
ONE DOLLAR
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RESERVE BANK OF INDIA
GUARANTEED BY THE CENTRAL GOVERNMENT
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Qatar Central Bank
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SUID-AFRIKAANSE RESERWEBANK
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Chapter

8

Solving Financial Problems Involving Exponential Functions

GOALS

You will be able to

- Understand the difference between simple interest and compound interest
- Solve problems involving compound interest and annuities

? Money makes the world go round. People all over the world save, borrow, and consume. In the next 15 years, what are some of the things you will need to save for? What options do financial institutions provide to help you save money?

8 Getting Started

WORDS You Need to Know

1. Match each term with the example that illustrates it.

a) interest **b)** interest rate **c)** loan **d)** investment

i) 5% per year
ii) Last month I earned \$12.50 in my savings account.
iii) Each month I deposit \$50 into a Registered Retirement Savings Plan.
iv) I bought a new TV and financed most of the cost.

SKILLS AND CONCEPTS You Need

Expressing Percent as a Decimal

A percent is a way of expressing a number as a fraction of 100 (*per cent* means "per hundred").

EXAMPLE

Convert each percent to a decimal.

a) 27% **b)** 104% **c)** 0.5%

Solution

$$27\% = \frac{27}{100} = 0.27$$

$$104\% = \frac{104}{100} = 1.04$$

$$0.5\% = \frac{0.5}{100} = 0.005$$

2. Convert each percent to a decimal.

a) 35% **b)** 67% **c)** 8.5% **d)** 2.75%

Calculating the Percent of a Number

To determine the percent of a number, change the percent into a decimal by dividing by 100 and multiply by the number you are finding the percent of.

EXAMPLE

Determine each amount.

a) 5% of 60 **b)** 3.5% of 220

Solution

$$5\% = 0.05 \times 60 = 3$$

$$3.5\% = 0.035 \times 220 = 7.7$$

3. Determine each amount.

a) 15% of 75
b) 75% of 68
c) 3.5% of 60
d) 7.25% of 2000

Evaluating Exponential Functions

When evaluating exponential functions, substitute the given value for x, then calculate the value of the function. Remember BEDMAS!

EXAMPLE

If $f(x) = 4(3)^{2x}$, determine $f(-2)$.

Solution

$$\begin{aligned} f(-2) &= 4(3)^{2(-2)} \\ &= 4(3)^{-4} \\ &= 4\left(\frac{1}{3^4}\right) \\ &= 4\left(\frac{1}{81}\right) \\ &= \frac{4}{81} \end{aligned}$$

4. **a)** If $f(x) = 3(4)^{2x}$, determine $f(3)$ and $f(-3)$.
b) If $f(x) = 5(2)^{3x}$, determine $f(2)$ and $f(-2)$.
c) If $f(x) = 6(1.5)^{x}$, determine $f(1)$ and $f(-1)$.
d) If $f(x) = 20\ 000(1.02)^{x}$, determine $f(10)$ and $f(20)$.

Relating Units of Time

One year is the length of time it takes for Earth to complete its orbit around the Sun. One day is the length of time for Earth to rotate once, 360°, on its axis.

1 year = 365 days or 52 weeks or 12 months

EXAMPLE

Express each of the following as a fraction of a year.

a) 4 weeks **b)** 3 months **c)** 200 days

Solution

a) 4 weeks
$= \frac{4}{52}$ of a year
$= \frac{1}{13}$ of a year

b) 3 months
$= \frac{3}{12}$ of a year
$= \frac{1}{4}$ of a year

c) 200 days
$= \frac{200}{365}$
$= \frac{40}{73}$ of a year

5. Express each of the following as a fraction of a year.
   a) 30 days
   b) 26 weeks
   c) 8 months
   d) 400 days
   e) 100 weeks
   f) 18 months

6. a) Determine the number of months in 3.5 years.
   b) Determine the number of weeks in 2.25 years.
   c) Determine the number of days in 5 years.
   d) Determine the number of months in 0.25 years.

PRACTICE

Study *Aid*

- For help, see Essential Skills Appendix.

Question	Appendix
10	A-7

7. Write each percent as a decimal.
   a) 35%
   b) 5%
   c) 14.6 %
   d) 115%
   e) $2\frac{3}{4}\%$
   f) $14\frac{3}{4}\%$

8. Evaluate.
   a) 3.25% of \$150
   b) 14% of \$28
   c) 110% of \$225
   d) 1.5% of \$2000
   e) 4% of \$75
   f) 400% of \$500

9. Evaluate to two decimal places.
   a) $(1.10)^{12}$
   b) $(1.03)^{-24}$
   c) $1000(1.10)^{4}$
   d) $5000(1.07)^{-12}$
   e) $100 \times \dfrac{(1.12)^{10} - 1}{0.12}$
   f) $575 \times \dfrac{1 - (1.08)^{48}}{0.08}$

10. Sketch the graph of each function.
   a) $f(x) = 4x - 5$
   b) $f(x) = -\dfrac{2}{3}x + 6$
   c) $f(x) = 1.5x + 3$
   d) $f(x) = -2.25x - 10$

11. Sketch the graph of each function.
   a) $f(x) = 2^{x}$
   b) $f(x) = \left(\dfrac{1}{3}\right)^{x}$
   c) $f(x) = 5(3)^{x}$
   d) $f(x) = 10\left(\dfrac{1}{2}\right)^{x}$

12. An antique postage stamp appreciates by 9% of its value each year. The stamp was worth \$0.34 in 1969. What will its value be in 2012?

13. A new car bought for \$25 600 loses 12% of its value each year. Determine its value at the end of 7 years.

APPLYING What You Know

Saving Money

Suppose you deposit $40 into a bank account at the beginning of January and continue to put $40 into the account at the beginning of every month for the rest of the year. The bank account earns 6% interest per year and the interest is paid into the account at the end of every month.

? How much money will be in the account at the end of the year?

A. Why is the monthly interest rate 0.5%? Why can you write it as 0.005?

B. The table shows the interest earned in January and the end-of-month balance. At the beginning of February, another $40 deposit is made. Copy the table and then enter the new balance at the start of February.

$40, 6% Annual Interest				
Month	**Deposit Made on First Day of Month ($)**	**Start-of-Month Balance ($)**	**Interest Earned During the Month ($)**	**End-of-Month Balance ($)**
January	40.00	40.00	$0.005 \times 40 = 0.20$	$40 + 0.20 = 40.20$
February	40.00			
March	40.00			
April	40.00			
May	40.00			
June	40.00			
July	40.00			
August	40.00			
September	40.00			
October	40.00			
November	40.00			
December	40.00			

C. Calculate the interest earned in February and the new balance at the end of the month. Enter these amounts into the table.

D. Repeat parts B and C for each of the remaining months in the table.

E. Determine the total amount of interest earned for the year.

F. Determine the amount in the account at the end of the year.

8.1 Investigating Interest and Rates of Change

YOU WILL NEED

- graphing calculator (optional)
- graph paper

GOAL

Identify the difference between simple interest and compound interest for a given principal.

INVESTIGATE the Math

The Canada Savings Bond (CSB) was created in 1946 by the government to help people reach their savings and investment goals. Regular-interest CSBs earn simple interest. The interest earned each year is deposited directly into the investor's bank account. Compound-interest CSBs also earn interest annually. However, the interest earned each year is reinvested, so that each year interest is earned not only on the principal, but also on the interest earned in previous years.

Sonia buys a $1000 regular-interest CSB. It has a 10-year term and earns 5% interest annually.

Zuhal invests $1000 in a compound-interest CSB. It has a 10-year term and earns 5% interest annually.

simple interest
interest earned or paid only on the original sum of money invested or borrowed

compound interest
interest calculated at regular periods and added to the principal for the next period

principal
a sum of money that is borrowed or invested

? How does the amount of interest earned on Canada Savings Bonds differ for simple interest and compound interest?

A. Copy and complete the savings tables for Sonia and Zuhal.

Year	Principal for Year ($)	Interest Earned ($)	Accumulated Interest ($)	Amount at End of Year ($)
1	1000	$0.05 \times 1000 = 50$	50	$1000 + 50 = 1050$
2	1000	$0.05 \times 1000 = 50$	$50 + 50 = 100$	$1050 + 50 = 1100$
3	1000	$0.05 \times 1000 = 50$	$50 + 100 = 150$	$1100 + 50 = 1150$
4	1000			
5	1000			
6	1000			
7	1000			
8	1000			
9	1000			
10	1000			

Communication | *Tip*

The duration of an investment, or the time required to pay off a loan, is called the **term**.

Zuhal's Table

Year	Principal for Year ($)	Interest Earned ($)	Accumulated Interest ($)	Amount at End of Year ($)
1	1000	$0.05 \times 1000 = 50$	50	$1000 + 50 = 1050$
2	1050	$0.05 \times 1050 = 52.50$	$50 + 52.50 = 102.50$	$1050 + 52.50 = 1102.50$
3	1102.50	$0.05 \times 1102.50 = 53.13$	$102.50 + 53.13 = 157.63$	$1102.50 + 53.13 = 1157.63$
4	1157.63			
5				
6				
7				
8				
9				
10				

Sonia's Table

B. On a single graph, plot the amount at the end of year versus time for both of the investments. Label the plots as simple interest and compound interest. How are the graphs the same? How are they different?

C. Create a table of first differences for the amounts at the end of the year for both types of savings bonds. Describe any patterns you see.

D. For each type of CSB, what kind of relationship exists between the year and the amount at the end of the year? Explain how you know.

E. If both investments were kept for another two years,
  a) how much simple interest would be earned in year 11 and year 12? Explain.
  b) how much compound interest would be earned in year 11 and year 12? Explain.

F. Which type of savings bond earns more interest by the end of year 10? Why?

Reflecting

G. Which type of Canada Savings Bond will double its value faster? Explain how you know.

H. Examine Sonia's table. How can you determine the amount of simple interest earned on a principal of P dollars invested at an interest rate of r% for t years?

APPLY the Math

EXAMPLE 1 Using a formula to determine the value of an investment earning simple interest

Kevin invested \$4500 in a two-year regular-interest CSB that earns $7\frac{1}{2}\%$ annually.

a) How much interest did he earn?

b) How much will his investment be worth at the end of the term?

Patricia's Solution

a) $I = Prt$

$P = \$4500$

$r = 7\frac{1}{2}\%$

$= \frac{7.5}{100}$

$= 0.075$

$t = 2$

$I = 4500 \times 0.075 \times 2$

$= 675$

Kevin earned \$675 in interest over two years.

The total interest earned over more than one year is the product of the yearly interest, *Pr*, and the number of years, *t*.

I used the values for *P*, *r*, and *t* for Kevin's situation. To describe the interest rate as a decimal, I divided the percent by 100. Then I substituted into the formula.

b) $A = P + I$

$= 4500 + 675$

$= 5175$

The savings bond is worth \$5175 at the end of two years.

I calculated the final **amount** of the investment by adding the principal to the interest he earned.

amount

the sum of the original principal and the interest; given by $A = P + I$, where A is the amount, P is the principal, and I is the interest

EXAMPLE 2 Selecting a strategy to determine the amount of simple interest earned for part of a year

Keila had a credit card balance of \$550 that was 31 days overdue. The annual interest rate on the card is 23.9%.

a) How much interest did Keila have to pay?

b) Explain why paying interest on an outstanding credit card balance is sometimes referred to as "the cost of borrowing money."

Eusebio's Solution

a) $P = \$550$

$r = 23.9\% = \frac{23.9}{100} = 0.239$

I knew I could use the formula $I = Prt$ to calculate the interest over t years. So I needed to identify P, r, and t.

$t = \frac{31}{365}$

Since the interest rate is expressed in terms of a year, the time must also be written in terms of years. There are 365 days in a year.

$I = Prt$

$= 550 \times 0.239 \times \frac{31}{365}$

I substituted the values for P, r, and t, and multiplied.

$\doteq 11.16$

Therefore, Keila paid \$11.16 in interest for the 31 days.

b) \$11.16 was the interest she owed for the late payment. It was like borrowing the money for the extra 31 days. The interest is what borrowing the money cost her.

If Keila had paid the credit card balance when it was due, the amount paid would have been \$550. Because she waited 31 days, she had to pay \$561.16.

EXAMPLE 3 Selecting a strategy to determine the principal of a simple-interest loan

Five years ago, Jason lent Matt money. Matt repaid Jason a total of \$2100, which included simple interest charged at 10%. How much did Jason originally lend Matt?

Kendra's Solution

$I = Prt$

$A = P + I$

$= P + Prt$

$= P(1 + rt)$

I wrote the formula for determining simple interest. I added the interest to the principal to determine the amount. I then factored out the common factor P.

Communication | Tip

A **Guaranteed Investment Certificate (GIC)** is an investment purchased from a bank, trust company, or credit union, which guarantees that a specified interest rate will be the same over a fixed period.

When an investment or a loan reaches the end of its term, it is said to have matured or to have reached **maturity**.

$A = \$2100$

$r = 10\% = 0.1$

$t = 5$ years

$2100 = P(1 + 0.1 \times 5)$

$2100 = 1.5P$

$P = \frac{2100}{1.5}$

$P = 1400.00$

Five years ago, Jason loaned $1400 to Matt.

