. a) Create a table of values for the function defined by $f(\theta) = \tan \theta$, where $0° \leq \theta \leq 360°$.

Rotation (θ)	0°	30°	60°	90°	120°	• • •	330°	360°
tan θ	0							

b) Plot these points, and draw a curve.
c) Is this a sinusoidal function? Explain.
d) Determine the period, the equation of the axis, and the amplitude of the function.

6.4 Comparing Sinusoidal Functions

YOU WILL NEED

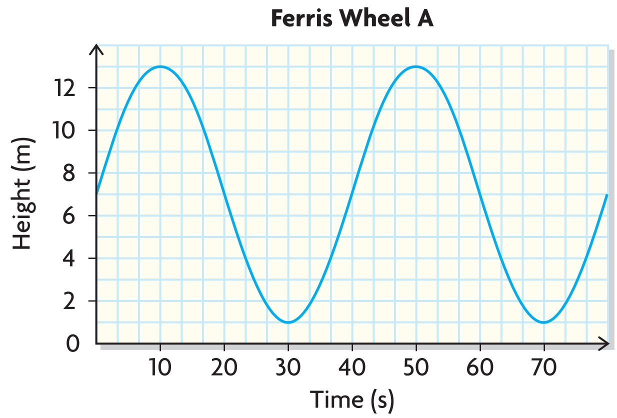
- graphing calculator

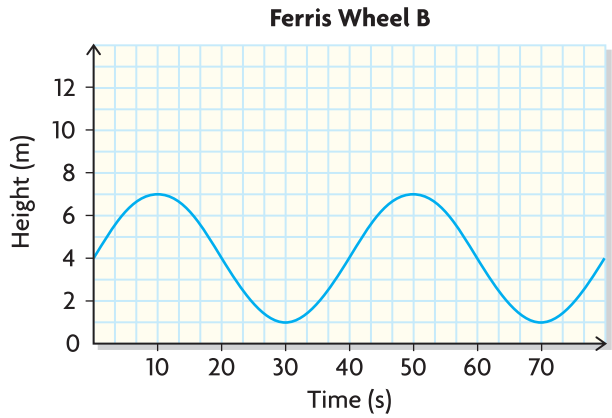
GOAL

Relate details of sinusoidal phenomena to their graphs.

LEARN ABOUT the Math

At an amusement park, a math teacher had different students ride two Ferris wheels. Thomas rode on Ferris wheel A, and Ryan rode on Ferris wheel B. The teacher collected data and produced two graphs.





? What information about the Ferris wheels can you learn from the graphs of these functions?

EXAMPLE 1 Connecting the features of the graph to the situation

Nathan's Solution: Comparing Periods in Sinusoidal Functions

In both graphs, the first peak is at 10 s and the second peak is at 50 s.

One of the easiest ways to get the period is to figure out how long it takes to go from one peak on the graph to the next peak. I read the times for the first two peaks.

$50 - 10 = 40$

The period of both Ferris wheels is 40 s.

Both Ferris wheels take 40 s to complete one revolution.

Leila's Solution: Comparing Equations of the Axes in Sinusoidal Functions

Equation of the axis for Graph A:

$$\frac{(13 + 1)}{2} = 7$$

I got the equation of the axis in each case by adding the maximum and the minimum and then dividing by 2, since the answer is halfway between these values.

The equation of the axis for Graph A is $h = 7$.

Equation of the axis for Graph B:

$$\frac{(7 + 1)}{2} = 4$$

The equation of the axis for Graph B is $h = 4$.

The axle for Ferris wheel A is 7 m above the ground, and the axle for Ferris wheel B is 4 m above the ground.

In both cases, the equation of the axis represents the distance from ground level to the middle of the Ferris wheel. This distance is how high the axle is above the ground.

Dave's Solution: Comparing Amplitudes in Sinusoidal Functions

Amplitude for Graph A:

$13 - 7 = 6$

I found the amplitude by getting the distance from the axis to a peak or maximum.

Amplitude for Graph B:

$7 - 4 = 3$

The radius of Ferris wheel A is 6 m.
The radius of Ferris wheel B is 3 m.

I found the diameters of the wheels by subtracting the minimum from the maximum value. I divided by 2 to find the radius, which is also the amplitude of the graph.

EXAMPLE 2 Comparing speeds

Wanda's Solution

Circumference of each wheel:

$C_A = 2\pi r_A$ $C_B = 2\pi r_B$

$C_A = 2\pi(6)$ $C_B = 2\pi(3)$

$C_A \doteq 37.7$ m $C_B \doteq 18.8$ m

The circumference of Ferris wheel A is about 37.7 m, and the circumference of Ferris wheel B is about 18.8 m.

I want to see how fast each Ferris wheel goes. Since speed is equal to distance divided by time, I figured out how far each student travelled around the wheel in completing one revolution by finding the circumference of each wheel.

Speed of each rider:

$s_A = \frac{d_A}{t}$ $s_B = \frac{d_B}{t}$

$s_A = \frac{37.7}{40}$ $s_B = \frac{18.8}{40}$

$s_A \doteq 0.94$ m/s $s_B \doteq 0.47$ m/s

To calculate the speed, I divided each circumference by the time taken to complete one revolution, which is 40 s.

Riders on Ferris wheel A are travelling twice as fast as those on Ferris wheel B.

Ferris wheel A is probably scarier than Ferris wheel B because you are travelling much faster on A.

Reflecting

A. How does changing the radius of the wheel affect the sinusoidal graph?

B. How does changing the height of the axle of the wheel affect the sinusoidal graph?

C. How does changing the speed of the wheel affect the sinusoidal graph?

D. What type of information can you learn by examining the graph modelling the height of a rider in terms of time?

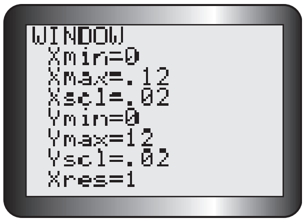
APPLY the Math

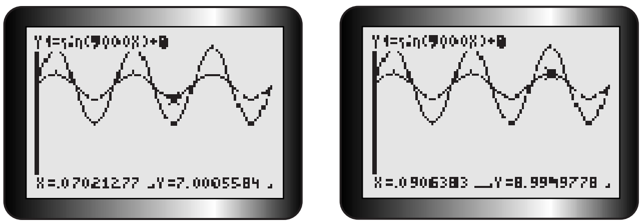
Comparing the differences between the maximum and minimum values of a sinusoidal function can give you insight into the periodic motion involved.

EXAMPLE 3 Comparing peaks and troughs

As the shaft on an electric motor rotates, the shaft vibrates. The vibration causes stress to the shaft. The relationship between the stress on the shaft, y, measured in megapascals, and time, x, measured in seconds, can be modelled with a sinusoidal function. Two different equations for two different motors are given. Using a graphing calculator and the WINDOW settings shown, graph both sinusoidal functions at the same time. What can you conclude?

Motor A: $y = \sin(9000x)^\circ + 8$
Motor B: $y = 3\sin(9000x)^\circ + 8$

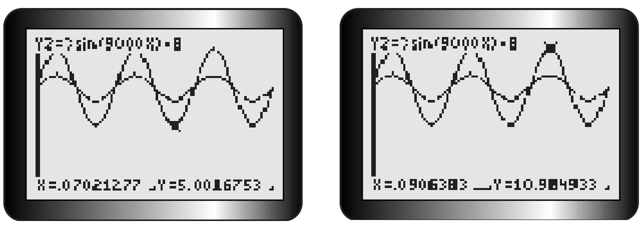




Using the Trace feature on the calculator, I found the high and low points on the graphs.

The troughs on the graph for Motor A are at 7, and the peaks are at 9.

The stress for the shaft in Motor A ranges between 7 and 9 MPa.



The troughs on the graph for Motor B are at 5, and the peaks are at 11.

The stress for the shaft in Motor B ranges between 5 and 11 MPa.

There are more extreme changes in stress to the shaft in Motor B.

In Summary

Key Idea

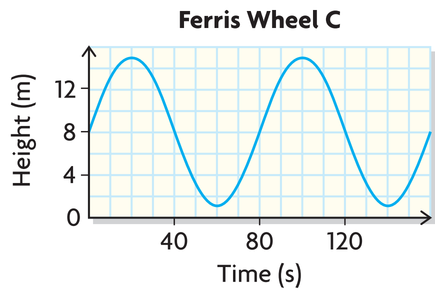
- Sinusoidal functions can be used as models to solve problems that involve circular or oscillating motion at a constant speed.

Need to Know

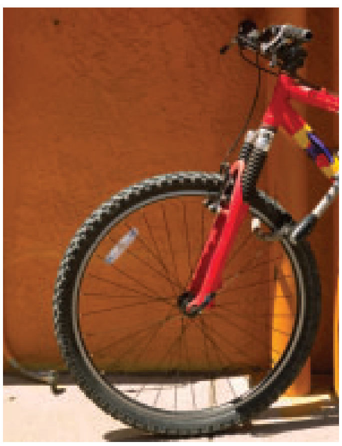
- One cycle of motion corresponds to one period of the sinusoidal function.
- The amplitude of the sinusoidal function depends on the situation being modelled.

CHECK *Your Understanding*

1. Tashina was on the same field trip as Thomas and Ryan. Tashina rode on Ferris wheel C. The resulting graph is shown. How does Ferris wheel C compare with Ferris wheels A and B in terms of maximum height, radius of the wheel, and speed?



2. Draw two sinusoidal functions that have the same period and amplitude but have different equations of the axes.



PRACTISING

3. Sketch a height-versus-time graph of the sinusoidal function that models each situation. Assume that the first point plotted on each graph is at the lowest possible height.

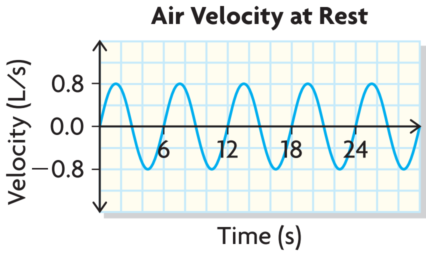
a) A Ferris wheel with a radius of 9 m, whose axle is 10 m above the ground, and that rotates once every 60 s

b) A water wheel with a radius of 2 m, whose centre is at water level, and that rotates once every 20 s

c) A bicycle tire with a radius of 35 cm and that rotates once every 30 s

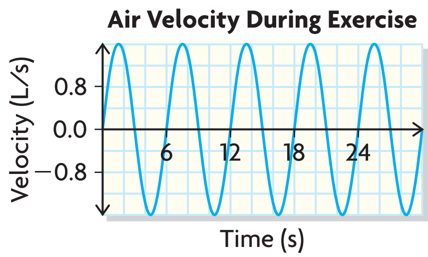
d) A girl lying on an air mattress in a wave pool that is 3 m deep, with waves 1 m in height that occur at 10 s intervals

4. When you breathe, the velocity of the air entering and exiting your lungs changes in terms of time. The air entering your lungs has a positive velocity, and the air exiting your lungs has a negative velocity. If a person is at rest, this relationship can be modelled using the graph shown.

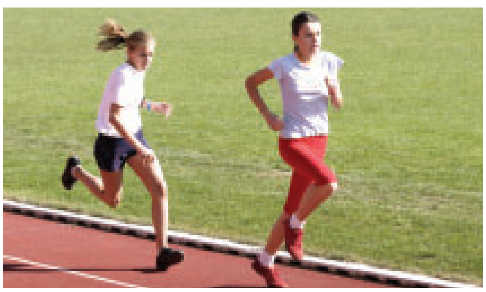


a) What is the equation of the axis, and what does it represent in this situation?
b) What is the amplitude of this function?
c) What is the period of the function, and what does it represent in this situation?
d) State the domain and range of the function.

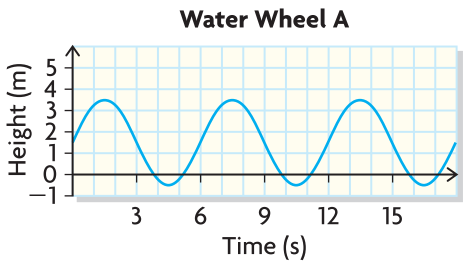
5. When you exercise, the velocity of air entering your lungs in terms of time changes. The graph models this relationship.