I substituted these values to calculate the principal, P.

EXAMPLE 4 Selecting a strategy to calculate compound interest

Mohsin bought a $500 Guaranteed Investment Certificate (GIC). It has a 3-year term and earns 3.25% compounded annually. How much interest will the GIC have earned at maturity?

Edwin's Solution

Year	Principal (Amount at Start of Year) ($)	Interest Earned ($)	Amount at the End of the Year ($)
1	500	$0.0325 \times 500 = 16.25$	$500 + 16.25 = 516.25$
2	516.25	$0.0325 \times 516.25 = 16.78$	$516.25 + 16.78 = 533.03$
3	533.03	$0.0325 \times 533.03 = 17.32$	$533.03 + 17.32 = 550.35$

The original principal is $P = \$500$.

The interest rate is 3.25%, or 0.0325 as a decimal.

The time is 3 years.

Since the interest is compounded annually, the interest earned will be added to the principal at the beginning of the next year. I set up a table to help me keep track of the principal, interest, and year-end amounts.

$550.35 - 500 = 50.35$

The interest earned in 3 years is $50.35.

Mohsin started with $500.00 and ended with $550.35. The interest he earned is the difference.

Check:

$16.25 + 16.78 + 17.32 = 50.35$

I checked my calculation. First I added the interest earned each year. Then I added the total interest earned to the original investment. The result was the same.

In Summary

Key Ideas

- Simple interest is calculated by applying the interest rate only to the original principal amount, resulting in linear growth.
- Compound interest is calculated by applying the interest rate to the original principal and any accumulated interest, resulting in exponential growth.

Need to Know

- The interest rate is converted to decimal form prior to calculating interest earned.
- Simple interest can be calculated with the formula $I = Prt$, where
  - I is the interest, earned in dollars
  - P is the principal invested or borrowed, in dollars
  - r is the annual interest rate, expressed as a decimal
  - t is the time, in years
- The amount that a simple-interest investment or loan is worth can be calculated with the formula $A = P + Prt$ or its factored form, $A = P(1 + rt)$, where
  - A is the final amount of the investment or loan, in dollars
  - P is the principal, in dollars
  - r is the annual interest rate, expressed as a decimal
  - t is the time, in years
- Tables are useful as tools for organizing calculations involving compound interest.

CHECK Your Understanding

1. Calculate the simple interest earned in 1 year.

a) \$100 invested at 3% **b)** \$100 invested at 4.5%

2. Each simple-interest investment matures in 2 years. Calculate the interest and the final amount.

a) \$675 invested at 7.25% **b)** \$4261 invested at 13.75%

3. Each investment matures in 3 years. The interest compounds annually. Calculate the interest and the final amount.

a) \$600 invested at 5% **b)** \$750 invested at $4\frac{3}{4}$%

PRACTISING

4. Calculate the simple interest earned or due and the amount at the end of each term.
   a) \$500 invested at 6% for 4 years
   b) \$2000 invested at 4.8% for 5 years
   c) \$1250 borrowed at 3% for 18 months
   d) \$1000 borrowed at 10% for 12 weeks
   e) \$5000 borrowed at $5\frac{1}{2}$% for 40 days

5. Copy and complete the tables for an investment of \$500.
K

	Regular-Interest CSB, $11\frac{1}{4}$%, 5 years		
Year	**Interest Earned (\$)**	**Accumulated Interest (\$)**	**Amount at End of Year (\$)**
1			
2			
3			
4			
5			

	Compound-Interest CSB, $11\frac{1}{4}$%, 5 years		
Year	**Interest Earned (\$)**	**Accumulated Interest (\$)**	**Amount at End of Year (\$)**
1			
2			
3			
4			
5			

6. Calculate the missing information in the table.

	Principal, *P* (\$)	Interest Rate, *r* (%)	Time, *t*	Simple Interest, *I* (\$)
a)	735.00	$5\frac{1}{2}$	27 days	
b)		8.25	240 days	138.25
c)	182.65	6.75		23.28
d)	260.00		2 months	16.50

7. An investment of \$1500 earned \$35.20 at a simple-interest rate of 5.5% per year. How long was the investment held?

8. Jennifer's investment account balance grew from \$400 to \$432.76 in 5 months. What annual rate of simple interest does her account pay?

9. Gordon has a Canada Savings Bond that pays \$150 in simple interest each year. The annual interest rate is 4.75%. What principal did Gordon invest in the bond?
A

10. Natasha borrows $3600 to help pay for university tuition. The annual interest rate is 8%. She will start to repay the loan in 4 years.

a) How much interest will have been added to the original amount if the interest is simple?

b) How much interest will have been added to the original amount if the interest is compound?

11. T Afzal can buy a $2000 GIC from a bank that earns 6.5% compounded annually for 5 years. At another bank, he can buy a $2000 GIC that earns 7% simple interest for 5 years.

a) Which GIC earns more interest?

b) How much more interest does it earn?

12. C Explain the difference between simple interest and compound interest. Use examples to support your explanation.

Extending

13. Tricia invested in a 90-day term deposit that earned 5.75% simple interest annually. When it matured, she received $760.63, which she reinvested in a 270-day term deposit so that it would earn 6.25% annually.

a) How much was originally invested?

b) How much will she receive when the second term deposit matures?

> **Communication *Tip***
>
> An investment purchased from a bank, trust company, or credit union for a fixed period or term is called a **term deposit**.

14. Barry deposited $9000 in an account that pays 10% each year, where interest is compounded 4 times a year at the end of each 3-month period. How much will be in the account at the end of 3 years?

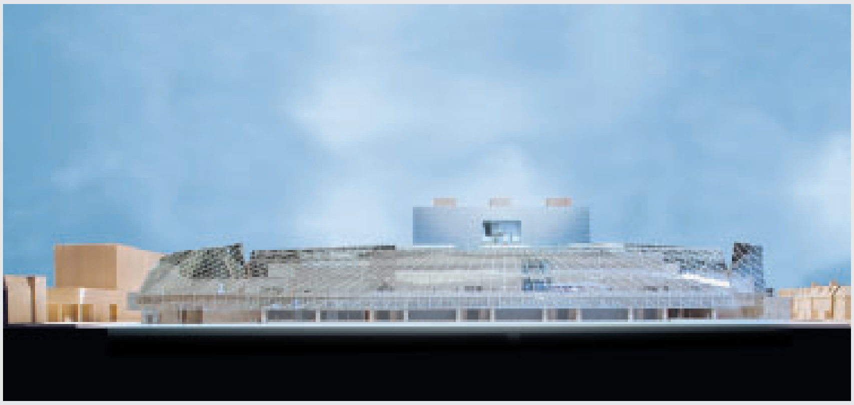
15. Azif has deposited $4800 into a savings account that pays 2.8% simple interest per year. When his balance passes $5000, the interest rate increases to 3.5% simple interest per year, with the amount of his investment so far becoming the new principal. If Azif leaves the money in the account for 5 years and makes no other deposits, how much will he have?

8.2 Compound Interest: Determining Future Value

GOAL

Solve problems that involve calculating the amount of an investment for a variety of compounding periods.

INVESTIGATE the Math



(Photo courtesy Art Gallery of Ontario/Sean Weaver)

This new frontage and hosting centre at Toronto's Art Gallery of Ontario is designed by Canadian-born Frank Gehry, one of the world's leading architects. It has been funded by governments, corporate sponsors, and many individuals.

The gallery's president, Charles Baillie, and his wife, Marilyn, personally pledged \$2 million, and later increased their donation to \$5 million. The Art Gallery received this money in early 2005, but construction was not set to be completed until late 2008, so they deposited it into a bank account. The new hosting centre has been named Baillie Court in their honour.

? How can you develop an expression to calculate what the Baillie's \$5 million new donation will be worth in 2008, if it earns interest at 10%/a compounded annually?

A. Copy and complete the investment table showing the growth of the donation to determine its **future value.**

Time Invested (years)	Calculation of Amount, Using Formula $A = P(1 + rt)$	Amount at End of Year, or Future Value ($)
1 (2005)	$A = 5\,000\,000(1 + 0.10(1))$ $= 5\,000\,000(1.1)$	5 500 000
2 (2005–2006)	$A = 5\,500\,000(1 + 0.10(1))$ $= 5\,000\,000(1.10)(1.10)$ $= 5\,000\,000(1.10)^2$	6 050 000
3 (2005–2007)		
4 (2005–2008)		

future value

the final amount (principal plus interest) of an investment or loan when it matures at the end of the investment or loan period

B. Examine the calculations for the amount at the end of each year. By what factor does the amount increase by each year? What expression could you use to represent this factor if i represents the interest rate, expressed as a decimal?

C. Consider how the investment would grow if the Art Gallery of Ontario decided to leave the money invested until 2014. Predict the amount of the investment at the end of 2014.

D. Check your prediction by extending the table you created in part A to show 6 more years of calculations.

E. Plot the amount at the end of each year versus time for the 10-year investment. What kind of growth does the graph suggest? Explain.

F. Examine the calculations for determining the future value at the end of each year. What is the expression that represents the future value at the end of 2008?

G. Calculate the future value at the end of 2008 from part F. How much of this amount is earned interest?

Reflecting

H. Write expressions that will represent the amount of the investment at the end of the 5th year, 10th year, and nth year.

I. Write a formula that will represent the amount of an investment earned on a principal invested for several years. Let
- A represent the amount or future value of the investment in dollars
- P represent the principal in dollars
- i represent the interest rate expressed as a decimal
- n represent the number of periods over which the investment is compounded

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine the amount for different compounding periods

Jayesh has $2000 to invest in a compound-interest account in which he would like to leave the money for 3 years. He considers three different **compounding periods**:

compounding period
each period over which compound interest is earned or charged in an investment or loan

Option A: 8%/a compounded annually

Option B: 8%/a compounded semi-annually

Option C: 8%/a compounded quarterly

a) For each option, what is the amount at the end of 3 years? What is the interest earned?

b) How does changing the compounding period affect the amount of the investment? Why?

c) For each option, create a timeline to show the annual value of Jayesh's investment.

Communication *Tip*

The expression 8%/*a* (read "8 percent per annum") is shorthand for "8% per year." So, for example, the quarterly interest rate is 8% divided by 4.

Soo-Lin's Solution

a) Option A:

$A = P(1 + i)^n$

$P = \$2000$

$i = 8\% = 0.08$

$n = 3 \times 1 = 3$

I wrote the formula for calculating the amount when a principal, *P*, earns compound interest at a rate of *i*% for each of *n* compounding periods.

The compounding period is 1 year.

I substituted these values into the formula.

$A = 2000(1 + 0.08)^3$

$= 2000(1.08)^3$

$\doteq 2519.42$

The amount after 3 years using Option A will be $2519.42.

Interest $= 2519.42 - 2000.00 = 519.42$

I subtracted the principal from the amount to get the interest.

The interest earned in 3 years is $519.42.

Option B:

$P = \$2000$

$i = \frac{0.08}{2} = 0.04$

Interest is compounded semi-annually. I had to divide the interest rate by 2 to get the rate for each compounding period.

$n = 3 \times 2 = 6$

I multiplied the number of years by 2 to calculate the number of compounding periods.

$$A = 2000(1.04)^6$$
$$\doteq 2530.64$$

The amount after 3 years using Option B will be $2530.64.

Interest $= 2530.64 - 2000.00 = 530.64$

The interest earned in 3 years is $530.64.

Option C:

$P = \$2000$

$$i = \frac{0.08}{4} = 0.02$$

Interest is compounded quarterly. I had to divide the interest rate by 4 to get the rate for each compounding period.

$$n = 3 \times 4 = 12$$

I multiplied the number of years by 4 to calculate the number of compounding periods.

$$A = 2000(1.02)^{12}$$
$$\doteq 2536.48$$

The amount after 3 years using Option C will be $2536.48.

Interest $= 2536.48 - 2000.00 = 536.48$

The interest earned in 3 years is $536.48.

b) The amount of an investment increases as the number of compounding periods increases. Since changing the compounding period changes both the interest rate and the number of periods, the amount, *A*, of the investment is also changed.

Changing the compounding period changes the interest, *i*, used in the formula. This is because the annual interest rate has to be divided by the fraction of the year the compounding period is. The compounding period also affects the number of compounding periods over the 3 years.

c)

Option	Amount after Year *t*	Timeline	Interest Earned ($)
A	$2000(1 + 0.08)^t$	0: 2000 1: 2160.00 2: 2332.80 3: 2519.42	519.42
B	$2000\left(1 + \frac{0.08}{2}\right)^{2t}$	0: 2000 1: 2163.20 2: 2339.72 3: 2530.64	530.64
C	$2000\left(1 + \frac{0.08}{4}\right)^{4t}$	0: 2000 1: 2164.86 2: 2343.32 3: 2536.48	536.48

EXAMPLE 2 Using a graph to represent and compare the amount of an investment over time

Alwynn invests \$500 in an account that earns 6%/a compounded monthly.
Peter invests \$500 at the same time, but in an account that earns 6%/a at simple interest.

a) Determine the difference between their investments at the end of the 5th year.

b) Using graphing technology, compare the balances in the accounts at the end of each month for 10 years.

c) Discuss how the two investments are different.

Ella's Solution

a) Alwynn's investment:

$A = P(1 + i)^n$

I used the formula for future value with compound interest.

$P = 500$

$i = \frac{0.06}{12} = 0.005$

I divided the annual rate by 12 to calculate the monthly rate.

$n = 5 \times 12 = 60$

I multiplied the number of years by 12 to calculate the number of compounding periods.

$A = 500(1.005)^{60}$

$\doteq 674.43$

Alwynn's investment is worth \$674.43 after 5 years.

Peter's investment:

$I = Prt$

I used the formula for simple interest.

$P = 500$

$r = 6\% = 0.06$

$t = 5$

$I = 500 \times 0.06 \times 5 = 150$

$A = P + I$

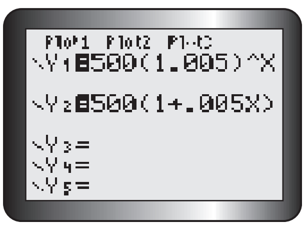
$= 500 + 150 = 650$

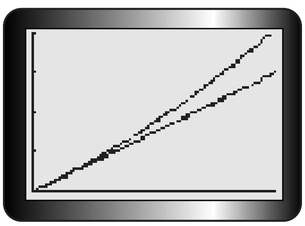
I added the interest to the principal to determine the amount.

Peter's investment is worth \$650 after 5 years.

The difference is $\$674.43 - \$650 = \$24.43$.

b)



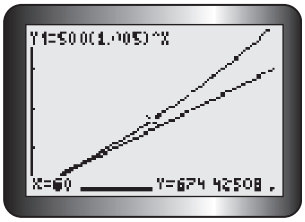


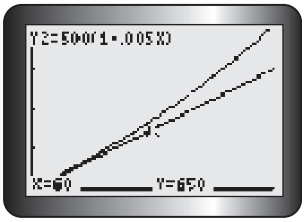
To compare the balances, I used the monthly interest rate of 0.005 for both investments.

For Alwynn, I graphed $A = 500(1.005)^n$ in Y_1.
For Peter, I graphed $A = 500(1 + 0.005n)$ in Y_2.

In both cases, I graphed values of n from 1 to 120 (10 years = 120 months).

c) From the graphs, Alwynn's investment is growing much faster than Peter's.





I used the value operation to compare the amounts of each investment after 5 years.

As time passes, the difference between the monthly balances will increase. Peter's monthly balance grows at a constant rate, but Alwynn's grows by more and more each month.

Tech Support

For help graphing functions and determining a value, see Technical Appendix, B-2 and B-3.

In Summary

Key Idea

- The formula for calculating the amount of an investment earning compound interest is $A = P(1 + i)^n$, where
  - A is the amount or future value, in dollars
  - P is the principal, in dollars
  - i is the interest rate per compounding period
  - n is the number of compounding periods

Need to Know

- The compounding frequency determines the number of compounding periods per year. The compounding period changes the total number of periods, n, over which the interest is compounded during the term of the investment. Changing the compounding period changes the interest, i, because the annual interest rate must be adjusted to the rate that would be used for each compounding period.

Annually	once per year	i = annual interest rate	n = number of years
Semi-annually	2 times per year	i = annual interest rate $\div$ 2	n = number of years $\times$ 2
Quarterly	4 times per year	i = annual interest rate $\div$ 4	n = number of years $\times$ 4
Monthly	12 times per year	i = annual interest rate $\div$ 12	n = number of years $\times$ 12
Daily	365 times per year	i = annual interest rate $\div$ 365	n = number of years $\times$ 365

- The amount, or future value, of an investment increases with the number of compounding periods. However, the amount of the increase is not usually significant when interest is compounded more often than monthly.
- A timeline is useful for organizing and visualizing the information required to solve a compound-interest problem.

CHECK Your Understanding

1. An investment earns 9%/a. Calculate i and n when the interest is compounded

a) annually for 4 years
b) semi-annually for 6 years
c) quarterly for 2 years
d) monthly for 3 years

2. Complete the table.

	Principal ($)	Annual Interest Rate (%)	Time (years)	Compounding Frequency	Rate for the Compounding Period, i (%)	Number of Compounding Periods, n	Amount ($)	Interest Earned ($)
a)	400.00	5	15	annually				
b)	750.00	13	5	semi-annually				
c)	350.00	2.45	8	monthly				
d)	150.00	7.6	3	quarterly				
e)	1000.00	4.75	4	daily				

PRACTISING

3. An investment earns 5.75%/a. Calculate i and n when the interest is compounded.

a) annually for 3 years
b) semi-annually for 5 years
c) quarterly for 3 years
d) monthly for 2 years

4. Complete the table.

	Principal ($)	Interest Rate (%)	Years	Compounding Frequency	i	n	Amount ($)	Interest Earned ($)
a)	800	8	10	annually				
b)	1500	9.6	3	semi-annually				
c)	700	$3\frac{1}{2}$	5	monthly				
d)	300	7.25	2	quarterly				
e)	2000	$4\frac{1}{4}$	$\frac{1}{2}$	daily				

5. K Calculate the amount you would end up with if you invested $5000 at 14.6%/a compounded annually for 10 years.

6. Mario neglected to pay a credit card bill of $1550 at 17%/a, compounded daily, for 2 weeks after it was due. What is the amount he must pay to settle the bill at the end of the 2 weeks?

7. Use a timeline to show the growth in value of a $300 bond at 9%/a for 1 year, compounded monthly. What is the interest earned?

8. If $350 grows to $500 in 3 years, what is the annual interest rate assuming that interest is compounded annually?

9. Wasantha deposits $750 into a savings account that pays compound interest annually. The table at the right shows his annual balance for this investment. What interest rate did the bank give Wasantha?

Year	Final Balance ($)
1	795.00
2	842.70
3	893.26
4	946.86
5	1003.67

10. In about how many years will $600 grow to $1000 if it is invested at 8%/a compounded annually?

11. A donor gives $50 000 to the high school he graduated from. The amount must be invested for 3 years, and the accumulated interest will be used to buy books for the school library. If the money earns 7.75%/a compounded monthly, how much will be available for the books?

12. A Josephine is purchasing a used car. Her bank has offered a loan of $5000 at 5%/a compounded monthly. The used-car dealer has offered a loan of 5.25%/a compounded semi-annually.
   a) What is the amount owing on each loan after 1 year?
   b) Which loan should Josephine take and why?

13. Create a financial problem whose solution could be represented by the function $f(x) = 500(1.01)^x$.

14. T a) $1000 is invested for 1 year at 10%/a. Copy and complete the table, showing the amounts that $1000 invested at 10%/a would grow to in 1 year for different compounding frequencies.

Compounding Frequency	Number of Compounding Periods per Year	Formula	Amount ($)
annually	1	$1000(1 + 0.10)$	
semi-annually	2	$1000(1 + 0.05)^2$	
quarterly			
monthly			
weekly			
daily			
hourly			

   b) How do the amounts change as the number of compounding periods per year increases?
   c) Is there a maximum amount that can be earned in 1 year? Explain.
   d) Why don't banks offer hourly compounding frequencies?

15. For each of the following expressions, identify the principal, compounding frequency, interest rate per compounding period, number of compounding periods, annual interest rate, and number of years. Evaluate the expression to determine the amount; then determine the interest.

	Formula	*P* ($)	Compounding Frequency	*i* (%)	*n*	Annual Interest Rate (%)	Number of Years	*A* ($)	*I* ($)
a)	$A = 145(1 + 0.0475)^{12}$								
b)	$A = 850(1 + 0.195)^{5}$								
c)	$A = 4500\left(1 + \frac{0.0525}{365}\right)^{1095}$								
d)	$A = 4500\left(1 + \frac{0.15}{12}\right)^{78}$								
e)	$A = 4500\left(1 + \frac{0.03}{4}\right)^{20}$								

16. Banks offer a variety of terms (for example, 90 days, 3 months, 1 year, 3 years, 10 years, where the interest rate is greater for larger terms) and a variety of compounding periods. What advantages does the variety of terms and compounding periods provide for bank customers? Use different scenarios to support your explanation.
C

Extending

17. In the 1980s, financial experts used to talk about doubling your investment in 7 years. This was related to the 1980 Canada Savings Bond, which had an interest rate that would double the investment in 7 years. Determine that interest rate.

18. Mustafa wants his investment to be worth $10 000 in 5 years. The bank will give him 6%/a interest compounded annually. How much does Mustafa have to invest now?

19. If $500 was invested at 8%/a, compounded annually, in one account, and $600 was invested at 6%/a, compounded annually, in another account, when would the amounts in both accounts be equal?

8.3 Compound Interest: Determining Present Value

GOAL

Solve problems that involve calculating the principal that must be invested today to obtain a given amount in the future.

LEARN ABOUT the Math

When Hua was born, her parents decided to invest some money so that she could have a gift of \$20 000 on her 16th birthday. They decided on a compound-interest government bond that paid 10% interest per year, compounded monthly. After the initial amount was invested, there would be no further transactions until the bond reached maturity.

? How much money must be invested today to guarantee Hua's future amount of \$20 000?

EXAMPLE 1 Selecting a strategy to determine the principal needed to grow to a given amount

a) What is the present value that Hua's parents must invest today to reach their savings goal of \$20 000 by her 16th birthday?

b) If Hua's parents decide to wait until she is 13 and then invest a lump sum to save for the gift of \$20 000, what is the present value if the investment earns the same rate of interest?

present value

the principal that must be invested today to obtain a given amount in the future

Martha's Solution: Using a Timeline

a)
$$A = P(1+i)^n$$
$$\frac{A}{(1+i)^n} = \frac{P(1+i)^n}{(1+i)^n}$$
$$\frac{A}{(1+i)^n} = P$$

I knew I could use the formula $A = P(1 + i)^n$ to find the amount of the principal. What I needed to find was the principal that must be invested now to get an amount of \$20 000. I saw that if I divided the future value of an investment by $(1 + i)^n$, I could work my way back to the present value, or principal, to be invested.

I rearranged the formula to solve for P, giving me an expression for the present value.

Age	16	15	14	13	…	2	1	0
Years before Age 16	0	1	2	3	…	14	15	16
	$\frac{20\,000}{(1.1)^0}$	$\frac{20\,000}{(1.1)^1}$	$\frac{20\,000}{(1.1)^2}$	$\frac{20\,000}{(1.1)^3}$		$\frac{20\,000}{(1.1)^{14}}$	$\frac{20\,000}{(1.1)^{15}}$	$\frac{20\,000}{(1.1)^{16}}$
	= 20 000	= 18 181.81	= 16 528.93	= 15 026.30		= 5266.63	= 4787.84	= 4352.58

I drew a timeline showing how the present value needed to reach \$20 000 decreases as the time before Hua's 16th birthday increases. It makes sense that if they invest it longer, they don't have to invest as much, since there would be more time to earn extra interest.

$$A = P(1 + i)^n$$

$$20\,000 = P(1 + 0.1)^{16}$$

$$P = \frac{20\,000}{(1 + 0.10)^{16}}$$

$$= \frac{20\,000}{(1.1)^{16}}$$

$$= 4352.58$$

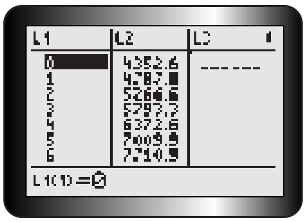
Hua's parents must invest \$4352.58 now.

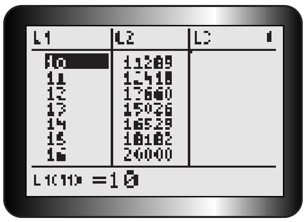
I divided each amount by $1 + 0.10 = 1.1$ to get the amount in the next column.

The future amount of money is $A = \$20\,000$.

The annual interest rate is $i = 10\%$, or 0.1.

The interest is compounded annually for $n = 16$ years.





I checked my answer by creating a table of values showing the year-end amount of investment, $A = 4352.58\,(1 + 0.1)^n$, for each year.

At the end of year 16, the initial investment of \$4352.58 is worth \$20 000.

This means that if the parents invest \$4352.58 today, their investment will be worth \$20 000 when Hua turns 16.

b) $A = P(1 + i)^n$

$$P = \frac{A}{(1 + i)^n}$$

$A = \$20\,000$

$i = 10\%$ or 0.1

$n = 3$

$$P = \frac{20\,000}{(1 + 0.10)^3}$$

$$= 15\,026.30$$

Hua's parents must invest \$15 026.30 when she turns 13.

To find the principal Hua's parents must invest, I rearranged the formula to solve for P.

I knew the future amount of money (A), the annual interest rate (i), and the number of years of compounding (n).

Communication ***Tip***

$A = P(1 + i)^n$ is sometimes written as $FV = PV(1 + i)^n$, where FV is future value and PV is present value.

Reflecting

A. What is the total interest earned over the 16 years of the investment in part (a)? What is the total interest over the 3 years of the investment in part (b)?

B. What are the advantages and disadvantages of investing earlier rather than later?

C. The formula $A = P(1 + i)^n$ can be used to calculate both present value and future value. State what you need to know and how you would use the formula to calculate the following.
i) future value **ii)** present value

APPLY the Math

EXAMPLE 2 Determining present value with a compounding period of less than one year

An investment earns $7\frac{3}{4}$%/a compounded semi-annually. Determine the present value if the investment is worth $800 five years from now.