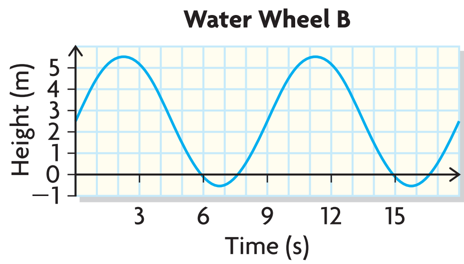


a) According to this exercise model, is the individual taking more breaths per minute or just deeper breaths than the individual in question 4? How do you know?
b) What property (period, equation of axis, or amplitude) of this graph has changed compared with the graph in question 4?
c) What is the maximum velocity of the air entering the lungs? Include the appropriate units of measure.



6. Maire is visiting a historical site that has two working water wheels. **K** She notices that there is a large spike on the circumference of each wheel. She studies the motion of each spike, collects data, and constructs the following graphs:





Analyze the graphs, and compare the water wheels. Refer to

- the radius of each wheel
- the height of the axle relative to the water
- the time taken to complete one revolution
- the speed of each wheel

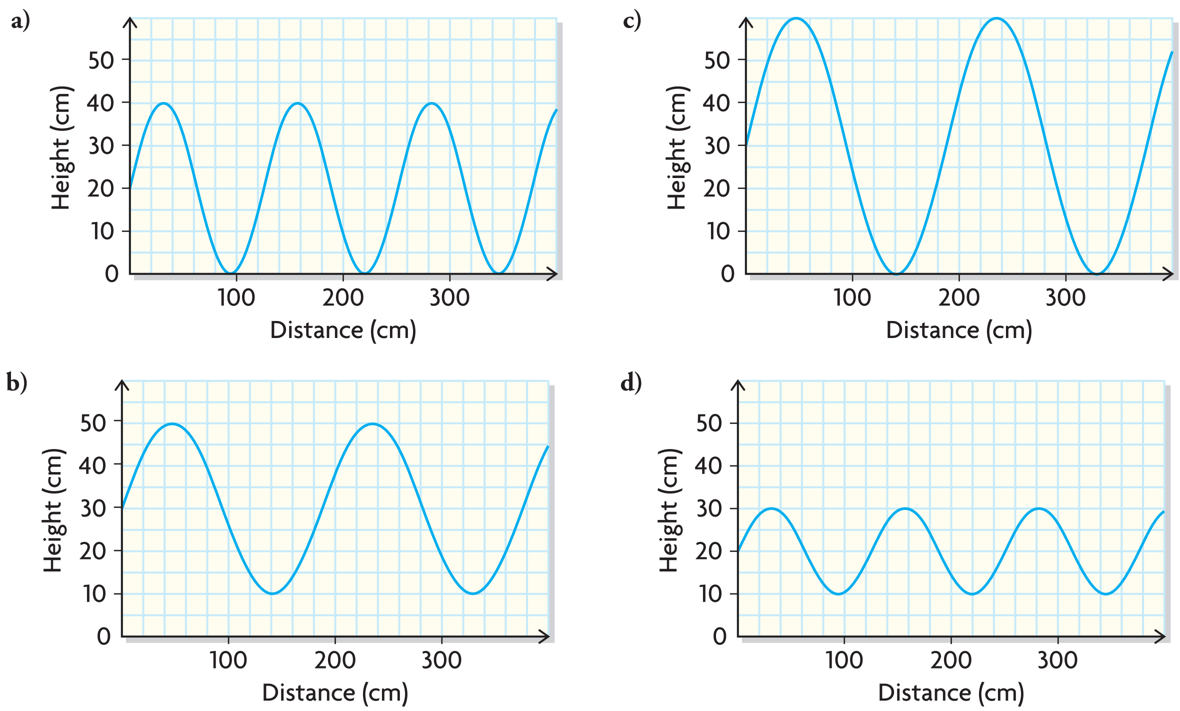
7. Marcus has two different-sized wheels, on which he conducted four similar experiments.

Experiment 1: Marcus took the larger wheel and stuck a piece of tape on the wheel's side so that the tape would eventually touch the ground as it rolled. He rolled the wheel and recorded the height of the tape above the ground relative to the distance the wheel travelled. He graphed his data.

Experiment 2: Marcus took the same wheel and stuck the tape on the side of the wheel 10 cm in from the edge so that the tape would never touch the ground. He rolled the wheel, collected his data, and graphed the data.

Experiment 3: Marcus took the smaller wheel and stuck the tape on the side of the wheel so that the tape would eventually touch the ground as it rolled. He rolled the wheel, collected his data, and graphed the data.

Experiment 4: Using the smaller wheel, Marcus stuck the tape on the side of the wheel 10 cm in from the edge so that the tape would never touch the ground. He rolled the wheel, collected his data, and graphed the data.



a) Match each graph to the appropriate experiment.
b) What is the equation of the axis for each graph? What does the equation of the axis represent for each graph?
c) What is the amplitude for each graph?
d) What is the period for each graph? What does the period represent in each graph?
e) What is the radius of the larger wheel?
f) What is the radius of the smaller wheel?
g) In Experiment 2, approximately how far above the ground is the tape after the wheel has travelled 100 cm from its initial position?
h) If the tape was placed on the centre of the larger wheel, what would the resulting graph look like?

8. A The tables show the length of daylight for Windsor, Ontario, which is at a latitude of 40°, and for Fort Smith, Northwest Territories, which is at a latitude of 60°. The hours of daylight are calculated as the interval between sunrise and sunset.

Windsor, latitude of 40°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	9.6	10.7	11.9	13.2	15.0	15.4	14.8	13.8	12.5	11.2	10.0	9.3

Fort Smith, latitude of 60°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	6.6	9.2	11.7	14.5	18.8	22.2	17.5	15.8	13.0	10.2	7.6	5.9

Tech Support

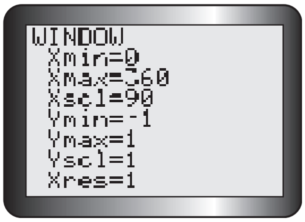
For help on changing the window settings and graphing functions, see Technical Appendix, B-2 and B-4.

a) Plot the data on separate coordinate systems, and draw a curve through each set of points.

b) Compare the two curves. Refer to the periods, the equations of the axes, and the amplitudes.

c) What might you infer about the relationship between hours of daylight and the latitude at which you live?

9. In high winds, the top of a flagpole sways back and forth. The distance the tip of the pole vibrates to the left and right of its resting position can be defined by the function $d(t) = 2\sin(720t)°$, where $d(t)$ is the distance in centimetres and t is the time in seconds. If the wind speed decreases by 20 km/h, the motion of the tip of the pole can be modelled by the function $d(t) = 1.5\sin(720t)°$. Using graphing technology in DEGREE mode and the WINDOW settings shown, plot and examine the two graphs, and discuss the implications of the reduced wind speed on the flagpole. Refer to the period, axis, and amplitude of each graph.



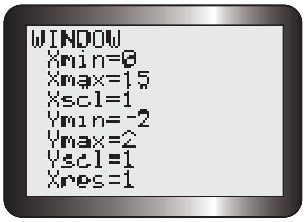
10. The height, $h(t)$, of a basket on a water wheel at time t can be modelled by $h(t) = \sin(6t)°$, where t is in seconds and $h(t)$ is in metres.

a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$. Sketch this graph.

b) How long does it take for the wheel to make a complete revolution? Explain how you know.

c) What is the radius of the wheel? Explain how you know.

d) Where is the centre of the wheel located relative to the water level? Explain how you know.



11. A buoy rises and falls as it rides the waves. The function $h(t) = 1.5\sin(72t)°$ models the position of the buoy, $h(t)$, on the waves in metres at t seconds.

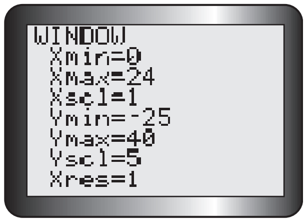
a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$. Sketch this graph.

b) How long does it take for the buoy to travel from the peak of a wave to the next peak? Explain how you know.

c) How many waves will cause the buoy to rise and fall in 1 min? Explain how you know.

d) How far does the buoy drop from its highest point to its lowest point? Explain how you know.

12. The average monthly temperature, $T(t)$, in degrees Celsius, in Sydney, Australia, can be modelled by the function $T(t) = -12\sin(30t)^\circ + 20$, where t represents the number of months. For $t = 0$, the month is January; for $t = 1$, the month is February, and so on.
    a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $T(t)$. Sketch the graph.
    b) What does the period represent in this situation?
    c) What is the temperature range in Sydney?
    d) What is the mean temperature in Sydney?

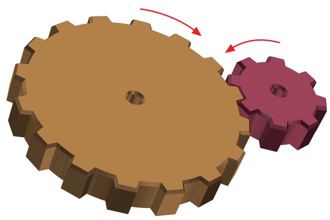


13. Explain how you would compare graphs of sinusoidal functions derived from similar real-world situations.

Extending

14. The diameter of a car's tire is 50 cm. While the car is being driven, the tire picks up a nail.
    a) How far does the tire travel before the nail returns to the ground for the first time?
    b) Model the height of the nail above the ground in terms of the distance the car has travelled since the tire picked up the nail.
    c) How high above the ground will the nail be after the car has travelled 235 cm?
    d) The nail reaches a height of 10 cm above the ground for the third time. How far has the car travelled?

15. A gear of radius 1 m turns in a counterclockwise direction and drives a larger gear of radius 3 m. Both gears have their central axes along the horizontal.
    a) In which direction is the larger gear turning?
    b) If the period of the smaller gear is 2 s, what is the period of the larger gear?
    c) Make a table in convenient intervals for each gear to show the vertical displacement, d, of the point where the two gears first touched. Begin the table at 0 s, and end it at 12 s. Graph vertical displacement versus time.
    d) What is the displacement of the point on the large gear when the drive gear first has a displacement of –1 cm?
    e) What is the displacement of the drive gear when the large gear first has a displacement of 2 m?
    f) What is the displacement of the point on the large gear at 5 min?



Curious Math

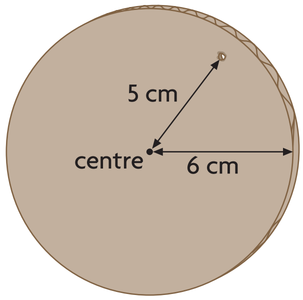
If It Rolls, Is It Sinusoidal?

If you roll an object, is the graph of its movement always a sinusoidal function?

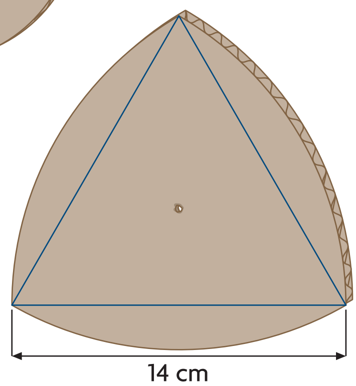
YOU WILL NEED

- ruler
- protractor
- corrugated cardboard
- graph paper
- tape
- compass
- scissors

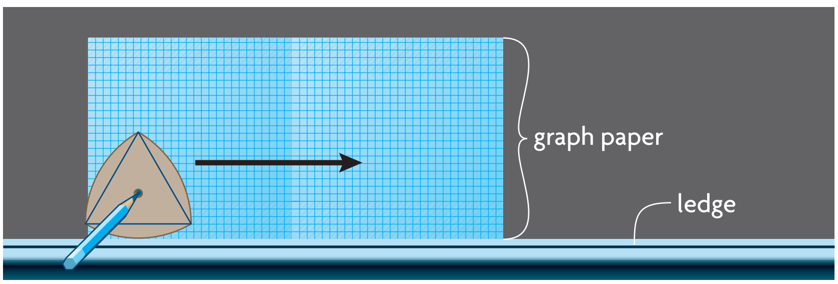
1. Draw a circle of radius 6 cm on cardboard and cut it out. Using a pencil, punch a small hole in the cardboard circle, 5 cm from its centre. The hole should be big enough so that the tip of the pencil can fit through.



2. Draw an equilateral triangle with sides of length 14 cm on cardboard. Using a pencil, punch a small hole in the centre of the triangle, large enough so that the tip of the pencil can fit through. Using a compass, draw an arc from one vertex of the triangle to another, using the third vertex as the centre of rotation. Continue this process until you have drawn three arcs. Cut out the figure.



3. Tape two sheets of graph paper together, short end to short end. Tape these two pieces of paper onto a wall or chalkboard so that there is a flat surface, such as a floor or chalk ledge, perpendicular to the paper.
4. Put a pencil through the hole in the cardboard circle, and rotate the circle along the flat surface. The pencil should be pressed against the graph paper so that it marks a curve on the paper as the circle rotates.
5. Repeat step 4 using the cardboard equilateral triangle.