Luc's Solution

$$P = \frac{A}{(1 + i)^n}$$
$$= A(1 + i)^{-n}$$

Since I was dividing by $(1 + i)^n$, I multiplied by the reciprocal, using a negative exponent.

$A = 800$

$i = \frac{0.0775}{2} = 0.038\ 75$

$n = 5 \times 2 = 10$

The amount of the investment is $800.

The annual interest rate is $7\frac{3}{4}\% = 0.0775$.

The semi-annual interest rate is $\frac{1}{2}$ of the annual rate.

I multiplied the number of years by 2 to calculate the number of compounding periods.

$$P = 800(1.038\ 75)^{-10}$$
$$= 546.99$$

I substituted the values of A, i, and n, and evaluated P.

The present value of the investment is $546.99.

EXAMPLE 3 Solving a problem involving present value

Tony has \$3000 in his savings account. He intends to buy a laptop computer and printer and invest the remainder for 2 years, compounding monthly at an annual interest rate of 3%. He wants to have \$2000 in his account 2 years from now. How much can he spend on the laptop and printer?

Martha's Solution

$$P = \frac{A}{(1 + i)^n}$$

$$A = 2000$$

$$i = \frac{0.03}{12} = 0.0025$$

$$n = 12 \times 2 = 24$$

$$P = \frac{2000}{(1 + 0.0025)^{24}}$$

$$= \frac{2000}{(1.0025)^{24}}$$

$$= 1883.67$$

The future value of the investment is \$2000.

The annual interest rate is $3\% = 0.03$.

The monthly rate of interest is $\frac{1}{12}$ of the annual rate.

I multiplied the number of years by 12 to calculate the number of compounding periods.

Then I substituted into the formula.

The present value is \$1883.67.

Amount that can be spent

= Amount in savings account − Present value

= \$3000.00 − \$1883.67

= \$1116.33

Tony can spend \$1116.33 on a laptop and printer.

I needed to determine the amount of Tony's savings that he needs to keep invested to reach \$2000 in 2 years. Whatever he has left after this amount is set aside is what he can spend.

I subtracted the present value from \$3000 to determine how much Tony can spend.

In Summary

Key Idea

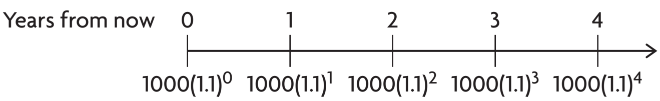
- The formula for calculating future value can be rearranged to give the present value of an investment earning compound interest. The rearranged formula is

$$P = \frac{A}{(1 + i)^n} \quad \text{or} \quad P = A(1 + i)^{-n}$$

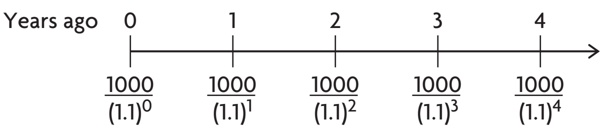
  where
  - A is the amount, or future value, in dollars
  - P is the principal, or present value, in dollars
  - i is the interest rate per compounding period
  - n is the number of compounding periods

Need to Know

- The amount, P, that must be invested now in order to grow to a specific amount later on can be calculated from the future value by dividing by $(1 + i)^n$, where
  - i is the interest rate per compounding period
  - n is the number of compounding periods
- Drawing a timeline can help you decide whether you need to determine the future value (or amount) or the present value (or principal) of an investment or loan.



The future value of \$1000 invested at 10%/a compounded annually for 4 years.



The present value of \$1000 invested at 10%/a compounded annually for 4 years.

- Interest earned can be calculated by subtracting the present value (principal) from the future value (amount):

$$I = FV - PV, \quad \text{or} \quad I = A - P$$

CHECK Your Understanding

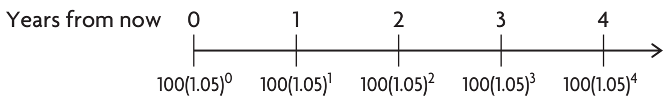
1. Solve for the principal, P.

a) $100 = P(1.05)^3$ **b)** $500 = P(1 + 0.00375)^{48}$

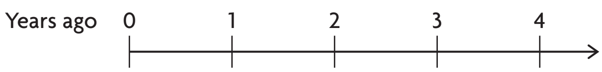
2. Copy and complete the table.

	Future Value ($)	Annual Interest Rate (%)	Time Invested (years)	Compounding Frequency	*i* (%)	*n*	Present Value ($)	Interest Earned ($)
a)	4 000	5	15	annually				
b)	3 500	2.45	8	monthly				
c)	10 000	4.75	4	daily				

3. The first timeline that follows visually represents the future value of $100 invested at 5%/a compounded annually for 4 years. Copy and complete the second timeline to show the calculations of present value in each year for an investment whose future value is $150.



The future value of $100 invested at 5%/a compounded annually for 4 years.



The present value of $150 invested at 5%/a compounded annually for 4 years.

PRACTISING

4. Copy and complete the table.

	Future Value ($)	Annual Interest Rate (%)	Time Invested (years)	Compounding Frequency	*i* (%)	*n*	Present Value ($)	Interest Earned ($)
a)	8000	10	7	annually				
b)	7500	13	5	semi-annually				
c)	1500	7.6	3	quarterly				

5. Use a timeline to illustrate the present value of an investment worth $5750 in 3 years at 12%/a compounded semi-annually.

6. How much should Jethro invest now to have $10 000 in 3 years' time?
K The money will be invested at 5%/a compounded monthly.

7. Tim has arranged to pay $2000 toward a debt now and $3000 two years from now. What amount of money would settle the entire debt today if the interest is 10.5%/a compounded semi-annually?

8. On Abby's 21st birthday, she receives a gift of $10 000, the accumulated amount of an investment her grandparents made for her when she was born. Determine the amount of their investment and the interest earned if the interest rate was 8.75%/a compounded

a) annually **b)** semi-annually

9. Daveed has a savings account that pays interest at 4.25%/a compounded monthly. She has not made any deposits or withdrawals for the past 6 months. There is $3542.16 in the account today. How much interest has the account earned in the past 6 months?

10. Jason borrowed money that he will pay back in 3 years' time. The interest rate was 5.25%/a compounded monthly. He will repay $3350 after 3 years. How much money did Jason borrow?

11. Betty plans to send her parents on a $15 000 vacation for their 30th
A wedding anniversary 10 years from now. She would like to invest the money today in a GIC term deposit earning 6%/a compounded semi-annually and split the cost of its purchase with her sister and brother. How much will each person contribute toward the purchase of the GIC?

12. Clem inherits $250 000. He wants to save $150 000 for college or
T university costs in 4 years.

a) How much should Clem invest in a GIC earning 10.5%/a compounded monthly to ensure that he has $150 000 in savings 4 years from now?

b) How much of Clem's inheritance remains after his investment?

c) How much interest would Clem's inheritance earn in 4 years if he invested the entire amount in the GIC now?

13. For each situation, determine

i) the present value **ii)** the interest earned

a) A loan of $21 500 is due in in 6 years. The interest rate is 8%/a, compounded quarterly.

b) A loan of $100 000 is due in 5 years. The interest rate is 5%/a, compounded semi-annually.

14. Copy the table and fill in the missing entries.

	Future-Value Formula	A ($)	Compounding Frequency	i (%)	n	Annual Interest Rate (%)	Number of Years	Present Value ($)
a)	$280\ 000 = P(1 + 0.0575)^{24}$		semi-annually					
b)	$16\ 000 = P(1 + 0.20)^{5}$		annually					
c)	$10\ 000 = P\left(1 + \frac{0.0425}{365}\right)^{1460}$							
d)	$9500 = P\left(1 + \frac{0.15}{12}\right)^{50}$							
e)	$1500 = P\left(1 + \frac{0.03}{4}\right)^{24}$							

15. C Marshall wants to have $5000 in 4 years. He has two options for investment: A savings account will pay 3.5%/a compounded monthly; a GIC will pay 3.4%/a compounded semi-annually. Write an explanation of which investment Marshall should pick and why.

Extending

16. A loan at 12%/a compounded semi-annually must be repaid in one single payment of $2837.04 in 3 years. What is the principal borrowed?

17. What equal deposits, one made now and another made one year from now, will accumulate to $2000 two years from now at 6.25%/a compounded semi-annually?

18. Gina agrees to pay $25 000 now and $75 000 in 4 years for a studio condominium. If she can invest at 10.5%/a compounded annually, what sum of money does she need now to buy the condominium?

8.4 Compound Interest: Solving Financial Problems

GOAL

Use the TVM Solver to solve problems involving future value, present value, number of payments, and interest rate.

YOU WILL NEED

- graphing calculator with TVM Solver program

LEARN ABOUT the Math

There are a variety of technological tools for calculating financial information involving compound interest. These include spreadsheets, calculators on websites of financial institutions, and graphing calculator programs.

Some graphing calculators include financial programs such as the Time Value of Money (TVM) Solver. This program can be used to quickly investigate and solve many compound-interest problems.

? How can the TVM Solver be used to solve problems involving compound interest, and how does it compare with using a formula?

EXAMPLE 1 Selecting a strategy to determine the amount of an investment

Peggy's employer has loaned her \$5000 to pay for university course tuition and textbooks. The interest rate of the loan is 2.5%/a compounded monthly, and the loan is to be paid back in one payment at the end of 2 years. How much will Peggy have to pay back?

Jeremy's Solution: Using a Formula

$A = P(1 + i)^n$

$P = 5000$

$i = \frac{0.025}{12}$

$n = 2 \times 12 = 24$

$A = 5000\left(1 + \frac{0.025}{12}\right)^{24}$

$A = 5256.09$

Peggy will pay \$5256.08 at the end of 2 years.

The principal is $P = \$5000$.

The annual interest rate is 2.5%, or 0.025. Since it is compounded monthly, I divided it by 12.

I multiplied the number of years by 12 to determine the number of compounding periods.

Tech Support

For help using the TVM Solver to solve problems involving compound interest, see Technical Appendix, B-15. When no payments are involved in solving compound-interest problems, **PMT** is set to 0. **P/Y**, the number of payments per year, is always set to 1 when there are no payments.

Mei-Mei's Solution: Using the TVM Solver

I needed to find the future value. Since it is unknown, I entered 0 for **FV**.

The term of the loan is 2 years, so **N** = 2.

The interest rate, **I**%, is 2.5%/a.

The present value of the loan is 5000. Since money has been received, the sign of **PV** is positive.

The interest is compounded monthly, so the number of compounding periods per year, **C/Y**, is 12.

I moved the cursor next to **FV** because that is the value to be calculated.

I pressed ALPHA ENTER to solve for **FV**.

The future value is negative, indicating that this is money to be paid out.

Peggy will pay \$5256.08 at the end of 2 years.

EXAMPLE 2 Selecting a strategy to determine the present value of an investment

How much was invested at 4%/a compounded semi-annually for 3 years if the final amount was \$7500?

Martin's Solution: Using a Formula

$A = 7500$

The future value of the amount of the investment is A = \$7500.

$$i = \frac{0.04}{2} = 0.02$$

The annual interest rate is 4%, or 0.04. Since it is compounded semi-annually, I divided it by 2.

$n = 3 \times 2 = 6$

$P = A(1 + i)^{-n}$

$$P = 7500\left(1 + \frac{0.04}{2}\right)^{-6}$$

$P = 6659.79$

The original present value was \$ 6659.79.

I multiplied the number of years by 2 to determine the number of compounding periods.

I substituted the values for *A*, ***i***, and ***n*** into the formula and evaluated *P*.

Rebecca's Solution: Using the TVM Solver

I needed to calculate the present value. Since it is unknown, I entered 0 for **PV**.

The future value is \$7500.

The investment earns interest for 3 years, so **N** = 3.

The interest rate, **I%**, is 4%.

Interest is compounded semi-annually, so **C/Y** = 2.

I moved the cursor next to **PV** because that was the value to be calculated.

I solved for **PV**.

The negative sign indicates an investment, or money paid (cash outflow). The future value was positive, indicating money received or earned (cash inflow).

The original present value of the investment was \$6659.79.

Reflecting

A. If you are using the TVM Solver, when is the present value entered as positive and when is it entered as negative? Explain, using examples.

B. How is using the TVM Solver to solve compound-interest problems similar to using the formula $A = P(1 + i)^n$? How is it different?

C. Which method do you prefer? Explain why.

APPLY the Math

EXAMPLE 3 Selecting a strategy to determine the annual interest rate

What annual interest rate was charged if an $800 credit card bill grew to $920.99 in 6 months and interest was compounded monthly?

Delacey's Solution: Using a Formula

$A = \$920.99$

$P = \$800$

The number of compounding periods is $n = 6$.

I listed the values I knew and substituted into the formula for A.

I needed to solve for i.

$$A = P(1 + i)^n$$

$$(1 + i)^n = \frac{A}{P}$$

$$(1 + i)^6 = \frac{920.99}{800}$$

$$\left((1 + i)^6\right)^{\frac{1}{6}} = \left(\frac{920.99}{800}\right)^{\frac{1}{6}}$$

To solve for i, I raised each side of the equation to the power of $\frac{1}{6}$.

$$1 + i = \sqrt[6]{\frac{920.99}{800}}$$

By using the power-of-a-power rule, I was able to get the exponent on $1 + i$ to be 1. To do this, I had to calculate the 6th root of the number on the right side.

$$i = \sqrt[6]{\frac{920.99}{800}} - 1$$

I solved for i by subtracting 1 from both sides.

$$i = 0.023\,75$$

$$i \times 12 = 0.023\,75 \times 12$$

$$= 0.285$$

$$= 28.5\%$$

Since i is the monthly interest rate, I multiplied it by 12 to determine the annual interest rate.

The annual interest rate is 28.5%.

Kara's Solution: Using the TVM Solver

I needed to determine the interest rate. I entered 0.5 for **N**, since the investment earns interest for 0.5 years. I entered 0 for **I%**, since the interest rate is unknown. The present value, **PV**, is 800. The future value, **FV**, is −920.99 because the money will eventually be paid out. The number of compounding periods per year, **C/Y**, is 12, because interest is compounded monthly.

I moved the cursor next to **I%** because that was the value to be calculated.

I solved for **I%**.

The interest rate is 28.5%/a.

$$A = P(1 + i)^n$$

$$= 800\left(1 + \frac{0.285}{12}\right)^6$$

$$= 920.99$$

I checked the answer with the formula $A = P(1 + i)^n$.

EXAMPLE 4 Selecting a strategy to determine the number of years required to double an investment

Approximately how long would it take for a \$15 000 investment to double if it earns 10%/a interest compounded semi-annually?

Marita's Solution: Using a Formula with Guess-and-Check

$$A = P(1 + i)^n$$

$$(1 + i)^n = \frac{A}{P}$$

The present value is P = \$15 000. The future value is A = \$30 000.

The annual interest rate is 10%. The semi-annual interest rate is $i = 0.05$.

$$1.05^n = \frac{30\,000}{15\,000}$$

$$1.05^n = 2$$

I substituted these values to get an equation involving n, the number of compounding periods.

$$1.05^2 = 1.1025$$

$$1.05^6 = 1.3400$$

$$1.05^{14} = 1.98$$

$$1.05^{15} = 2.08$$

Since n is an exponent, I tried different values of n to solve the equation.

I started with 2, then 6, but I got values that were too low. 14 was really close and 15 was too high.

The number of compounding periods is approximately 14.

n must be a number between 14 and 15, but closer to 14.

The number of years is $\frac{14}{2} = 7$.

I divided 14 by 2 because each year has 2 compounding periods.

It will take approximately 7 years for the investment to double in value.

Samir's Solution: Using the TVM Solver

I needed to find the number of years, **N**. I entered 0 for **N**, since it is unknown. I entered 10 for **I%**, since it is 10%, and 2 for **C/Y** because the interest is compounded semi-annually.

I entered −15 000 for **PV** and 30 000 for **FV**. I knew that **PV** must be negative because it is the amount paid in to the investment and **FV** will be money paid out at a later date.

I moved the cursor next to **I%** because that was the value to be calculated.

I solved for **N**.

It will take about 7 years for the investment to double in value.

In Summary

Key Ideas

- The TVM Solver is a program on some graphing calculators. It can be used to investigate and solve financial problems involving compound interest.
- For compound-interest problems, you can use the two forms of the compound-interest formula

$$A = P(1 + i)^n \quad \text{and} \quad P = A(1 + i)^{-n}$$

either as an alternative to using the TVM Solver or as a check.

(continued)

Need to Know

- When entering the values for present value or future value in the TVM Solver, consider whether the money is paid out (cash outflow) or received (cash inflow). Money paid out, such as a loan repayment or the principal of an investment, is negative. Money received, such as the final amount of an investment, is positive.
- Enter the values of all of the program variables except for the one you want to calculate. The remaining variable has a value of 0 because its value is unknown. The actual value of the remaining variable is then calculated and displayed.
- You can also calculate an interest rate by guess-and-check, if you know the number of compounding periods per year, the number of years, and the future and present values.

CHECK Your Understanding

1. Copy the table that follows. For each of problems (a) through (d), record the values you would enter for the known TVM Solver variables. Record 0 for the unknown. Then solve the problem and indicate the solution by marking it with *.

a) Determine the amount of an investment if $600 is invested at 4.5%/a interest for 8 years, compounded quarterly.

b) How long would it take $6000 to grow to $8000 if it is invested at 2.5%/a compounded semi-annually?

c) What interest rate is needed for $20 000 to double in 5 years if interest is compounded quarterly?

d) What amount needs to be invested at 6%/a interest compounded weekly if you want to have $900 after 1 year?

	N	I%	PV	PMT	FV	P/Y	C/Y
a)							
b)							
c)							
d)							

PRACTISING

2. Guo is a civic employee. His last contract negotiated a 2.75% increase each year for the next 4 years. Guo's current salary is $48 500 per year. What will his salary be in 4 years?

3. Beverley plans to invest $675 in a GIC for 2 years. She has researched two plans: Plan A offers 5.9%/a interest compounded semi-annually. Plan B offers 5.75%/a interest compounded monthly. In which plan should Beverley invest to earn the most?

4. Determine the future value of an investment of $10 000 compounded annually at 5%/a for

a) 10 years **b)** 20 years **c)** 30 years

5. For each situation, determine both the present value and the earned interest.

a) An investment that will be worth $5000 in 3 years. The interest rate is 4%/a compounded annually.

b) An investment that will be worth $13 500 in 4 years. The interest rate is 6%/a compounded monthly.

c) A loan repayment of $11 200 paid after 5 years, with interest of 4.4%/a compounded monthly.

d) An investment that will be worth $128 500 in 8 years. The interest rate is 6.5%/a compounded semi-annually.

e) A loan repayment of $850 paid after 400 days, with interest of 5.84%/a compounded daily.

f) An investment that will be worth $6225 in 100 weeks. The interest rate is 13%/a compounded weekly.

6. At what interest rate will an investment compounded annually for 12 years double in value?

7. How long does it take for an investment to triple in value at 10%/a interest compounded monthly?

8. When Ron was born, a $5000 deposit was made into an account that pays interest compounded quarterly. The money was left until Ron's 21st birthday, when he was presented with a cheque for $12 148.79. What was the annual interest rate?

9. **A** Shirley redeemed a $2000 GIC and received $2220. The GIC paid interest at 5.25%/a compounded quarterly. For how long was the money invested?

10. A $3000 GIC pays 5%/a interest compounded annually for a 3-year term. At maturity, the accumulated amount is reinvested in another GIC at 6.5%/a compounded annually for 5 years. What is the final amount when the second investment matures?

11. Today Sigrid has $7424.83 in her bank account. For the last 2 years, her account has paid 6%/a compounded monthly. Before then, her account paid 6%/a compounded semi-annually for 4 years. If she made only one deposit 6 years ago, determine the original principal.

12. On June 1, 2001, Anna invested $2000 in a money market fund that paid 6%/a compounded monthly. After 5 years, her financial advisor moved the accumulated amount to a new account that paid 8%/a compounded quarterly. Determine the balance in her account on January 1, 2013.

13. **T** On the day Sarah was born, her grandparents deposited $500 in a savings account that earns 4.8%/a compounded monthly. They deposited the same amount on her 5th, 10th, and 15th birthdays. Determine the balance in the account on Sarah's 18th birthday.

14. **K** Tresha paid for household purchases with her credit card. The credit card company charges 18%/a compounded monthly. Tresha forgot to pay the monthly bill of $465 for 3 months after it was due to be paid.

a) How much does Tresha owe at the end of each of the 3 months?

b) How much of each amount in part (a) is interest?

15. **C** Do an Internet search of the phrase "compound interest calculators." Try out two different online calculators. In what ways are they similar to the TVM Solver? In what ways are they different?

Extending

16. Asif bought an oil painting at a yard sale for $15 in 2001. Five years later, he took the painting to have it appraised. To his surprise, it was worth between $20 000 and $30 000. What annual interest rate corresponds to the growth in the value of his purchase?

17. A used car costs $32 000. The dealer offers a finance plan at 2.4%/a compounded monthly for 5 years with monthly payments. If you pay cash for the car, its cost is $29 000. The bank will loan you cash for 5.4%/a compounded monthly, also with monthly repayments for 5 years. Should you finance the purchase of the car through the dealer or through the bank? Explain.

18. Barry bought a boat 2 years ago, paying $10 000 toward the cost. Today he must pay the $7500 he still owes, which includes the interest charge on the balance due. Barry financed the purchase at 6.2%/a compounded semi-annually. Determine the purchase price of the boat.

8 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: What is the difference between simple interest and compound interest?

A: Simple interest is calculated only on the original principal. The formulas used are $I = Prt$ and $A = P(1 + rt)$, where

- I is the interest earned, in dollars
- P is the principal invested, in dollars
- r is the annual interest rate, expressed as a decimal
- t is the time, in years
- A is the final amount earned, in dollars

Simple-interest investments grow at a constant, or linear, rate over time.

Compound interest is calculated at regular periods, and the interest is added to the principal for the next period. The formula used is $A = P(1 + i)^n$, where

- A is the amount, or future value, of the investment, in dollars
- P is the principal, in dollars
- i is the interest rate, expressed as a decimal
- n is number of compounding periods

Compound-interest investments grow exponentially, as a function of the number of compounding periods.

Study *Aid*

- See Lesson 8.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 to 3.

Q: What is the difference between future value and present value, and how are they calculated for situations involving compound interest?

A: When an investment matures, both the principal and the interest are paid to the investor. This total amount is called the amount of the investment, or the future value of the investment.

The formula $A = P(1 + i)^n$ can be used to determine the future value. This formula is sometimes written as $FV = PV(1 + i)^n$, where

- FV is the future value, or amount of the investment, in dollars
- PV is the present value, or principal, in dollars

To see how this formula comes about, consider this example: Marina invests \$5000 in a savings account that pays 5.25%/a, compounded annually. In this case, $r = 5.25\%$ or 0.0525 and $P = 5000$. The amount at the end of the first year is

$$A = P + I$$
$$= P + Prt$$
$$= P(1 + rt)$$

Study *Aid*

- See Lesson 8.2, Examples 1 and 2, and Lesson 8.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 4 to 10.

$$\begin{aligned} A &= 5000(1 + 0.0525 \times 1) \\ &= 5000(1.0525)^1 \\ &= 5262.50 \end{aligned}$$

This amount becomes the new principal at the beginning of the second year and earns interest at the same rate. The amount at the end of the second year is

$$\begin{aligned} A &= 5262.50(1.0525) \\ &= 5000(1.0525)^2 \\ &= 5538.78 \end{aligned}$$

The amount at the end of the third year is

$$\begin{aligned} A &= 5538.78(1.0525) \\ &= 5000(1.0525)^3 \\ &= 5829.57 \end{aligned}$$

The general term is

$$A = 5538.78(1.0525)^n$$

The terms are $5000(1.0525)^1$, $5000(1.0525)^2$, $5000(1.0525)^3$, ….

Therefore, the sequence of year-end amounts is \$5262.50, \$5538.78, \$5829.57, ….

Present value is the amount of money that must be invested today at a given rate and compounding frequency in order to provide for a given amount in the future. The formula $A = P(1 + i)^n$ can be rearranged to $P = A(1 + i)^{-n}$ or $PV = FV(1 + i)^{-n}$ to determine present value.

Study Aid

- See Lesson 8.4, Examples 1 to 4.
- Try Mid-Chapter Review Questions 11 to 14.

Q: What are the advantages and disadvantages of using the TVM Solver to solve compound-interest problems?

A: Advantages: The TVM Solver is useful for quickly calculating values of unknown financial variables, such as present value, future value, interest rate, and number of payments. These equations are handled easily on the TVM Solver.

Disadvantages: You need to decide whether present value or future value should be entered as a negative or positive number. If you make a data entry error, it is not always easy to identify the specific error. You need to remember what each of the variables in the TVM Solver means.

PRACTICE Questions

Lesson 8.1

1. Copy and complete the table.

	Principal ($)	Annual Interest Rate (%)	Time	Simple Interest Paid ($)	Amount ($)
a)	250	2	3 years		
b)		2.5	200 weeks	38.46	
c)	1000	3.1	18 months	46.50	
d)	5000	5	30 weeks		
e)		4.2	5 years	157.50	
f)		3	54 months	202.50	

2. Tami earned \$20.64 in simple interest by investing a principal of \$400 in a Treasury bill. If the interest rate was 1.72%/a, for how many years did she have her investment?

3. You invest \$500 for 10 years at 10%/a simple interest. Your friend invests \$500 for 10 years at 10%/a interest compounded annually. Copy and complete the tables to compare the investments.

Year	Your Investment (10% Simple Interest)	
	Interest Earned ($)	Accumulated Interest ($)
1		
2		
3		
⋮		
10		

Friend's Investment (10% Compound Interest)	
Interest Earned ($)	Accumulated Interest ($)
⋮	

Lesson 8.2

4. Copy and complete the table.

	Principal ($)	Annual Interest Rate (%)	Years Invested	Compounding Frequency	Amount ($)	Interest Earned ($)
a)	400	5	15	annually		
b)	350	2.45	8	monthly		
c)		3.5	5	quarterly	500	
d)	120		7	semi-annually	150	
e)	2 500	7.6		monthly		350
f)	10 000	7.5	3	quarterly		

5. Sam invests $800 at 6%/a compounded annually for 5 years. What is the total interest earned?

6. $1000 is invested at 8%/a compounded daily for 10 years. What is the total interest earned?