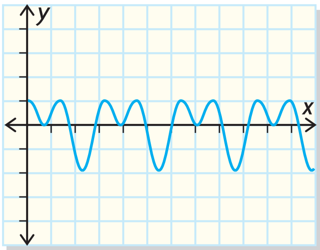
a) Does each curve represent a periodic function? Why or why not?
b) Does each curve represent a sinusoidal function? Why or why not?
c) Will rolling an object always result in a sinusoidal function?

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: What are periodic functions?

A: Periodic functions repeat at regular intervals. As a result, their graphs have a repeating pattern.



Study *Aid*

- See Lesson 6.2, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: What are the characteristics of a sinusoidal function?

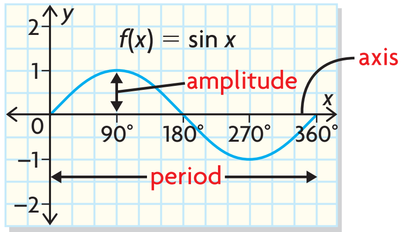
A: Sinusoidal functions, like other periodic functions, repeat at regular intervals. Unlike other periodic functions, sinusoidal functions form symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve. The three most important characteristics of a sinusoidal function are the period, the equation of the axis, and the amplitude.

Study *Aid*

- See Lesson 6.3, Examples 1 and 2.
- Try Mid-Chapter Review Questions 4 and 5.

Period	Equation of the Axis	Amplitude
The period is the change in x corresponding to one cycle. (A cycle of a sinusoidal function is a portion of the graph from one point to the point at which the graph starts to repeat.) One way to determine the period is to look at the change in x's between two maximum values.	The equation of the axis is the equation of the line halfway between the maximum and minimum values on a sinusoidal function. It can be determined from the following formula: $y = \frac{(\text{maximum value} + \text{minimum value})}{2}$	The amplitude is the vertical distance from the function's axis to the minimum or maximum value.

EXAMPLE



For the function $f(x) = \sin x$, the period is 360°, the equation of the axis is $y = 0$, and the amplitude is 1.

The domain is $\{x \in \mathbf{R}\}$, and the range is $\{f(x) \in \mathbf{R} \mid -1 \leq f(x) \leq 1\}$.

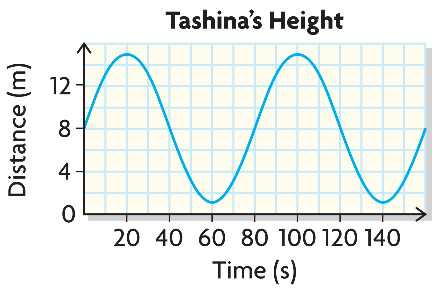
Q: Why do you learn about sinusoidal functions?

A1: Many situations can be modelled using sinusoidal functions. Examples are:

- the motion of objects in a circular orbit
- the motion of a pendulum
- the motion of vibrating objects
- the number of hours of sunlight for a particular latitude
- the phase of the Moon
- the current for an AC circuit

A2: When the graph of a sinusoidal function models a repeating situation, the graph can be used to make predictions.

EXAMPLE



The graph represents Tashina's ride on a Ferris wheel. According to the graph,

- it takes 80 s to complete one revolution (the period)
- the axle is 7 m above the ground ($y = 7$, the equation of the axis)
- the radius of the Ferris wheel is 6.5 m (the amplitude)
- we can predict that at 200 s, Tashina's height on the Ferris wheel will be 8 m

PRACTICE Questions

Lesson 6.2

1. Which of the following situations would produce a periodic graph?

a) Angelo is bouncing a tennis ball in the air with his racket. He strikes the ball with the same force each time such that the ball reaches the same maximum height.
- independent variable: time
- dependent variable: height of the ball

b) A super ball is released from a third-storey window. The ball bounces back up to 80% of its previous height on each bounce.
- independent variable: time
- dependent variable: height of the ball

c) A police cruiser is parked on the street with its siren on.
- independent variable: time
- dependent variable: intensity of the sound coming from the siren

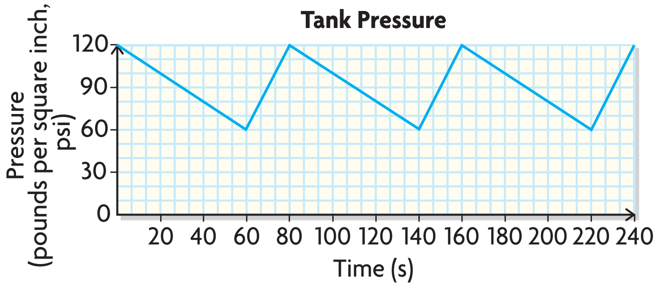
d) Alicia's investment fund doubles every eight years.
- independent variable: time
- dependent variable: total amount of money in the fund

e) Lexi is driving through a parking lot that has speed bumps placed at regular intervals.
- independent variable: the distance Lexi travels
- dependent variable: the force exerted on the shock absorbers in her vehicle

2. Explain what each characteristic means for a periodic curve. Show each on a labelled diagram.

a) cycle
b) period
c) amplitude
d) equation of the axis
e) maximum and minimum

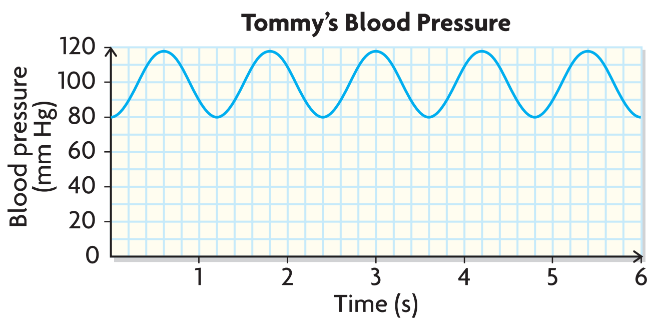
3. A power nailer on an assembly line fires continuously. The compressed air that powers the nailer is contained in a large tank, and the pressure in this tank changes as the nailer is fired. A pump maintains a certain level of pressure in the tank. The pressure in the tank in terms of time can be represented by the graph shown.



a) Is this function periodic?
b) At what pressure does the pump turn on?
c) At what pressure does the pump turn off?
d) What is the period of the function? Include the units of measure.
e) How long does the pump work at any one time?

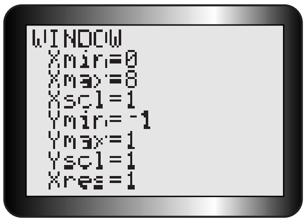
Lesson 6.3

4. Tommy's blood pressure in terms of time can be modelled by a sinusoidal function. The graph shown represents this relationship.



a) What is the period of the function?
b) How many times does Tommy's heart beat each minute?
c) What is the range of the function? Explain the meaning of this range in terms of Tommy's blood pressure.

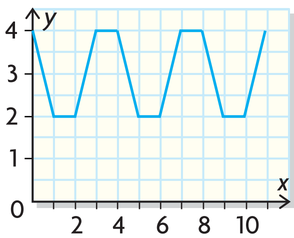
5. The pendulum on a grandfather clock swings uniformly back and forth. For a particular clock, the distance the pendulum moves to the left and right of its resting position in terms of time can be modelled by the function $d(t) = 0.25 \sin (180t)°$. The distance is measured in metres, and time is measured in seconds. Using graphing technology, in DEGREE mode, with the WINDOW settings shown, answer the following questions.

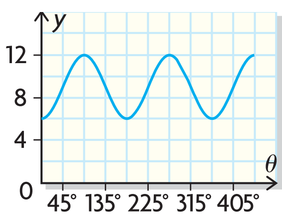


a) What is the period of the function, and what does it represent in this situation? (*Hint:* The period for this function is going to be quite short.)
b) What is the equation of the axis, and what does it represent in this situation?
c) What is the amplitude of the function, and what does it represent in this situation?
d) What will be the distance of the pendulum from its resting position at 10.2 s?

6. Sketch three cycles of a sinusoidal function that has a period of 30, an amplitude of 6, and whose equation of the axis is $y = 5$.

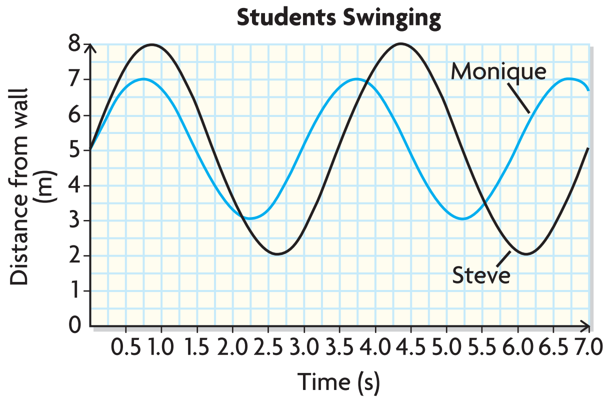
7. State the period, amplitude, and the equation of the axis for each function.

a)


b)


Lesson 6.4

8. Steve and Monique are swinging on separate swings beside a school. The lengths of the ropes on each swing differ. Their distances from one wall of the school in terms of time can be modelled by the graphs shown.



a) Compare the two curves. Refer to the periods, amplitudes, and the equations of the axes.
b) Compare Monique's motion on the swing with Steve's motion.
c) State the range of each function.

9. A Ferris wheel at the county fall fair has a radius of 12 m and rotates once every 60 s. At its lowest point, a rider is 2 m above the ground. Another Ferris wheel at an amusement park has a radius of 15 m and rotates once every 75 s. On this ride, the highest point a passenger reaches is 33 m above the ground.
a) On the same graph, sketch the height of a passenger above the ground for two complete revolutions of both wheels.
b) Compare the period, amplitude, and the equation of the axis of both graphs.
c) Which Ferris wheel is travelling faster? Explain how you know.

6.5 Transformations of the Sine Function: $f(x) = \sin(x - c)$ and $f(x) = \sin x + d$

GOAL

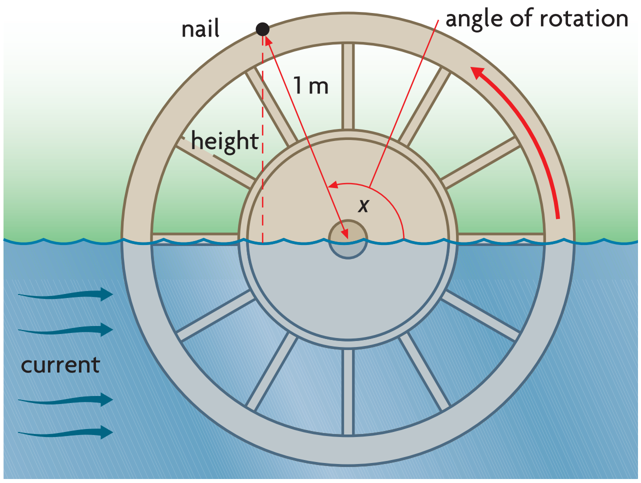
Determine how the values c and d affect the functions $f(x) = \sin x + d$ and $f(x) = \sin(x - c)$.

YOU WILL NEED

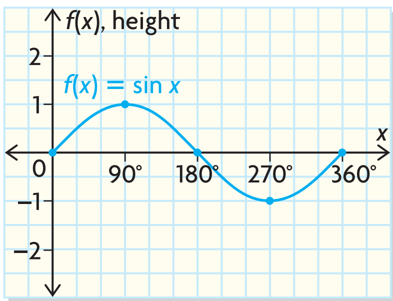
- graphing calculator

INVESTIGATE the Math

The table and graph show the relationship between the height of a nail on a water wheel with a radius of 1 m relative to the water level and the angle of rotation of the wheel.



Five Key Points for $f(x) = \sin x$	
x, Angle of Rotation	$f(x)$, Height (m)
0°	0
90°	1
180°	0
270°	−1
360°	0

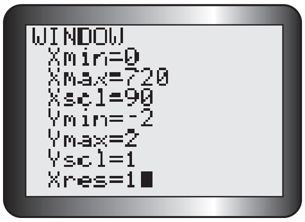


? What characteristics of the water wheel are directly related to c and d in the graphs of $f(x) = \sin(x - c)$ and $f(x) = \sin(x) + d$, and how does changing these values affect the graph of $f(x) = \sin x$?

A. Predict what will happen to the graph of $f(x) = \sin x$ when different values of d are used in functions of the form $f(x) = \sin x + d$.