7. You have inherited $30 000 and want to invest it for 20 years. You have two options: a Treasury bill that earns 3.48%/a interest compounded monthly and a GIC that earns 3.5%/a compounded semi-annually. Determine which investment is the better choice.

Lesson 8.3

8. Anthony wants to have $10 000 in 5 years. His bank will pay him 6%/a interest compounded monthly. How much does he have to invest now?

9. How much should be invested at $12\frac{1}{2}$%/a compounded semi-annually to amount to $1150 in $3\frac{1}{2}$ years?

10. Determine the present value of $3500 if the original deposit can earn 8%/a compounded quarterly.

Lesson 8.4

11. Steve's investment doubled from $1000 to $2000 over 8 years. He knows that the interest was compounded quarterly. What annual rate did he get on his investment?

12. How many years will it take to see a $500 investment grow to $937.70 if the annual interest rate is 4.5%/a compounded monthly? Round to the nearest year.

13. How long will it take an amount of money invested at 5%/a compounded annually to grow to five times as large as the principal? Round your answer to the nearest year.

14. Sarah has saved $500 from babysitting. She would like to put this money into a savings account. Bank A offers an account that pays 6 %/a compounded monthly while Bank B offers an account that pays 7% compounded semi-annually. Which bank will provide her with the most interest over a 2-year period?

8.5 Regular Annuities: Determining Future Value

GOAL

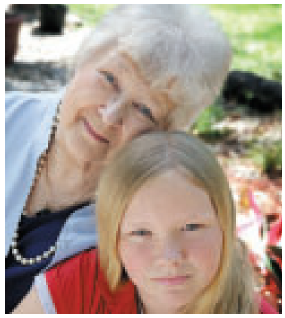
Solve future-value problems involving regular payments or deposits.

YOU WILL NEED

- graphing calculator with TVM Solver
- spreadsheet software

INVESTIGATE the Math

When Jessica turned 13 years old, her grandmother gave her \$500. Jessica deposited the money into a savings account that paid 4.5%/a compounded annually. Since then, Jessica's grandmother has made an automatic deposit of \$500 into Jessica's account on her birthday each year. Each payment earns a different amount of interest because it compounds for a different amount of time.



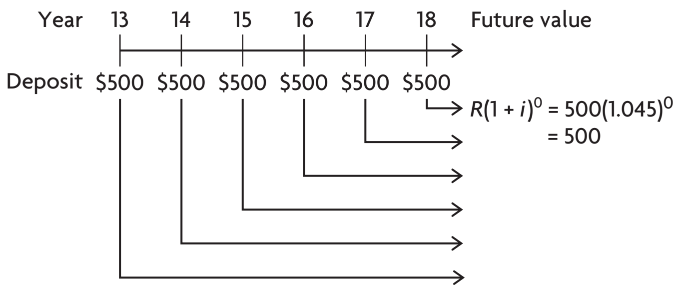
? How much money is in Jessica's bank account on her 18th birthday, and how much interest has it earned?

A. The timeline shows the deposits of the annuity made into Jessica's account. For how many compounding periods has the \$500 deposit on her 17th birthday earned interest? For how many compounding periods has each of the 16th, 15th, 14th, and 13th birthday deposits earned interest?

annuity

a series of equal deposits or payments made at regular intervals; a **simple** annuity is an annuity in which the payments coincide with the compounding period, or *conversion* period; an **ordinary** annuity is an annuity in which the payments are made at the end of each interval; unless otherwise stated, each annuity in this chapter is a simple, ordinary annuity

Future value of Jessica's annuity



B. Complete the timeline by calculating the amount, or future value, of each \$500 birthday deposit. The deposit made on Jessica's 18th birthday has not earned any interest. (In the diagram, R is like the variable P you have been using in earlier sections.)

C. What is the total amount of the annuity on Jessica's 18th birthday?

D. How much of the annuity is interest earned?

Reflecting

E. In what ways is calculating the amount, or future value, of an annuity the same as calculating the amount, or future value, of a single deposit? In what ways is it different?

F. Why can't the formula $A = 5[500(1 + 0.03)^5]$ be used to calculate the amount of Jessica's annuity?

G. Create an expression that represents the sum of the amounts, or future value, of the six deposits. Use expressions of the form $R(1 + i)^n$, where
- R is the regular deposit or payment, in dollars
- i is the interest rate per compounding period, expressed as a decimal
- for each deposit, n is the number of compounding periods

APPLY the Math

EXAMPLE 1 Selecting a strategy to calculate the amount of an annuity

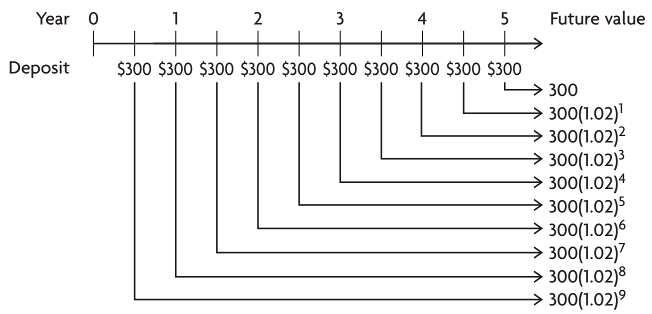
Steve makes deposits of \$300 semi-annually into an account that pays 4%/a interest compounded semi-annually.

a) How much money will be in the account after 5 years?

b) How much interest will Steve have earned over the 5-year term?

Marta's Solution: Using a Timeline

a)



The amount of the regular payment is R = \$300.

The annual interest rate is 4%, or 0.04, so the semi-annual interest rate is $i = 0.02$.

I created a timeline to organize the calculations of the future value of each deposit.

Future value of Steve's annuity

Total amount

$= 300 + 300(1.02)^1 + 300(1.02)^2 + \dots + 300(1.02)^9$

$= 300 + 306 + 312.12 + 318.36 + 324.73$
$\quad + 331.22 + 337.85 + 344.61 + 351.50 + 358.53$

$= 3284.92$

The total of the individual future values is the amount in Steve's account at the end of year 5.

Steve will earn a total of \$3284.92.

b) $3284.92 - (10 \times 300) = 284.92$

\$284.92 of the amount is interest earned.

The total of the deposits is 10 × \$300. The difference between the deposits and the total amount is the interest earned.

Joseph's Solution: Using a Spreadsheet

Formulas (as entered)

	A	B
1	**Deposit**	**Future Value**
2	0	=300*(1.02)^A2
3	=1+A2	=300*(1.02)^A3
		⋮
12	Sum	=SUM(B2:B11)
13	Interest	=B12 − (10*300)

Values (as displayed)

	A	B
1	**Deposit**	**Future Value**
2	0	300.00
3	1	306.00
4	2	312.12
	⋮	
11	9	358.53
12	Sum	3284.92
13	Interest	284.92

I entered the deposit numbers into the first column. In the second column, I used the formula $300(1.02)^n$ to calculate the future value of each deposit.

I used the Fill Down command to complete and display the future values.

To determine the future value of the annuity, I calculated the total of all these amounts. I then subtracted the amount of the actual deposits to calculate the interest.

a) Steve will earn a total of $3284.92.

b) $284.92 of the amount is interest earned.

Sergei's Solution: Use the TVM Solver

a)

I entered 10 beside **N** for the 10 compounding periods (not the number of years) and 4 beside **I** for 4% annual interest. **PV** is not required, so it is 0. I entered −300 beside **PMT** because this is money Asif must pay. **P/Y** and **C/Y** are both 2 because the payments are made and compounded semi-annually.

I wanted to solve for the future value, so I entered 0 for **FV**. I placed the cursor beside **FV** because I was solving for it.

I solved for **FV**. The future value is positive because it is money that Asif will receive.

Steve will earn a total of $3284.92.

b) $3284.92 - (10 \times 300) = 284.92$

$284.92 of the amount is interest earned.

I calculated the interest earned by subtracting the total of the payments, $10 \times \$300$, from the future value.

If technology is not available to help you calculate the future value of an annuity, you can do so using a formula.

EXAMPLE 2 Using a formula to determine the future value of an annuity

Jay deposits \$1500 every 3 months for 2 years into a savings account that earns 10%/a compounded quarterly. How much money will have accumulated at the end of 2 years?

Measha's Solution

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

I wrote the formula for calculating the future value of an annuity, where

- A is the amount, or future value, in dollars
- R is the regular deposit, or payment, in dollars
- i is the interest rate per compounding period, expressed as a decimal
- n is the total number of deposits

$$R = \$1500$$

$$\begin{aligned} i &= 10\% \div 4 \\ &= 2.5\% = 0.025 \end{aligned}$$

The interest rate is 10% compounded quarterly. To determine i, I divided it by 4.

$$n = 2 \times 4 = 8$$

I multiplied the number of years by 4 to determine the number of compounding periods.

$$\begin{aligned} A &= \frac{1500[(1 + 0.025)^8 - 1]}{0.025} \\ &= 13\,104.17 \end{aligned}$$

Jay will have \$13 107.17 at the end of 2 years.

EXAMPLE 3 Selecting a strategy to calculate the regular payment

An investor wants to retire in 25 years with \$1 000 000 in savings. Her current investments are earning, on average, 12%/a compounded annually.

a) What regular annual deposit must she make to have the required amount at retirement?

b) How much of the \$1 000 000 is interest earned?

Andrea's Solution: Using a Formula

a) $$A = \frac{R[(1 + i)^n - 1]}{i}$$

I wrote the formula for calculating the future value of an annuity.

$$1\,000\,000 = \frac{R[(1 + 0.12)^{25} - 1]}{0.12}$$

The future value is $A = \$1\,000\,000$.

The annual interest rate is $i = 0.12$.

Regular deposits are made every year, so $n = 25$.

$$120\,000 = R[(1.12)^{25} - 1]$$

I needed to solve for R, so I multiplied each side by 0.12.

$$R = \frac{120\,000}{[(1.12)^{25} - 1]}$$

$$= 7499.97$$

I isolated R by dividing both sides by the expression in square brackets.

I solved for R by simplifying the right side.

The investor needs to deposit $7499.97 annually.

Pradesh's Solution: Using the TVM Solver

a)

I wanted to solve for the payment, so I entered 0 for **PMT**.

I entered 25 beside **N** for 25 compounding periods and 12 beside **I%** for an annual interest of 12%.

PV is not required, so it is 0. I entered 1 000 000 beside **FV** for the desired savings after 25 years.

Since the payments are made once a year and compounded annually, **P/Y** and **C/Y** are 1.

I placed the cursor beside **PMT** because I was solving for the payments.

I solved for **PMT**. The payments are negative because the investor pays them.

b) $25 \times \$7499.97 = 187\,499.25$

Over 25 years, the deposits total $187 499.25.

$$1\,000\,000 - 187\,499.25 = 812\,500.75$$

$812 500.75 of the $1 000 000 is interest earned.

There are 25 deposits of $7499.97. I calculated the total amount deposited and subtracted it from the final amount of $1 000 000.

In Summary

Key Ideas

- Since an annuity is a series of equal deposits made at regular intervals, the amount, or future value, can be found by determining the sum of all the future values for each regular payment.
- The amount, or future value, of an annuity is the sum of all deposits and the accumulated interest and can be found with the formula

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

where

- A is the amount, or future value, in dollars
- R is the regular deposit, or payment, in dollars
- i is the interest rate per compounding period, expressed as a decimal
- n is the total number of deposits

Need to Know

- Problems involving annuities can be solved with a formula, spreadsheet software, or financial software such as the TVM Solver.
- When the TVM Solver is used to solve problems that involve regular payments, the present value, **PV**, and future value, **FV**, are set to 0.

CHECK Your Understanding

1. Draw a timeline representing an annuity of semi-annual payments of \$450 for 3 years at 12%/a compounded semi-annually. Use the timeline to show how the future value of each payment contributes toward the future value of the annuity.

2. Geoff and Marilynn are each investing in a 3-year Registered Retirement Savings Plan (RRSP) fund at 6%/a compounded quarterly. Geoff will make one deposit of \$3600 at the beginning of the first year. Marilynn will make a \$300 deposit at the end of March and will continue to contribute \$300 every quarter until the end of the third year. Determine the difference in the future values.

PRACTISING

3. Determine the amount of each annuity.

a) Regular deposits of \$500 every 6 months for 4 years at 8%/a compounded semi-annually

b) Regular deposits of \$200 every month for 8 years at 10%/a compounded monthly

4. For each situation, identify R, i, and n. Then use the formula

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

to determine the amount of the annuity.

	Payment ($)	Interest Rate	Compounding Period	Term of Annuity	Amount ($)
a)	1000	8%/a	annually	3 years	
b)	500	$7\frac{1}{2}$%/a	quarterly	8.5 years	
c)	200	3.25%/a	monthly	5 years	

5. Calculate the regular deposit made twice a year for 5 years at 6%/a compounded semi-annually to accumulate an amount of $4000.