Tech | *Support*

If you are graphing the functions on a graphing calculator, make sure that the calculator is in DEGREE mode and that you are using the following WINDOW settings:



To change the table setting, press 2nd WINDOW.

To view the table, press 2nd GRAPH.

B. Graph $f(x) = \sin x$ in Y1 using the calculator settings in the margin. Investigate the two functions in the chart below by graphing each function in Y2, one at a time. Compare the graphs of the two functions with the graph of $f(x) = \sin x$. Compare the table of values of both graphs, using the settings given in the chart. Use this information to sketch the transformed function, and then describe how the characteristics (i.e., amplitude, range, and so on) have changed.

Equation	Table Settings	Sketch of the New Graph	Description of How the Graph Has Changed (i.e., amplitude, range, and so on)
$g(x) = \sin x + 0.5$	TblStart = 0 ΔTbl = 90		
$h(x) = \sin x - 0.5$	TblStart = 0 ΔTbl = 90		

C. Were your descriptions in the last column what you predicted? For each new function, describe the type of transformation that produced it.

D. Predict what will happen to the graph of $f(x) = \sin x$ when different values of c are used to graph functions of the form $f(x) = \sin(x - c)$.

E. Repeat part B and create a new table using the functions $g(x) = \sin(x - 90°)$ and $h(x) = \sin(x + 90°)$.

F. Were your descriptions in the last column what you predicted? For each new function, describe the type of transformation that produced it.

G. For the functions $f(x) = \sin x + 0.5$ and $f(x) = \sin(x - 90)$, how would the characteristics of the water wheel or its placement be different?

Reflecting

H. When you compare a function of the form $g(x) = \sin x + d$ with $f(x) = \sin x$, does the amplitude, period, or equation of the axis change from one function to the other?

I. When you compare a function of the form $h(x) = \sin(x - c)$ with $f(x) = \sin x$, does the amplitude, period, or equation of the axis change from one function to the other?

J. Suppose only 0.25 m of the water wheel was still exposed, due to high floodwaters. What would the graph of the sinusoidal function describing the height of the nail in terms of the rotation look like? What transformation of $f(x) = \sin x$ would you be dealing with? Sketch the situation.

K. Discuss how the domain and range change when
  a) a horizontal translation is applied
  b) a vertical translation is applied

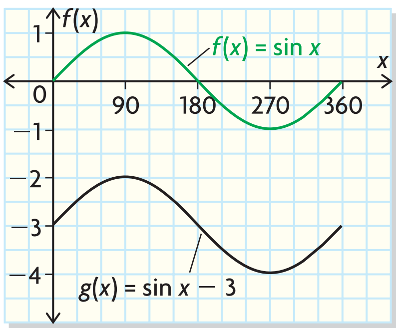
L. Explain how you would change the equation $f(x) = \sin x$ so that the graph moves up or down the y-axis as well as left or right along the x-axis.

APPLY the Math

EXAMPLE 1 Reasoning about a vertical translation of $f(x) = \sin x$

a) How does the function $g(x) = \sin x - 3$ transform the function $f(x) = \sin x$?
b) What are the domain and range of the new function?
c) State the equation of the axis, amplitude, and period of the new function.

Anne's Solution



I started by graphing $f(x) = \sin x$ (in green). The function $g(x) = \sin x - 3$ has undergone a vertical translation of -3. So I slid the graph of $f(x)$ down 3 units to get the graph of $g(x) = \sin x - 3$ (in black). The equation of the axis is $y = -3$, but the period and amplitude are the same.

a) The -3 moves the graph down 3 units. The domain is $\{x \in \mathbf{R}\}$.

Anytime I add a value to or subtract a value from the sine function, the graph moves up or down.

b) The domain is $\{x \in \mathbf{R}\}$. The range is $\{y \in \mathbf{R} \mid -4 \leq y \leq -2\}$.

The graph continues to the left and the right, covering all positive and negative values of x. The largest y-value is -2, and the smallest y-value is -4.

c) The equation of the axis is $y = -3$. The amplitude is 1. The period is 360°.

The axis is halfway between the maximum and minimum. It is also equal to the vertical translation. The amplitude is the distance from the axis to the maximum. The distance from one maximum to another gives the period.

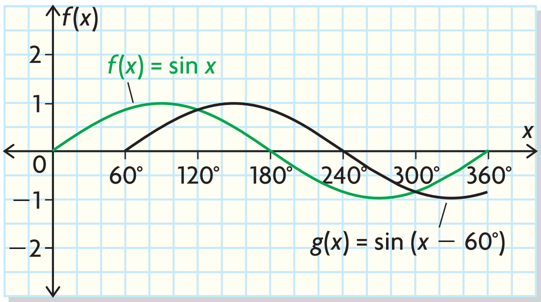
EXAMPLE 2 Reasoning about a horizontal translation of $f(x) = \sin x$

a) How does the function $g(x) = \sin(x - 60°)$ transform the function $f(x) = \sin x$?

b) What are the domain and range of the new function?

c) State the equation of the axis, amplitude, and period of the new function.

Teja's Solution



I started by graphing $f(x) = \sin x$ (in green). The function $f(x) = \sin(x - 60°)$ has undergone a horizontal translation of 60°. So I slid the graph of $g(x)$ right by 60° to get the graph of $g(x) = \sin(x - 60°)$ (in black). The equation of the axis is $y = 0$, and the period and amplitude are the same.

a) The graph moves right 60°.

b) The domain is $\{x \in \mathbf{R}\}$. The range is $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$.

c) The axis is $y = 0$. The amplitude is 1. The period is 360°.

Anytime I add or subtract a number from the independent variable in the sine function, the graph moves right or left.

The graph continues to the left and the right, covering all positive and negative values of x. The largest y-value is 1, and the smallest y-value is -1.

Both a vertical and a horizontal translation can be applied to the sine function in the same manner you have seen with quadratic functions.

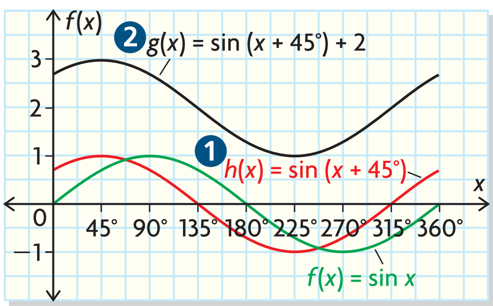
EXAMPLE 3 Reasoning about translations of $f(x) = \sin x$

a) How does the function $g(x) = \sin(x + 45°) + 2$ transform the function $f(x) = \sin x$?

b) What are the domain and range of the transformed function?

c) State the equation of the axis, amplitude, and period of the new function.

David's Solution



I began by graphing $f(x) = \sin x$, my green graph. The function given has two transformations, so I dealt with them one at a time.

1. I slid the graph left 45° to show the horizontal translation, my red graph, $h(x)$.
2. Then I slid this graph up 2 units to show the vertical translation, my black graph, $g(x)$.

I checked my work on my graphing calculator, and I'm confident that I did it right.

a) The graph is moved to the left 45° and up 2 units.

b) The domain is $\{x \in \mathbf{R}\}$. The range is $\{y \in \mathbf{R} \mid 1 \leq y \leq 3\}$.

c) The equation of the axis is $y = 2$. The period is 360°, and the amplitude is 1.

The graph continues to the left and the right, covering all positive and negative values of *x*. The largest *y*-value is 3, and the smallest *y*-value is 1.

The vertical translation of 2 affected the axis. The horizontal translation doesn't affect the period or the amplitude.

EXAMPLE 4 Representing an equation from the description of a transformation of $f(x) = \sin x$

The graph of $f(x) = \sin x$ has been translated to the right 30° and up 4 units. Write the new equation.

Ryan's Solution: Without Graphing

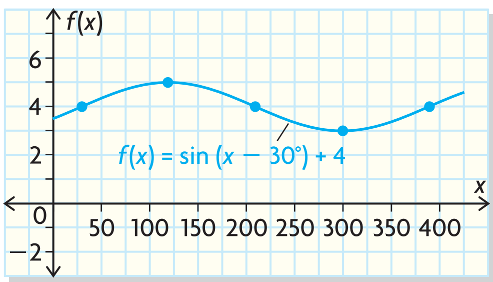
Horizontal translation: 30°
vertical translation: +4
$f(x) = \sin(x - 30°)$

If the graph's been translated to the right 30°, then there's been a horizontal translation of 30°. So there should be an $x - 30°$ in the equation.

$f(x) = \sin(x - 30°) + 4$

If the graph is slid up 4 units, then there's been a vertical translation of +4. I'll add 4 to my equation.

Brianna's Solution: By Graphing



$f(x) = \sin x$ has five key points: (0°, 0), (90°, 1) (180°, 0), (270°, −1), and (360°, 0). These points are going to move when transformations are applied to the equation.

Since there is a horizontal translation of 30°, all the *x*-values will be increased by 30. By adding 30 to the *x*-coordinate of each of my five key points, they change to (30°, 0), (120°, 1), (210°, 0), (300°, −1), and (390°, 0).

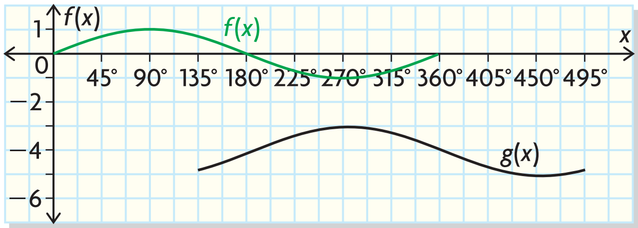
Since there's a vertical translation of 4, all the *y*-values for my new key points will be increased by 4. By adding 4 to the *y*-coordinate of each of my five key points, they change to (30°, 4), (120°, 5), (210°, 4), (300°, 3), and (390°, 4).

I plotted my new points and sketched the curve.

From the graph, I determined the equation to be $f(x) = \sin(x - 30°) + 4$.

EXAMPLE 5 Identifying transformations of $f(x) = \sin x$ from a graph

Explain what transformations were applied to $f(x) = \sin x$ (green curve) to create the black curve, $g(x)$.



Sasha's Solution

$270° - 90 = 180°$

horizontal translation: $+180°$

$f(x) = \sin(x - 180°)$

There are two transformations; I'll start with the horizontal one and then move on to the vertical transformation. I looked at the x-coordinate of the first peak of the original graph; it was $x = 90°$. Then I looked at the x-coordinate of the first peak of the transformed graph; it was $x = 270°$. So the graph has moved to the right by 180°. I must have a horizontal translation of 180°.

vertical translation: -4

$f(x) = \sin(x - 180°) - 4$

I looked at the y-coordinate of the first peak of the original graph; it was $y = 1$. Then I looked at the y-coordinate of the first peak of the transformed graph; it was $y = -3$. So the graph has moved down by 4 units. I must have a vertical translation of -4.

In Summary

Key Idea

- The graph of the function $f(x) = \sin(x - c°) + d$ is congruent to the graph of $f(x) = \sin x$. The differences are only in the placement of the graph.

Need to Know

- Transformations that move a function left/right as well as up/down are called translations.
- The graph of the function $f(x) = \sin(x - c°) + d$ can be drawn from the graph of $f(x) = \sin x$ by applying the appropriate translations one at a time to the key points of $f(x) = \sin x$.
  - Move the graph of $f(x) = \sin x$ $c°$ to the right when $c > 0$.
  - Move the graph of $f(x) = \sin x$ $c°$ to the left when $c < 0$.
  - Move the graph of $f(x) = \sin x$ d units down when $d < 0$.
  - Move the graph of $f(x) = \sin x$ d units up when $d > 0$.
- A vertical translation affects the equation of the axis and the range of the function, but has no effect on the period, amplitude, or domain. A horizontal translation slides a graph to the left or right, but has no effect on the period, amplitude, equation of the axis, domain, or range.

CHECK *Your Understanding*

1. State the transformations for each function, and determine the domain and range.

a) $f(x) = \sin(x + 40°)$

b) $f(x) = \sin x + 8$

c) $f(x) = \sin(x - 60°)$

d) $f(x) = \sin x - 5$

2. Sketch $f(x) = \sin(x - 90°) - 5$, and verify with graphing technology.

3. a) The graph of $f(x) = \sin x$ has been translated to the left 70° and up 6 units. Write the new equation.

b) State the amplitude, period, and equation of the axis of the new function.

c) State the domain and range of the new function.