6. Carollynne has found her dream home in Pictou, Nova Scotia. It is selling for $500 000. When she retires 2 years from now, she plans to sell her present house for $450 000 and move. She decides to set aside $900 every two weeks until she retires in a fund earning 10.5%/a, compounded every second week. What is the difference between the future value of Carollynne's investment and the extra $50 000 she needs for her dream home?

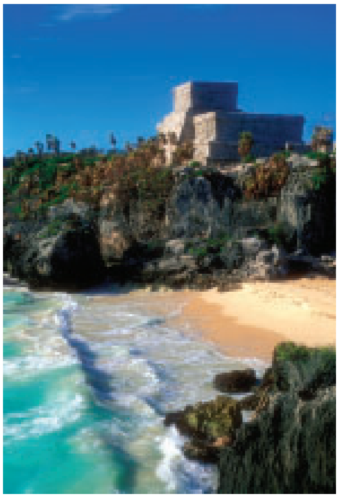
7. Yanmei has contributed $250 to an RRSP at the end of each 3-month period for the past 35 years. During this time, the RRSP has earned an average of 11.5%/a compounded quarterly.

a) How much will the RRSP be worth at maturity?

b) How much of the investment will be interest earned over the 35 years?

8. Miguel wants to buy an entertainment system as a gift for his sister's wedding. He estimates that when she marries 1 year from now, the system will cost $2499, plus GST (government sales tax) at 6% and PST (provincial sales tax) at 8%. He knows he can deposit $225 a month into an account earning 3.5%/a compounded monthly. Will he have enough money to buy the gift? Explain.

9. T At the end of every 6 months, Marcia deposits $100 in a savings account that pays 4%/a compounded semi-annually. She made the first deposit when her son was 6 months old, and she made the last deposit on her son's 21st birthday. The money remained in the account until her son turned 25, when Marcia gave it to him. How much did he receive?



10. Marcel would like to take a vacation to Mexico during March break, 6 months from today. The trip will cost \$3600. Marcel deposits \$195 into an account at the end of each month for the next 8 months at 9%/a compounded monthly. Will he have enough money to pay for his trip? Explain.

11. Mario deposits \$25 at the end of each month for 4 years into an account that pays 9.6%/a compounded monthly. He then makes no further deposits and no withdrawals. Determine the balance 10 years after his last deposit.

12. A Darcey would like to accumulate \$80 000 in savings before she retires 20 years from now. She intends to make the same deposit at the end of each month in an RRSP that pays 6.3%/a compounded monthly.

a) Draw a timeline to represent the annuity.

b) What regular payment will let Darcey reach her goal?

c) Suppose Darcey decides to wait 5 years before starting her deposits. What regular payment would she have to make to reach the same goal?

13. Describe the payments, interest rates, and type of compounding necessary for a 15-year annuity with a future value between \$10 000 and \$12 000. Use two different compounding periods, each at a different interest rate, to modify the amounts shown.

14. K Which annuity will earn the greater amount at the end of 2 years? Justify your answer.

a) \$50 at the end of every week at 5%/a compounded weekly

b) \$2600 at the end of every year at 5%/a compounded annually

15. C Explain why the formula for calculating the accumulated value of a simple annuity would not work if the interest-compounding period did not coincide with the payment interval.

Extending

16. Byron has just bought a car for \$25 000. He plans to replace it with a similar car in 3 years. At that time, his current car will be worth about one-third of its current value, and he will trade it for the new car. He will start saving for the rest of the cost by investing every month into an account paying 4.5%/a compounded monthly. How much should each payment be so that he can pay cash for the new car?

17. Nastassia borrowed \$4831 at 7.5%/a compounded monthly. She has made 30 monthly payments of \$130 each. She is now in a position to pay off the balance. What is that balance?

8.6 Regular Annuities: Determining Present Value

GOAL

Find the present value when payments or deposits are made at regular intervals.

YOU WILL NEED

- graphing calculator with TVM Solver
- spreadsheet software

LEARN ABOUT the Math

Harry has money in an account that pays 9%/a compounded annually. One year from now he will go to college. While Harry attends college, the annuity must provide him with 4 equal annual payments of $5000 for tuition.

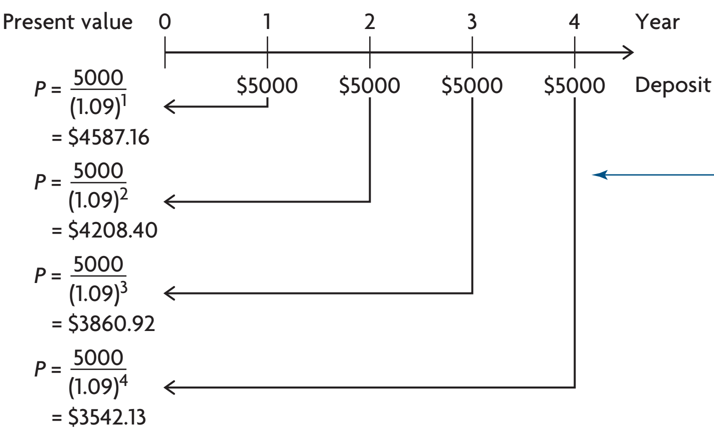
? How much must be in Harry's account now if the first payment starts in a year?

EXAMPLE 1 Selecting a strategy to determine the present value of an annuity

Determine the present value of Harry's annuity.

Rahiv's Solution: Using a Timeline

Present Value of Annuity at 9%/a Compounded Annually



I used a timeline to organize the solution.

The annual interest rate is 9%, or 0.09. At the end of 1 year, the first payment's present value will have earned interest for 1 year: $n = 1$.

The second payment's present value will have earned interest for 2 years: $n = 2$.

The third payment's present value will have earned interest for 3 years: $n = 3$.

The fourth payment's present value will have earned interest for 4 years: $n = 4$.

$$PV = \frac{5000}{(1.09)^1} + \frac{5000}{(1.09)^2} + \frac{5000}{(1.09)^3} + \frac{5000}{(1.09)^4}$$

$$= 4587.16 + 4208.40 + 3860.92 + 3542.13$$

$$= 16\ 198.61$$

The present value is the total of the present values of all the payments.

To have four equal annual payments of \$5000 starting 1 year from now, Harry needs \$16 198.61 in his account now.

Tamika's Solution: Use a Spreadsheet

Formulas (as entered)

	A	B
1	End of Year	Present Value
2	1	$= \frac{5000}{(1 + 0.09)^\wedge A2}$
3	= A2 + 1	$= \frac{5000}{(1 + 0.09)^\wedge A3}$
4	= A3 + 1	$= \frac{5000}{(1 + 0.09)^\wedge A4}$
5	= A4 + 1	$= \frac{5000}{(1 + 0.09)^\wedge A5}$
6	Total PV	= SUM(B2:B5)

Values (as displayed)

	A	B
1	End of Year	Present Value
2	1	4587.16
3	2	4208.40
4	3	3860.92
5	4	3542.13
6	Total PV	16 198.60

The annual interest rate is 9%, or 0.09.

The formulas I used in the spreadsheet are in columns A and B. I knew that the amount paid at the end of each year had earned interest from the beginning of the annuity. So I used the year number, located in column A, in the formula to calculate the present value of each payment.

I read the total present value of the annuity from cell B6.

Reflecting

A. In calculating the present value of an annuity, the amount of each payment is divided by a factor of $(1 + i)$ for each additional compounding period. Why does this factor increase the distance in the future the payment is made?

B. Compare the methods used in the two solutions. What are the advantages and disadvantages of each?

C. If Harry were to receive payments every month instead of every year, which method would you use? Explain.

APPLY the Math

If technology is not available to help you calculate the present value of an annuity, then you can also use a formula.

EXAMPLE 2 Using a formula to determine the present value of an annuity

Roshan has set up an annuity to help his son pay living expenses over the next 5 years. The annuity will pay \$50 a month. The first payment will be made 1 month from now. The annuity earns 7.75%/a compounded monthly.

a) How much money did Roshan put in the annuity?

b) How much interest will the annuity earn over its term?

c) Verify your results using the TVM solver.

Tim's Solution

a) $PV = \dfrac{R[1 - (1 + i)^{-n}]}{i}$

$R = 50$

I wrote the formula for calculating the present value of an annuity, where
- *PV* is the present value, in dollars
- *R* is the regular payment, in dollars
- *i* is the interest rate per compounding period, expressed as a decimal
- *n* is the total number of payments

$i = \dfrac{0.0775}{12} = 0.006\,458\,333\,3$

$n = 5 \times 12 = 60$

The annual interest rate is $7\frac{3}{4}$%. The monthly interest rate is $\frac{1}{12}$ of the annual rate, so I divided it by 12. I multiplied the number of years by 12 to determine the number of compounding periods.

$PV = \dfrac{50[1 - (1 + 0.006\,458\,333\,3)^{-60}]}{0.006\,458\,333\,3}$

I substituted the values for *R*, *i*, and *n*, and calculated *PV*.

$PV = 2480.53$

Roshan put \$2480.53 in the annuity.

b) $60 \times 50 = 3000$

$3000 - 2480.53 = 519.47$

The annuity earned \$519.47 in interest over its term.

I determined the total interest earned over the term of the annuity by subtracting the present value of the annuity from the total value of payments.

c)

I entered 60 beside **N** for the 60 compounding periods. I entered 7.75 beside **I%** for 7.75% annual interest. The regular payment of \$50 means that Roshan has paid this amount out to his son, so I entered −50 beside **PMT**. **FV** is not required, so I entered a value of 0.

I entered 12 for both **P/Y** and **C/Y** because the payments are made and compounded monthly.

I placed the cursor beside **PV** because I was solving for the present value.

I solved for **PV**. The present value is positive because it is money that is deposited into the bank at the beginning of the annuity to provide for the monthly withdrawls.

I could have also used **PMT** $= +50$, and the **PV** would have been $-2480.532\ 809$.

Roshan put \$2480.53 in the annuity.

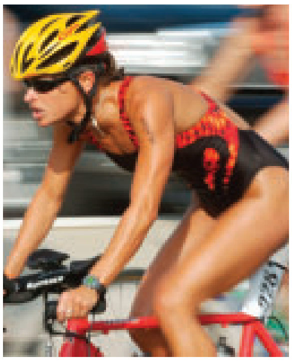
I used the TVM Solver's interest function and entered 1 for the starting payment number and 60 for the ending payment number.

The annuity earned \$519.47 in interest.

Tech Support

Before using the **ΣInt** function, make sure that values have been entered into the TVM Solver for **N**, **I%**, **PV**, **PMT**, **P/Y**, **C/Y**, and **PMT:END**. For more help with using the TVM Solver to solve problems involving compound interest, see Technical Appendix, B-15.

EXAMPLE 3 Selecting a strategy that uses present value to calculate the payment of an annuity



Robin bought a bicycle for \$1500. She arranged to make a payment to the store at the end of every month for 1 year. The store is charging 11%/a interest compounded monthly.

a) How much is each monthly payment?

b) How much interest is Robin paying?

Leshawn's Solution: Using a Formula

a) $PV = \dfrac{R[1 - (1 + i)^{-n}]}{i}$

I used the formula for present value so that I could solve for the payment.

$PV = 1500$

$i = \dfrac{11}{12}\% = 0.009\ 166\ 666\ 7$

$n = 12$

The present value is $PV = \$1500$. The annual interest rate is 11%, so the monthly interest rate is $\frac{1}{12}$ as much.

She makes 12 payments in a year.

$$1500 = \frac{R[1 - (1 + 0.009\ 17)^{-12}]}{0.009\ 17}$$

$$\frac{0.009\ 17(1500)}{1 - (1 + 0.009\ 17)^{-12}} = R$$

$$132.57 = R$$

I multiplied both sides of the equation by 0.009 17, then I divided both sides by $1 - (1 + 0.009\ 17)^{-12}$ to solve for R.

Robin makes monthly payments of $132.57.

b) $132.57 \times 12 = 1590.84$

$1590.84 - 1500.00 = 90.84$

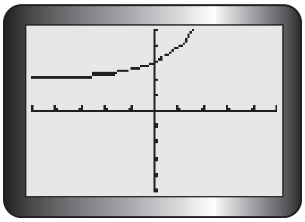
To calculate the interest paid, I found the total of the payments and subtracted the cost of the bicycle. The difference was the interest paid to the store.

The interest paid to the store is $90.84.

Henrique's Solution: Using the TVM Solver

a)

I entered 12 beside **N** for the 12 compounding periods and 11 beside **I%** for 11% annual interest. The bicycle cost $1500, so I entered −1500 beside **PV**, because this is what she paid for the bike out of her bank account. **FV** is not required, so I entered 0. I entered 12 for both **P/Y** and **C/Y** because the payments are made and compounded monthly.



I placed the cursor beside **PMT** because I was solving for the payment.

I solved for **PMT**. The payment is positive because the money is paid into the account.

Robin makes monthly payments of $132.57.

b)

I used the TVM Solver's interest function and entered 1 for the starting payment number and 12 for the ending payment number.

The interest paid to the store is $90.87.