PRACTISING

4. State the transformations for each sinusoidal function, and then sketch its graph.
K

a) $f(x) = \sin(x - 20°)$

b) $f(x) = \sin x + 5$

c) $f(x) = \sin(x - 150°) - 6$

d) $f(x) = \sin(x + 40°) - 7$

e) $f(x) = \sin(x + 30°) - 8$

f) $f(x) = \sin(x + 120°) + 3$

5. For question 4, what feature or features of the sinusoidal functions are alike? Choose from the period, amplitude, equation of the axis, domain, and range.

6. a) The function $f(x) = \sin x$ undergoes a horizontal translation of 15° and a vertical translation of 4. Write the new equation.

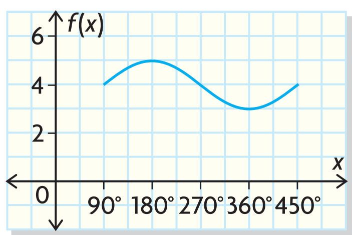
b) The function $f(x) = \sin x$ undergoes a vertical translation of -7 and a horizontal translation of 60°. Write the new equation.

c) The graph of $f(x) = \sin x$ has been translated to the right 45° and up 3 units. Write the new equation.

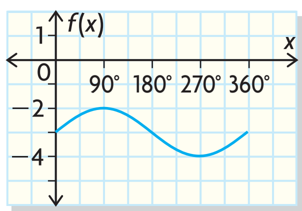
7. The graph of $f(x) = \sin x$ has been translated down 5 units and to the left 30°. Write the new equation.

8. For each graph, determine whether a horizontal or vertical translation (or both) has occurred to the graph of $f(x) = \sin x$. If so, indicate how much the graph has been translated.

a)



b)



9. Sketch each sinusoidal function, and verify your answers using graphing technology.

a) $f(x) = \sin x + 5$

b) $f(x) = \sin(x - 120°)$

c) $f(x) = \sin(x - 30°) + 4$

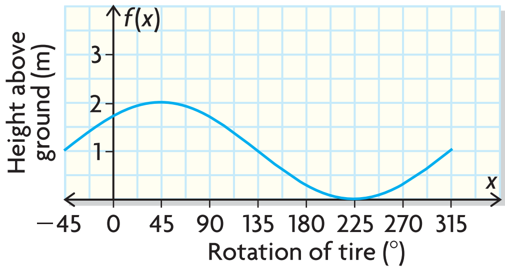
d) $f(x) = \sin(x - 60°) + 1$

e) $f(x) = \sin(x - 90°) - 2$

f) $f(x) = \sin(x + 30°) - 1$

10. A tire has a pebble stuck in its tread. The height of the pebble above the ground in terms of the rotation of the tire in degrees can be modelled by the graph shown.

Height of a Pebble



a) What is the period of this function?

b) What is the equation of the axis? What does it represent in this situation? What transformation is characterized by the equation of the axis?

c) What is the amplitude of the function? What does it represent in this situation?

d) Besides the transformation identified in part (b), what other transformation has occurred?

11. Two tires have pebbles stuck in their treads. The heights of the pebbles above the ground, in terms of the rotation of the tires, can be modelled by the following functions, where the dependent variable is the height in metres and the independent variable is the rotation in degrees:

Tire 1: $f(x) = \sin(x + 45°) + 1$

Tire 2: $g(x) = \sin(x - 90°) + 1$

a) What transformations do these functions share?

b) What transformations are different?

c) How are the real-world situations represented by the two functions the same? How are they different?

12. Determine the equations of three sinusoidal functions that would have the range $\{y \in \mathbf{R} \mid 3 \le y \le 5\}$.

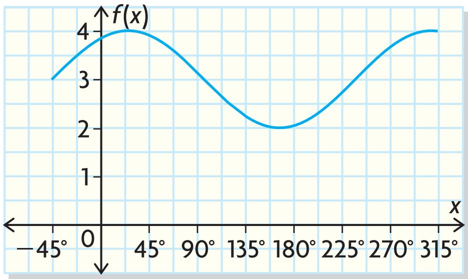
13. Create a chart that compares and contrasts horizontal and vertical translations.

14. C Explain how you can determine from the equation of the function whether a sinusoidal function has undergone a vertical translation and a horizontal translation.

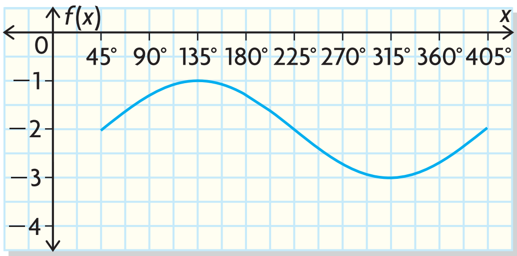
Extending

15. Write the equation for each sinusoidal function.

a)



b)

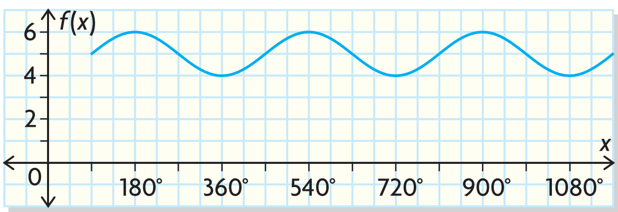


16. Three students were asked to determine the equation of the sinusoidal function shown. Whose answer is correct? Explain.

Student 1: $y = \sin(x - 90°) + 5$

Student 2: $y = \sin(x - 450°) + 5$

Student 3: $y = \sin(x - 810°) + 5$



6.6 More Transformations of sin x: $f(x) = a \sin x$

YOU WILL NEED

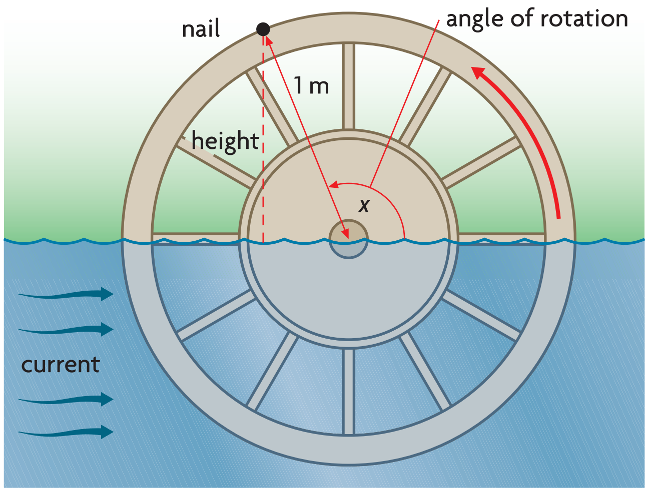
- graphing calculator

GOAL

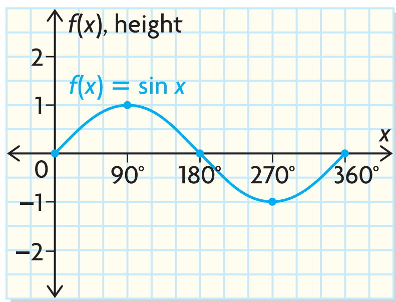
Determine how the value a affects the function $f(x) = a \sin x$.

INVESTIGATE the Math

The graph of $f(x) = \sin x$, based on the water wheel, relates the height of a point on the wheel to the water level and the angle of rotation.



5 Key Points for $f(x) = \sin x$	
x, Angle of Rotation	$f(x)$, Height (m)
0°	0
90°	1
180°	0
270°	−1
360°	0



? What characteristic of the water wheel is directly related to a in the graph of $f(x) = a \sin x$, and how does changing this value affect the graph of $f(x) = \sin x$?

A. Predict how the graph of $f(x) = \sin x$ will change if $\sin x$ is multiplied by
- a positive integer
- a positive fraction (less than 1)
- a negative integer

Explain your predictions.

B. Graph $f(x) = \sin x$ in Y1, using the calculator settings in the margin. Investigate the three functions in the chart by graphing each function in Y2, one at a time. Compare the graph of each with the graph of $f(x) = \sin x$. Compare the table of values of both graphs using the settings given in the chart. Use this information to sketch the transformed function, and then describe how the characteristics (i.e., amplitude, range, and so on) have changed.

Equation	**Table Settings and Values**	**Sketch of the New Graph**	**Description of How the Graph Has Changed (i.e., amplitude, range, and so on)**
$g(x) = 2 \sin x°$	TblStart = 0 ΔTbl = 90		
$h(x) = 0.5 \sin x°$	TblStart = 0 ΔTbl = 90		
$j(x) = -\sin x°$	TblStart = 0 ΔTbl = 90		

C. Were your descriptions in the last column what you predicted?

D. For each new function, describe the type of transformation that produced it.

E. For the functions $f(x) = 2 \sin x$ and $f(x) = 0.5 \sin x$, how would the characteristics of the water wheel or its placement be different?

Reflecting

F. What characteristic (amplitude, period, or equation of the axis) of the graph is affected by multiplying or dividing the value of $\sin x$ by 2? Why might you call one transformation a stretch and the other a compression?

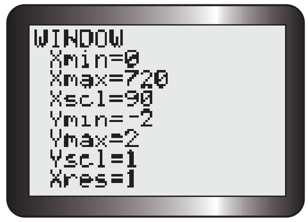
G. How would the function change for a water wheel of radius 3 m? What would the graph look like?

H. If $g(x) = a \sin x$, what effect does the value of a have on the function $f(x) = \sin x$ when

a) $a > 1$? **b)** $0 < a < 1$? **c)** $a = -1$?

Tech Support

If you are graphing the functions on a graphing calculator, make sure that the calculator is in DEGREE mode and that you are using the following WINDOW settings:



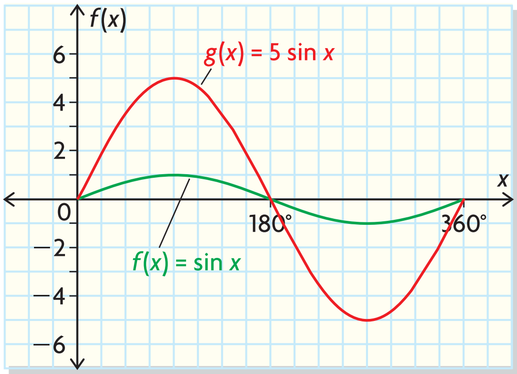
For help on using and changing the table settings, see Technical Appendix, B-6.

APPLY the Math

EXAMPLE 1 Connecting the effect of a vertical stretch on the function $f(x) = \sin x$

How does the value 5 in the function $g(x) = 5 \sin x$ change $f(x) = \sin x$?

Liz's Solution



I studied the equation and looked at the parameter *a*.

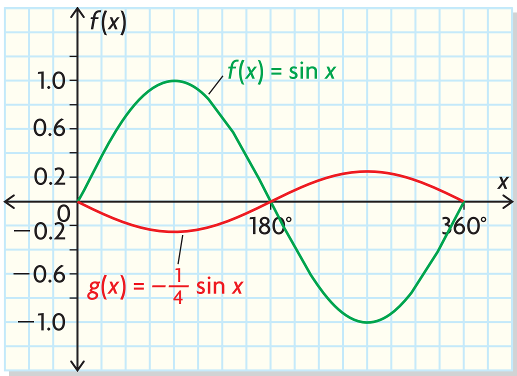
There's a vertical stretch of a factor of 5 because sin *x* is multiplied by 5. Normally my amplitude is 1; now my amplitude is 5 because all the *y*-coordinates of the points on $f(x)$ have been multiplied by 5. The zeros of the function are not affected by the stretch.

Multiplying by 5 has changed the amplitude. The equation of the axis is still $y = 0$. The period is unchanged and is still 360°. The domain is unchanged, $\{x \in \mathbf{R}\}$, but the range has changed to $\{y \in \mathbf{R} \mid -5 \leq y \leq 5\}$.

EXAMPLE 2 Connecting the effect of a vertical compression and reflection on the function $f(x) = \sin x$

How does the value $-\frac{1}{4}$ in the function $g(x) = -\frac{1}{4} \sin x$ change $f(x) = \sin x$?

Liz's Solution



I studied the equation and looked at the parameter *a*.

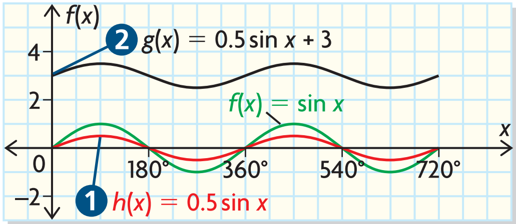
I can tell that there's a vertical compression of a factor of $\frac{1}{4}$ and a reflection in the *x*-axis because sin *x* is multiplied by $-\frac{1}{4}$. Normally my amplitude is 1; now my amplitude is $\frac{1}{4}$ because all the *y*-coordinates of the points on $f(x) = \sin x$ have been divided by −4 (or multiplied by $-\frac{1}{4}$). The zeros of the function are not affected by the compression.

Multiplying by $-\frac{1}{4}$ has changed the amplitude and caused a reflection in the *x*-axis. The equation of the axis is still $y = 0$. The period is unchanged and is still 360°. The domain is unchanged, $\{x \in \mathbf{R}\}$, but the range has changed to $\{y \in \mathbf{R} \mid -\frac{1}{4} \leq y \leq \frac{1}{4}\}$.

EXAMPLE 3 Connecting the effect of a vertical stretch on the function $f(x) = \sin x$

How do the values 0.5 and 3 in the function $g(x) = 0.5 \sin x + 3$ change $f(x) = \sin x$?

Anne's Solution



I studied the equation and looked at each parameter, one at a time.

1. There's a vertical compression by a factor of 0.5 because sin x is multiplied by 0.5. Normally, my amplitude is 1; now my amplitude is 0.5. This results in the graph of $h(x) = 0.5 \sin x$ (in red).

2. There's a vertical translation of 3 because 3 is added to the height of each point, so I have to slide the graph up 3 units. This results in the graph of $g(x) = 0.5 \sin x + 3$ (in black).

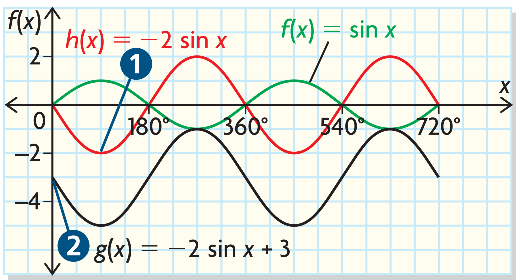
My answer looks reasonable because the period of my graph is still 360°, the amplitude is 0.5, and the equation of the axis is $y = 3$.

Multiplying by 0.5 has changed the amplitude, and adding 3 has changed the equation of the axis to $y = 3$. The period is unchanged and is still 360°. The domain is unchanged, $\{x \in \mathbf{R}\}$, but the range has changed to $\{y \in \mathbf{R} \mid 2.5 \leq y \leq 3.5\}$.

EXAMPLE 4 Connecting the effect of a reflection on the function $f(x) = \sin x$

How do the values -2 and -3 in the function $g(x) = -2 \sin x - 3$ change $f(x) = \sin x$?

Colin's Solution



I studied the equation and looked at each parameter, one at a time.

1. The negative sign in front of the 2 tells me there is a reflection in the x-axis, so my graph starts by going down, rather than up. There is a vertical stretch of 2. That means that my graph has an amplitude of 2. This results in the graph of $h(x) = -2 \sin x$ (in red).

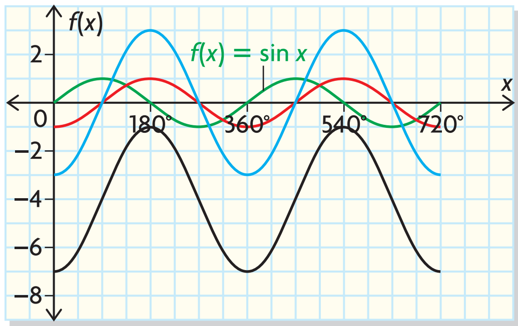
2. There is also a vertical translation of -3, so I have to slide the graph down 3 units. This results in the graph of $g(x) = -2 \sin x - 3$ (in black).

I think my graph is correct. The period is 360°, the amplitude is 2, and the equation of the axis is $y = -3$.

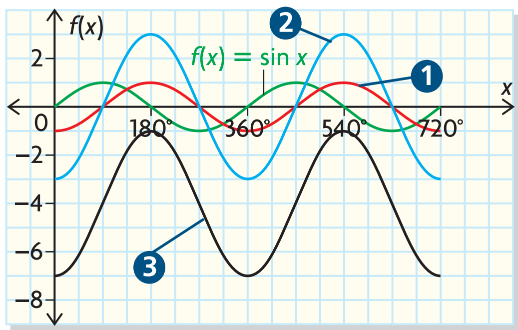
Multiplying by -2 has changed the amplitude and caused a reflection in the x-axis. Subtracting 3 has changed the equation of the axis to $y = -3$. The period is unchanged and is still 360°. The domain is unchanged, $\{x \in \mathbf{R}\}$, but the range has changed to $\{y \in \mathbf{R} \mid -5 \leq y \leq -1\}$.

EXAMPLE 5 Representing the equation of a transformed sine function

The green curve $f(x) = \sin x$ has been transformed to the black curve by means of three transformations. Identify them, and write the equation of the new function.



Ryan's Solution



I'm dealing with three transformations. There's a horizontal translation of 90° to the right (red curve), followed by a vertical stretch by a factor of 3 (blue curve), and then a vertical translation of 4 downward (black curve).

1. The equation for the red curve is $f(x) = \sin(x - 90°)$, since the green curve has been moved 90° to the right.

2. The equation for the blue curve is $f(x) = 3\sin(x - 90°)$, since the red curve has been stretched by a factor of 3.

3. The equation for the black curve is $f(x) = 3\sin(x - 90°) - 4$, since the blue curve has been shifted down 4 units.

The equation for the black curve is $f(x) = 3\sin(x - 90°) - 4$.

In Summary

Key Idea

- The graph of the function $f(x) = a \sin(x - c°) + d$ looks periodic in the same way the graph of $f(x) = \sin x$ does. The differences are only in the placement of the graph and how stretched or compressed it is.

Need to Know

- If $f(x) = a \sin x$, the value of a has the following effect on the function $f(x) = \sin x$:
  - When $a > 1$, the function is stretched vertically by the factor a.
  - When $0 < a < 1$, the function is compressed vertically by the factor a.
  - When $a < -1$, the function is stretched vertically by the factor a and reflected across the x-axis.
  - When $-1 < a < 0$, the function is compressed vertically by the factor a and reflected across the x-axis.
- A function of the form $f(x) = a \sin(x - c°) + d$ results from applying transformations to the graph of $f(x) = \sin x$ in the following order:
  1. Horizontal translations: determined by the value of c
  2. Stretches/compressions: determined by the value of a; Reflections: necessary only when $a < 0$
  3. Vertical translations: determined by the value of d

CHECK Your Understanding

1. State the transformations that are applied to $f(x) = \sin x$.

a) $f(x) = 3 \sin x$

b) $f(x) = -2 \sin x$

c) $f(x) = 0.1 \sin x$

d) $f(x) = -\frac{1}{3} \sin x$

2. The graph of $f(x) = \sin x$ has been compressed by a factor of 5 and reflected in the x-axis. Write the new equation.

3. a) Sketch the sinusoidal function $f(x) = 4 \sin x$. Verify your answer with graphing technology.

b) State the period, amplitude, and the equation of the axis.

c) State the domain and range.

PRACTISING

4. State whether a vertical stretch or a compression results for each of the following.

a) $f(x) = 2 \sin x$

b) $f(x) = 0.5 \sin x$

c) $f(x) = -\frac{1}{4} \sin x$

d) $f(x) = -4 \sin x$

e) $f(x) = \frac{1}{3} \sin x$

f) $f(x) = 10 \sin x$

5. State the transformations for each sinusoidal function.

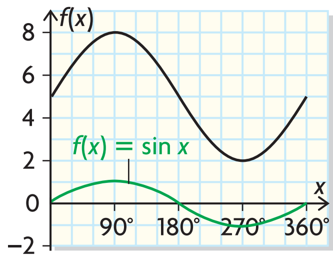
   a) $f(x) = 3 \sin (x + 20°)$
   b) $f(x) = -\sin x - 3$
   c) $f(x) = 5 \sin(x - 50°) - 7$
   d) $f(x) = -2 \sin x + 6$
   e) $f(x) = -7 \sin (x + 10°)$
   f) $f(x) = -0.5 \sin (x - 30°) + 1$

6. For each of the functions in question 5:

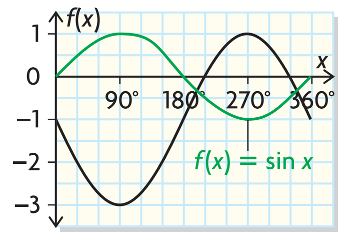
   a) State the amplitude, period, and equation of the axis.
   b) State the domain and range.

7. For each graph, determine whether a horizontal translation, a vertical stretch and/or a reflection in the x-axis, or a vertical translation has occurred to the graph of $f(x) = \sin x$. Describe the effect of each transformation.

   a)



   b)



8. Complete the following statements: If a sinusoidal function is of the form $f(x) = -a \sin (x - c°) + d$, then
K

   a) the negative sign in front of the a means that a ________ has occurred
   b) the a means that a ________ has occurred
   c) the c means that a ________ has occurred
   d) the d means that a ________ has occurred

9. Write the new equation for each function.

   a) The graph of $f(x) = \sin x$ has been translated to the right 135° and up 5 units.
   b) The graph of $f(x) = \sin x$ has been translated to the left 210° and down 7 units.

10. The function $f(x) = \sin x$ is stretched by a factor of 4 and then translated down 5 units.

    a) Sketch the resulting graph.
    b) What is the resulting equation of the function?

11. The function $f(x) = \sin x$ is reflected across the x-axis and translated up 4 units.

    a) Sketch the resulting graph.
    b) What is the resulting equation of the function?

12. Predict the maximum and the minimum values of $f(x)$ for each sinusoidal function. Verify your answers using graphing technology.

a) $f(x) = 3 \sin x$ **c)** $f(x) = 4 \sin x + 6$

b) $f(x) = -2 \sin x$ **d)** $f(x) = 0.5 \sin x - 3$

13. Sketch each sinusoidal function. Verify your answers using graphing technology.

a) $f(x) = 4 \sin x + 5$ **d)** $f(x) = 2 \sin x - 3$

b) $f(x) = -3 \sin x + 6$ **e)** $f(x) = 3 \sin (x - 45°) + 2$

c) $f(x) = -0.5 \sin x$ **f)** $f(x) = -4 \sin (x + 90°) - 5$

14. What characteristics (amplitude, period, or equation of the axis) do all the sinusoidal functions in question 13 have in common?

15. A Suppose the height of a Ferris wheel is modelled by the function $h(\theta) = 8 \sin (\theta - 45°) + 10$, where $h(\theta)$ is the height in metres and θ is the number of degrees one has rotated from the boarding position.

a) What does the 10 mean in terms of a Ferris wheel?

b) What does the 10 mean in terms of transformations?

c) Sketch the function. Verify your answer using graphing technology.

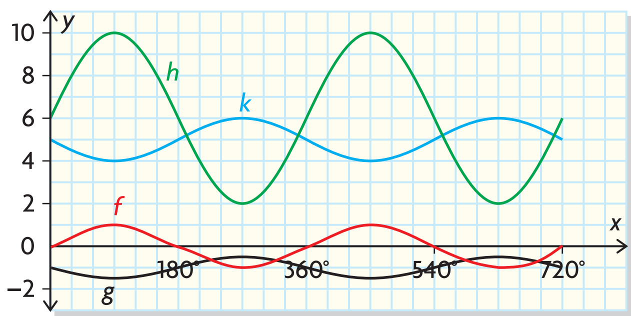
d) Determine the range of the function.

e) What is the amplitude of the function, and what does it represent in this situation?

f) When an individual has rotated 200° from the boarding position, how high above the ground is the individual?

16. T Draw a diagram of the Ferris wheel that could have formed the function $h(\theta) = -20 \sin (\theta + 90°) + 21$, where $h(\theta)$ is the height in metres and θ is the number of degrees the rider has rotated from the boarding position. Include all relevant measurements on your diagram, and indicate the direction the wheel is rotating.

17. **a)** Determine the equation of each function in the graph shown.



b) If the period of each function were changed from 360° to 180°, what type of transformation would you be dealing with?

18. Explain how you can graph a sinusoidal function that has undergone **C** more than one transformation.

Extending

19. State the period of each function.

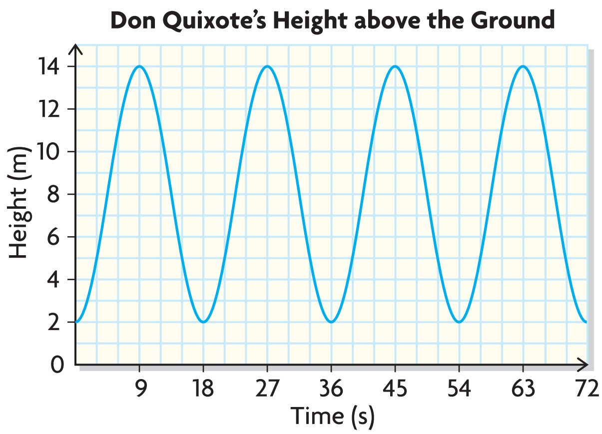
a) $f(x) = \sin\left(\frac{1}{2}x\right)$

b) $f(x) = \sin\left(\frac{1}{4}x\right)$

c) $f(x) = \sin(2x)$

d) $f(x) = \sin(10x)$

20. Don Quixote, a fictional character in a Spanish novel, attacked windmills, thinking they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above the ground in terms of time.



a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

c) What is the period of the function, and what does it represent in this situation?

d) What transformation would generate the period for this graph?

e) Determine the equation of the sinusoidal function.

f) If the wind speed decreased, how would that affect the graph of the function?

g) If the axle for the windmill were 1 m higher, how would that affect the graph of the function?

6 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you transform a sinusoidal function?

A:

Function	Type and Description of Transformations
$f(x) = \sin(x) + d$	The value d represents a vertical translation. If d is positive, then the graph shifts up the y-axis by the amount d. If d is negative, then the graph shifts down the y-axis by the amount d. This type of transformation will affect the equation of the axis and the range of the function.
$f(x) = \sin(x - c°)$	The value c represents a horizontal translation. If c is positive, then the graph shifts to the left on the x-axis by the amount c. If c is negative, then the graph shifts to the right on the x-axis by the amount c.
$f(x) = a \sin x$	The value a represents the vertical stretch/compression, which changes the amplitude of the sine function. If a is negative, it also represents a reflection of the function in the x-axis.

Study *Aid*

- See Lesson 6.5, Examples 1, 2, and 3.
- See Lesson 6.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 4, 5, and 9.

Q: How do I determine the domain and range of a sinusoidal function from its equation?

A: Sinusoidal functions cover all positive and negative values of x. Therefore, the domain of any sinusoidal function is $\{x \in \mathbf{R}\}$. To determine the range of a sinusoidal function, you must calculate the equation of the axis, based on the vertical translation, and then the amplitude, based on the vertical stretch. Determine the equation of the axis, and then go above and below that value an amount equivalent to the amplitude. For example, if the equation of the axis is $y = 7$, and the amplitude is 3, then you would go 3 units above and below 7. That would mean that the maximum value for y is 10 $(7 + 3)$, and the minimum value for y is 4 $(7 - 3)$.

Study *Aid*

- See Lesson 6.5, Example 1.
- Try Chapter Review Questions 6, 11, and 13.

PRACTICE Questions

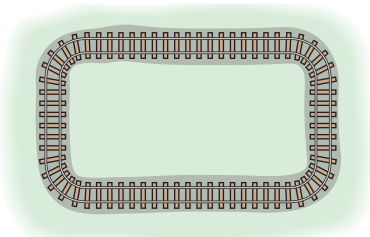
Lesson 6.2

1. Which of the following situations would produce periodic graphs?
   a) You are looking at a lighthouse.
   - independent variable: time
   - dependent variable: the intensity of the light from your perspective

   b) You are going to a ride a Ferris wheel.
   - independent variable: time
   - dependent variable: your height above the ground

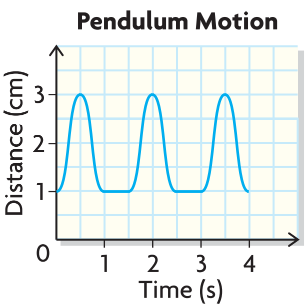
   c) Travis has a floating dock on an ocean.
   - independent variable: time
   - dependent variable: the height of the dock relative to the ocean floor

   d) You are watching a train that is travelling at a constant speed around the track shown.
   - independent variable: time
   - dependent variable: the distance between you and the train



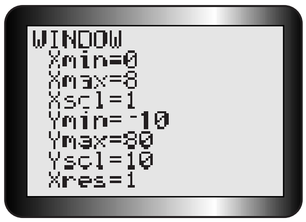
Lesson 6.3

2. A pendulum is swinging back and forth. However, the motion of the pendulum stops at regular intervals. The distance in terms of time between the pendulum and an interior wall of the device can be represented by this graph.



   a) Is this function periodic, sinusoidal, or both?
   b) How long is the pendulum at rest at any one time?
   c) What is the farthest distance the pendulum will be from the interior wall of the device?
   d) What is the period of the function? Include the units of measure.
   e) If the pendulum was momentarily stopped for a longer period, how would the graph change?

3. A machine used to remove asphalt from roads has a large rotating drum covered with teeth. The height, $f(t)$, in terms of time of one of these teeth relative to the ground can be modelled by the function $f(t) = 35 \sin (360t)° + 30$. The height is measured in centimetres, and time is measured in seconds. Using graphing technology with the WINDOW settings shown, answer the following questions.



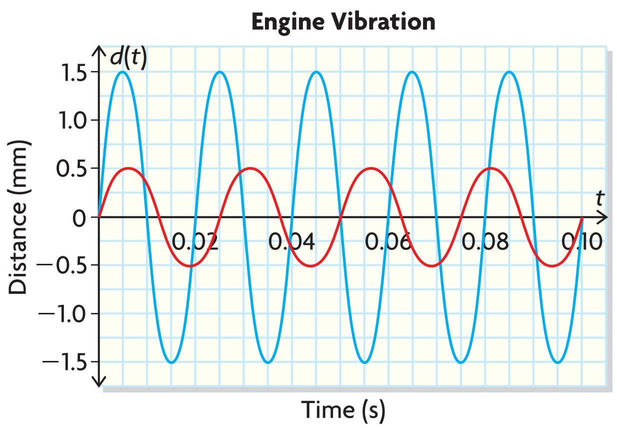
   a) What is the minimum height of the tooth? Does this number make sense? Explain.
   b) What is the equation of the axis, and what does it represent in this situation?
   c) What is the period of the function, and what does it represent in this situation? (*Hint:* The period of this function is going to be quite short.)
   d) What is the amplitude of the function, and what does it represent in this situation?

4. Sketch three cycles of a sinusoidal function that has a period of 360°, an amplitude of 3, and whose equation of the axis is $y = -1$.

5. Sketch two cycles of a sinusoidal function that has a period of 360° and an amplitude of 2 and whose equation of the axis is $y = 3$.

Lesson 6.4

6. Two different engines idling at different speeds are vibrating on their engine mounts. The distance, $d(t)$, in terms of time, t, that the engines vibrate to the left and right of their resting position can be modelled by the graphs shown.



a) Determine the range of each function.
b) What is the period of each function? What does the period tell you about the engines?
c) What is the equation of the axis for each function, and what does it represent in this situation?
d) What is the amplitude of each function? What does this tell you about the engines?

Lesson 6.5

7. State the amplitude, period, equation of the axis, and maximum and minimum values of $f(x)$ for each sinusoidal function.
a) $f(x) = \sin x + 3$
b) $f(x) = \sin(x + 60°) - 2$

8. Sketch each sinusoidal function in question 7, and state the domain and range. Verify your answers using graphing technology.

Lesson 6.6

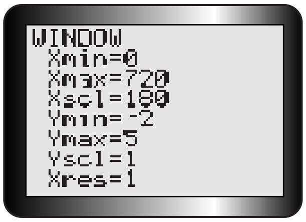
9. a) The function $f(x) = \sin x$ undergoes a reflection in the x-axis, a vertical stretch of 0.5, and a vertical translation of 4. What is the equation of the resulting function?
b) Sketch the resulting graph.

10. State the amplitude, period, equation of the axis, and maximum and minimum values of $f(x)$ for each sinusoidal function. Verify your answers using graphing technology.
a) $f(x) = 3 \sin x$
b) $f(x) = -2 \sin x$
c) $f(x) = 4 \sin x + 6$
d) $f(x) = -0.25 \sin x$
e) $f(x) = 3 \sin(x + 45°)$

11. Sketch each sinusoidal function, and state the domain and range. Verify your answers using graphing technology.
a) $f(x) = 3 \sin x + 5$
b) $f(x) = -\sin x - 2$
c) $f(x) = 0.5 \sin x - 1$
d) $f(x) = 2 \sin(x - 90°)$
e) $f(x) = -0.5 \sin(x + 90°) + 3$

12. The height of a Ferris wheel is modelled by the function $h(\theta) = 6 \sin(\theta - 45°) + 7$, where $h(\theta)$ is in metres and θ is the number of degrees the wheel has rotated from the boarding position of a rider.
a) Sketch the graph of the function, and verify your answer using graphing technology.
b) Determine the range of the function.
c) What is the amplitude of the function, and what does it represent in this situation?
d) When the rider has rotated 400° from the boarding position, how high above the ground is the rider?

6 Chapter Self-Test



1. Graph each sinusoidal function. If you are using a graphing calculator, set it to DEGREE mode and use the WINDOW settings shown at the left. State the amplitude, period, equation of the axis, domain, and range for each function.

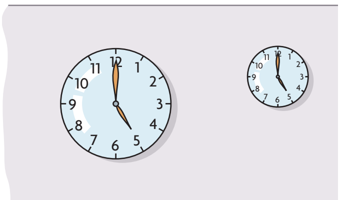
a) $f(x) = \sin x + 4$

b) $f(x) = \sin(x + 30°)$

c) $f(x) = 2 \sin x$

d) $f(x) = -0.5 \sin x$

2. Sketch three cycles of a sinusoidal function that has a period of 30, an amplitude of 6, and whose axis is $y = 11$.



3. The hands on a wall clock move in a predictable manner. As time passes, the distance between the tip of the minute hand and the ceiling changes. Suppose we have two different wall clocks. The two graphs model the relationship between distance and time.

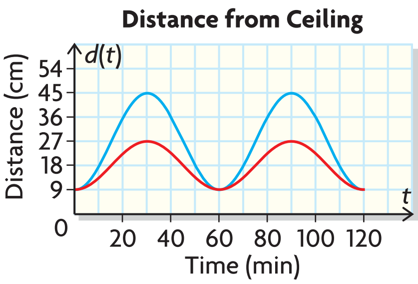
a) Determine each of the following and explain what they represent in this situation: period, equation of the axis, and amplitude.

b) What was the initial position of the tip of the minute hand?

c) What is the range for each function?

d) Approximate the distance between the minute hand and the ceiling for each clock at $t = 80$ min.

e) Draw a diagram showing the position of the clocks relative to the ceiling. Include all relevant numbers on your diagram.



4. The function $f(x) = \sin x$ undergoes a reflection in the x-axis and a vertical stretch of 2.

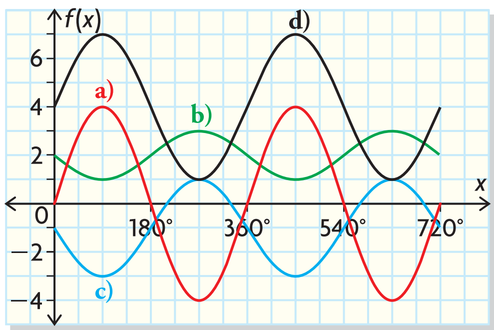
a) Write the resulting equation of the function.

b) Sketch the resulting graph.

c) State the amplitude, equation of the axis, period, domain, and range.

5. Sketch the sinusoidal function $f(x) = \sin(x + 90°) + 5$. Verify your answer using graphing technology.

6. Determine the equation of each function in the graph shown.



6 Chapter Task

Designing a Ferris Wheel

The London Eye, the world's largest Ferris wheel, is located in London, England. Completed in March 2000, it stands 135 m high and has 32 passenger capsules. Each capsule weighs 10 tonnes, carries 25 passengers, and takes 30 min to complete one revolution.

In 2008, Shanghai, China, expects to complete an even larger wheel. The wheel will have a diameter of 170 m and sit atop a 50 m entertainment complex. Each of the 36 passenger capsules will carry 30 passengers. Like the London Eye, it will take 30 min to complete one revolution.

? How can you design a Ferris wheel larger than the Shanghai wheel?

A. Draw a scale diagram of your Ferris wheel beside a scale diagram of the London Eye and the Shanghai wheel.

B. On the same coordinate system, draw three graphs modelling the height of a passenger in terms of time for all three wheels.

C. Determine the speed that a passenger would travel around your wheel in metres per minute.

D. Determine the period, amplitude, and equation of the axis for each graph.

E. Determine the range for each graph.

F. Explain the process you used in designing your wheel.

Task *Checklist*

- ✔ Do the numbers that you are using seem reasonable?
- ✔ Is your diagram of the Ferris wheels drawn to scale?
- ✔ Are the patrons riding the Ferris wheel travelling at a reasonable speed?
- ✔ Have you shown all necessary calculations and provided written explanations so that your teacher can understand your reasoning?

Cumulative Review

Multiple Choice

1. A T-ball player hits a baseball from a tee that is 1 m tall. The flight of the ball can be modelled by $h(t) = -5t^2 + 10t - 1$, where $h(t)$ is the height in metres and t is the time in seconds. When does the ball reach its maximum height?

a) 0.5 s
b) 1.00 s
c) 1.60 s
d) 1.5 s

2. A rock is dropped from the edge of a 180 m cliff. The function $h(t) = -5t^2 - 5t + 180$ gives the approximate height of the rock, $h(t)$, in metres t seconds after it was released. How long does it take for the rock to reach a ledge 80 m above the base of the cliff?

a) 5 s
b) 6 s
c) 3 s
d) 4 s

3. The function $f(x) = -5x^2 + 20x + 2$ in vertex form is

a) $f(x) = -5(x - 2)^2 + 18$
b) $f(x) = -5(x - 2)^2 + 22$
c) $f(x) = -5(x + 2)^2 - 22$
d) $f(x) = -5(x + 2)^2 - 18$

4. Which of the following quadratic equations has no solution?

a) $2x^2 - 4x = 2x - 3$
b) $2x^2 - 15x - 8 = 0$
c) $16(x + 1)^2 = 0$
d) $3(x + 5)^2 + 7 = 0$

5. Identify which parabola does not intersect the x-axis.

a) $f(x) = x^2 - 6x + 7$
b) $f(x) = 9 - x^2$
c) $f(x) = (4 + x)^2$
d) $f(x) = -2(x - 1)^2 - 1$

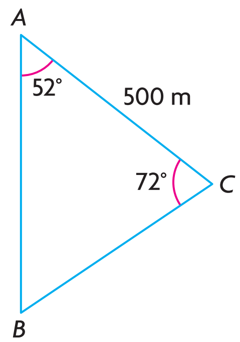
6. Two support wires are fastened to the top of a TV satellite dish tower from two points on the ground, A and B, on either side of the tower. One wire is 18 m long, and the other is 12 m long. The angle of elevation of the longer wire is 28°. How tall is the satellite dish tower?

a) 24.41 m
b) 18.2 m
c) 6 m
d) 8.45 m

7. Two airplanes leave the same airport in opposite directions. At 2:00 p.m., the angle of elevation from the airport to the first plane is 48° and to the second plane 59°. The elevation of the first plane is 5.5 km, and the elevation of the second plane is 7.2 km. Determine the air distance between the two airplanes to the nearest tenth of a kilometre.

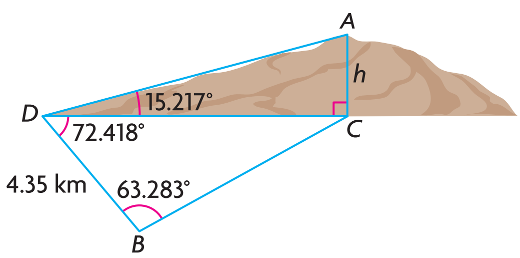
a) 9.4 km
b) 8.5 km
c) 15.7 km
d) 7.8 km

8. Determine the length of AB to the nearest metre.



a) 524 m
b) 544 m
c) 574 m
d) 564 m

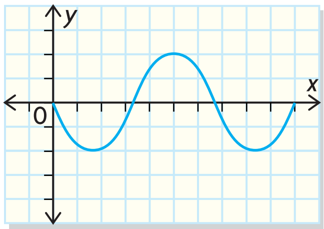
9. Use this diagram to determine the height, h, of the mountain.



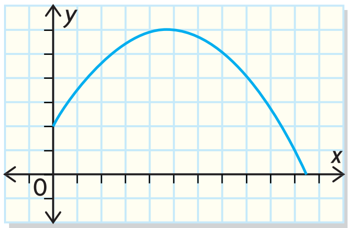
a) 2.5 km
b) 1.98 km
c) 1.5 km
d) 1.87 km

10. Which function is both periodic and sinusoidal?

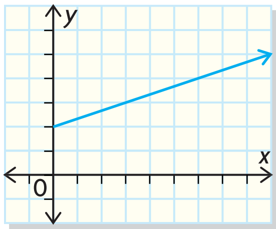
a)



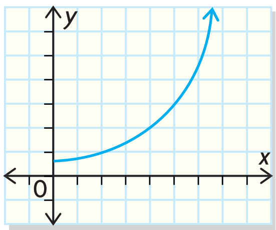
b)



c)



d)

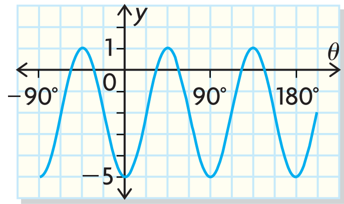


a) (b) **c)** (b) and (c)
b) (a) and (d) **d)** (a)

11. Identify the amplitude and equation of the axis of $f(x) = 2 \sin x + 5$.
a) amplitude: 2; equation of axis: $y = 5$
b) amplitude: 5; equation of axis: $y = 2$
c) amplitude: 2; equation of axis: $y = -5$
d) amplitude: 5; equation of axis: $y = 2$

12. What is the range of $f(x) = -4 \sin x - 2$?
a) $\{y \in \mathbf{R} \mid -6 \leq y \leq 2\}$
b) $\{y \in \mathbf{R} \mid -2 \leq y \leq 6\}$
c) $\{y \in \mathbf{R} \mid -4 \leq y \leq 4\}$
d) $\{y \in \mathbf{R} \mid -4 \leq y \leq 2\}$

13. Identify the correct amplitude and period.



a) amplitude: 1; period: 360°
b) amplitude: 3; period: 90°
c) amplitude: −5; period: 180°
d) amplitude: 3; period: 180°

14. Identify the transformations you would apply to $f(x) = \sin x$ to graph $f(x) = 0.5 \sin(x - 30°)$.
a) vertical stretch by 0.5, shift left 30°
b) vertical compression by 0.5, shift left 30°
c) vertical stretch by 0.5, shift right 30°
d) vertical compression by 0.5, shift right 30°

15. Identify the transformations you would apply to $f(x) = \sin x$ to graph $f(x) = -\sin x + 3$.
a) reflection in the x-axis, shift down 3
b) reflection in the y-axis, shift up 3
c) reflection in the x-axis, shift up 3
d) reflection in the y-axis, shift down 3

16. The function $f(x) = -2(x - 3)^2 + 5$ in standard form is
a) $f(x) = -2x^2 + 12x - 10$
b) $f(x) = -2x^2 + 6x + 10$
c) $f(x) = -2x^2 + 12x + 18$
d) $f(x) = -2x^2 + 12x - 13$

17. For the parabola defined by $f(x) = -3(x + 1)^2 - 4$, which of the following statements is not true?
a) The vertex is $(-1, -4)$.
b) The axis of symmetry is $x = 1$.
c) The parabola opens down.
d) The domain is $\{x \in \mathbf{R}\}$.

18. Given $y = -x^2 + 12x - 16$, state the coordinates of the vertex and the maximum or minimum value of y.
   a) vertex (6, 20), maximum 20
   b) vertex (−6, 20), minimum −6
   c) vertex (6, 20), minimum 6
   d) vertex (6, −20), maximum −20

19. The profit function for a new product is given by $P(x) = -4x^2 + 28x - 40$, where x is the number sold in thousands. How many items must be sold for the company to break even?
   a) 2000 or 5000
   b) 2000 or 3500
   c) 5000 or 7000
   d) 3500 or 7000

20. Which of the following statements is not true for the equation of a quadratic function?
   a) In standard form, the y-intercept is clearly visible.
   b) In vertex form, the break-even points are clearly visible.
   c) In factored form, the x-intercepts are clearly visible.
   d) In vertex form, the coordinates of the vertex are clearly visible.

21. Which of the following is not a step required to complete the square for $y = 7x^2 + 21x - 2$?
   a) $7(x^2 + 3x) - 2$
   b) $7x(x + 3) - 2$
   c) $7\left(x^2 + 3x + \frac{9}{4}\right) - \frac{71}{4}$
   d) $7\left(x^2 + 3x + \frac{9}{4} - \frac{9}{4}\right) - 2$

22. Which of the following statements is not true for a given quadratic function?
   a) The y-coordinate of the vertex represents the minimum or maximum value.
   b) The axis of symmetry is given by the x-coordinate of the vertex.
   c) The axis of symmetry is given by the y-coordinate of the vertex.
   d) The midpoint between the x-intercepts is the x-coordinate of the vertex.

23. A quadratic function in standard form will have two distinct real roots when
   a) $b^2 - 4ac < 0$
   b) $a^2 - 4bc > 0$
   c) $b^2 = 4ac$
   d) $b^2 - 4ac > 0$

24. Which value of k will produce one root for $y = -2(x + 7)^2 - k$?
   a) $k = 1$
   b) $k = 0$
   c) $k = -1$
   d) $k = -2$

25. The period of a periodic graph is
   a) the length of one cycle
   b) the distance from the maximum to the minimum value of the relation
   c) the same as the domain
   d) the same as the range

26. The equation of the axis of a periodic graph is
   a) $y = b$
   b) $y = mx + b$
   c) $y = \dfrac{\text{maximum value} + \text{minimum value}}{2}$
   d) $y = \dfrac{\text{maximum value} - \text{minimum value}}{2}$

27. Parallelogram $ABCD$ has sides of length 35 cm and 27 cm. The contained angle is 130°. The length of the longer diagonal is
   a) 27.2 cm
   b) 56.3 cm
   c) 22.5 cm
   d) 20.7 cm

28. In $\triangle ABC$, $\angle A = 85°$, $c = 10$ cm, and $b = 15$ cm. The height of $\triangle ABC$ is
   a) 17.3 cm
   b) 8.6 cm
   c) 13.8 cm
   d) 12.5 cm

29. In $\triangle PQR$, $\angle P = 70°$, $r = 5$ cm, and $q = 8$ cm. The area of $\triangle PQR$ is
   a) 13 cm^2
   b) 20 cm^2
   c) 18.8 cm^2
   d) 19.2 cm^2

Investigations

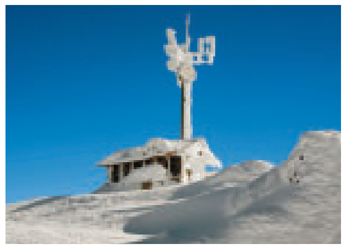
30. Designing a Football Field

Have you ever walked on a football field covered with artificial turf? If so, you probably noticed that the field is not flat. The profile of the surface is arched and highest in the centre, permitting rainwater to drain away quickly.

a) The diagram shows the profile of an actual field, viewed from the end of the field. Assuming that the cross-section is a parabola, determine the algebraic model that describes this shape.

b) Use your equation to determine the distance from the sidelines where the field surface is 20 cm above the base line.

31. How High Is the Tower?



A skier sees the top of a communications tower, due south of him, at an angle of elevation of 32°. He then skis on a bearing of 130° for 560 m and finds himself due east of the tower. Calculate

a) the height of the communications tower

b) the distance from the tower to the skier

c) the angle of elevation of the top of the tower from the new position

32. Amusement Rides

A popular ride at an amusement park is called the "Ring of Terror." It is like a Ferris wheel but is inside a haunted house. Riders board on a platform that is level with the centre of the "ring," and the ring moves counterclockwise. When a rider is moving above the platform, he or she meets flying creatures. When the seat descends to a level below the platform, creatures emerge from a murky, slimy pit. The radius of the ring is 6 m.

a) Graph the height of a rider with respect to the platform through three revolutions of the ride.

b) Determine the amplitude, period, equation of the axis, and the range of your graph.

c) Discuss how your graph would change if the rider got on the ride in the pit, at the bottom of the ring.