In Summary

Key Ideas

- The present value of an annuity is (1) the amount that must be invested now to provide payments of a specific amount at regular intervals over a certain term or (2) the amount borrowed or financed now that must be paid for by deposits of a specific amount at regular intervals over a certain term.
- The present value of an annuity is the sum of the present values of all of the regular payments.
- The formula for calculating the present value of an annuity is

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

where
- PV is the present value, in dollars
- R is the regular payment, in dollars
- i is the interest rate per compounding period, expressed as a decimal
- n is the number of payments or withdrawals

Need to Know

- Problems involving annuities can be solved with a formula, spreadsheet software, or financial software such as the TVM Solver.

CHECK Your Understanding

1. Draw a timeline to represent an annuity of semi-annual payments of \$300 for 3 years at 8%/a compounded semi-annually. Use the timeline to organize a solution that shows how the present value of each payment contributes toward the present value of the annuity.

2. For each situation, identify R, i, and n in the formula

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}.$$

Then determine the present value of the annuity.

	Withdrawal (\$)	Annual Interest Rate (%)	Compounding Period	Term of Annuity
a)	750	8	annual	3 years
b)	450	$7\frac{1}{2}$	quarterly	8.5 years
c)	225	3.25	monthly	5 years

3. Solve for each unknown.

a) $PV = \dfrac{450(1 - 1.055^{-8})}{0.055}$

b) $40\ 000 = \dfrac{R(1 - 1.006^{-12})}{0.006}$

PRACTISING

4. Kevin came into an inheritance of \$36 000, to be paid in equal monthly instalments for the next 5 years, starting 1 month from now. The money earns 7.5%/a compounded monthly. Determine the monthly instalment.

5. Mary needs \$750 a year for 3 years to buy textbooks. She will start university in 1 year. Her savings account pays 4%/a compounded annually. How much needs to be in her account now to pay for the books?

6. Your grandmother has set up an annuity of \$4000 in an account that pays 5.2%/a compounded monthly. What equal monthly payments will the annuity provide for in the next 4 years?

7. May Sum has saved \$125 000 in an investment account. She will use it to buy an annuity that pays 6.5%/a compounded quarterly. She will receive quarterly payments for the next 25 years. The first payment will be made 3 months from now.

a) What is the quarterly payment she will receive?

b) What is the interest earned over the duration of the annuity?

8. Claire buys a snowboard for \$150 down and pays \$35 at the end of each month for 1.5 years. If the finance charge is 16%/a compounded monthly, determine the selling price of the snowboard.

9. Felix's family has decided to deposit \$350 into an annuity every 3 months for 4 years. The account will earn 3.75%/a compounded quarterly. Starting 3 months after the last deposit, Felix will withdraw the money every 3 months in equal payments for 2 years. What is the amount of each withdrawal?

10. **K** Nick plans to buy a used car today. He can afford to make payments of \$250 each month for a maximum of 3 years. The best interest rate he can find is 9.8%/a compounded monthly. What is the most he can spend?

11. **A** Shimon wants to buy a speedboat that sells for \$22 000, including all taxes. The dealer offers either a \$2000 discount, if Shimon pays the total amount in cash, or a finance rate of 2.4%/a compounded monthly, if Shimon makes equal monthly payments for 5 years.

a) Determine the monthly payment that Shimon must make if he chooses the second offer.

b) What is the total cost of the dealer's finance plan for the speedboat?

c) To pay for the boat with cash now, Shimon can borrow the money from the bank at 6%/a, compounded monthly, over the same 5-year period. Which offer should Shimon choose, the bank's or the dealer's? Justify your answer.

12. René buys a computer system for $80 down and 18 monthly payments of $55 each. The first payment is due next month.

a) The interest rate is 15%/a compounded monthly. What is the selling price of the computer system?

b) What is the finance charge?

13. Betty is retiring. She has $100 000 in savings. She is concerned that she will not have enough money to live on. She would like to know how much an annuity, compounded monthly, will pay her each month for a variety of interest rates. She needs to know the monthly payments over the next 10 years, starting next month. Use a spreadsheet and different annual interest rates to prepare three different schedules of payments for Betty.

14. T The present value of the last payment of an annuity is $2500(1.05)^{-36}$.

a) Describe two annuities, with different compounding periods, that can be represented by the present value of the last payment.

b) Calculate the present values of the total payments for each annuity in part (a).

15. C The screens shown were obtained from the TVM Solver. Write a problem that corresponds to the information from each screen.

a)

b)

Extending

16. Do the following situations double the amount of an annuity at maturity?

a) Double the duration of the annuity.

b) Double each payment made.

Use examples to support your explanation.

17. Rudi deposited $100 at the end of each month into an annuity that paid 7.5%/a compounded monthly. At the end of 6 years, the interest rate increased to 8.5%/a . The deposits were continued for another 5 years.

a) What is the amount of the annuity on the date of the last deposit?

b) What is the interest rate earned on the annuity over the 11 years?

18. Kyla must repay her $17 000 student loan. She can afford to make monthly payments of $325. The bank's interest rate is 7.2%/a compounded monthly. Determine how long it will take Kyla to repay her loan.

8.7 Saving Plans and Loans: Creating Amortization Tables

GOAL

Examine how changing the conditions of an annuity affects the interest earned or paid.

YOU WILL NEED

- graphing calculator with TVM Solver
- spreadsheet software

INVESTIGATE the Math

One day you may wish to buy a home or a car. You might not be able to pay for it all at once. However, you might be able to make smaller payments toward its purchase over time. You can take out a loan or mortgage from a financial institution. You sign a contract agreeing to repay, or amortize, the value of the loan, plus interest, by making regular equal payments for the term, or **amortization** period, of the loan. A portion of each payment is interest. The rest of the payment is applied to reduce the principal, or amount borrowed.

amortization

the process of gradually reducing a debt through instalment payments of principal and interest

Tyler bought a used car for $10 000. He made a **down payment** of $3500. The car dealer offered to finance the rest of the purchase with a 1-year loan at 8%/a compounded monthly.

down payment

the partial amount of a purchase paid at the time of purchase

? What conditions affect the monthly payment and the amount of interest paid to the car dealer?

A. Study amortization table A, on the next page, which shows the repayment schedule of the $6500 loan over the period of 1 year. Determine

- the payment made each month
- total payments made to the car dealership, including down payment
- the total interest paid to the car dealership

B. Tyler thinks he can negotiate a 6%/a interest rate. Using the amortization spreadsheet, he replaces the annual rate of 0.08 with 0.06. The spreadsheet automatically recalculates information. Study amortization table B, on the next page. What information is recalculated? How does the lower interest rate affect the interest paid to the car dealer and the total cost of the car?

A.

Cost of Car	$10 000	Annual Rate	0.08	
Down Payment	$3500	Monthly Rate	0.0067	
Loan	$6500	Number of		
Payment	$565.42	Payments	12	
Payment Number	**Payment ($)**	**Interest Paid ($)**	**Principal Paid ($)**	**Outstanding Balance ($)**
0				6500.00
1	565.42	43.33	522.09	5977.91
2	565.42	39.85	525.57	5452.34
3	565.42	36.35	529.08	4923.26
4	565.42	32.82	532.60	4390.66
5	565.42	29.27	536.15	3854.50
6	565.42	25.70	539.73	3314.78
7	565.42	22.10	543.33	2771.45
8	565.42	18.48	546.95	2224.50
9	565.42	14.83	550.59	1673.91
10	565.42	11.16	554.27	1119.64
11	565.42	7.46	557.96	561.68
12	565.42	3.74	561.68	0.00
Totals	6785.04	285.04	6500.00	

B.

Cost of Car	$10 000	Annual Rate	0.06	
Down Payment	$3500	Monthly Rate	0.0050	
Loan	$6500	Number of		
Payment	$559.43	Payments	12	
Payment Number	**Payment ($)**	**Interest Paid ($)**	**Principal Paid ($)**	**Outstanding Balance ($)**
0				6500.00
1	559.43	32.50	526.93	5973.07
2	559.43	29.87	529.57	5443.50
3	559.43	27.22	532.21	4911.29
4	559.43	24.56	534.88	4376.41
	⋮			⋮
10	559.43	8.31	551.12	1110.53
11	559.43	5.55	553.88	556.65
12	559.43	2.78	556.65	0.00
Totals	6713.16	213.16	6500.00	

C. To change the payments so that the loan is amortized over 2 years, Tyler enters 24 beside **Number of Payments**. What information is recalculated? How does the extended term affect the interest paid to the car dealer and the total cost of the car?

Cost of Car	$10 000	Annual Rate	0.06
Down Payment	$3500	Monthly Rate	0.005
Loan	$6500	Number of	
Payment	$288.08	Payments	24

Payment Number	Payment ($)	Interest Paid ($)	Principal Paid ($)	Outstanding Balance ($)
0				6500.00
1	288.08	32.50	255.58	6244.42
2	288.08	31.22	256.86	5987.55
3	288.08	29.94	258.15	5729.41
4	288.08	28.65	259.44	5469.97
	⋮			⋮
22	288.08	4.28	283.81	571.88
23	288.08	2.86	285.22	286.65
24	288.08	1.43	286.65	0.00
Totals	6913.92	413.92	6500.00	

D. Tyler changes the spreadsheet by increasing the down payment and changing the interest and payment periods back to 1 year at 8%/a compounded monthly. How does increasing the down payment affect the interest paid and total cost of the car as determined in part A?

Cost of Car	$10 000	Annual Rate	0.08
Down Payment	$5000	Monthly Rate	0.0067
Loan	$5000	Number of	
Payment	$434.94	Payments	12

Payment Number	Payment ($)	Interest Paid ($)	Principal Paid ($)	Outstanding Balance ($)
0				5000.00
1	434.94	33.33	401.61	4598.39
2	434.94	30.66	404.29	4194.10
	⋮			⋮
11	434.94	5.74	429.20	432.06
12	434.94	2.88	432.06	0.00
Totals	5219.28	219.28	5000.00	

Tech Support

For help creating amortization schedules using a spreadsheet, see Technical Appendix, B-18.

Reflecting

E. What are the advantages and disadvantages of using an amortization table that has been created with spreadsheet software?

F. The formulas that are placed in the cells of the spreadsheet amortization table are shown here. For each quantity, locate the formula and describe its calculation.

a) present value
b) interest rate for compounding period
c) payment
d) interest paid

	A	B	C	D	E
1	Cost of Car		10 000	Annual Rate	0.08
2	Down Payment		3500	Monthly Rate	=E1/E3
3	Loan		=C1−C2	Number of Payments	12
4	Payment		=E2*C3/(1−(1+E2)^(−E3))		
5	Payment	Payment	Interest	Principal	Outstanding
6	Number		Paid	Paid	Balance
7	0				=C3
8	=A7+1	=$C$4	=E7*E$2	=B8−C8	=E7−D8
9	=A8+1	=$C$4	=E8*E$2	=B9−C9	=E8−D9
	⋮				⋮
18	=A17+1	=$C$4	=E17*E$2	=B18−C18	=E17−D18
19	=A18+1	=$C$4	=E18*E$2	=B19−C19	=E18−D19
20	Totals	=SUM(B8:B19)	=SUM(C8:C19)	=SUM(D8:D19)	

APPLY the Math

EXAMPLE 1 Selecting a strategy to compare the growth of annuities

Compare the amounts at age 65 that an RRSP at 6%/a compounded annually would earn under each option.

Option 1: making an annual deposit of $1000 starting at age 20
Option 2: making an annual deposit of $3000 starting at age 50

What is the total of the deposits in each situation?

Raina's Solution: Using Amortization Tables

Option 1:

Annual Rate	0.0600	Start Year	20
Rate per Period	0.0600	End Year	65
Compounding Periods/Year	1	Final Year End	45
Contribution	$1000		

End of Year	Interest ($)	Payment Made ($1000/year + interest) ($)	New Balance ($)
1		1000.00	1000.00
2	60.00	1060.00	2060.00
	⋮		⋮
44	11 250.45	12 250.45	199 758.03
45	11 985.48	12 985.48	212 743.51

From this amortization table, after 45 years of investing $1000, the balance of the RRSP is $212 743.51. The interest earned over 45 years is $167 743.51, on a total deposit of $45 000.

Option 2:

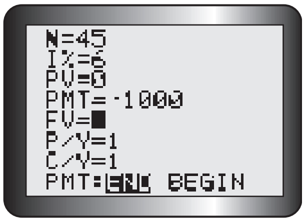
Annual Rate	0.0600	Start Year	50
Rate per Period	0.0600	End Year	65
Compounding Periods/Year	1	Final Year End	15
Contribution	$3000		

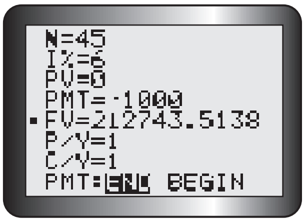
End of Year	Interest ($)	Payment Made ($3000/year + interest) ($)	New Balance ($)
1		3000.00	3000.00
2	180.00	3180.00	6180.00
	⋮		⋮
14	3398.78	6398.78	63 045.20
15	3782.71	6782.71	69 827.91

If the start year is changed to 50 and the contribution is changed to $3000, the balance of the RRSP is $69 827.91 and the interest earned over 15 years is $24 827.91, on a total deposit of $45 000.

Derek's Solution: Using the TVM Solver

Option 1:





There are 45 annual deposits from age 20 to age 65, so I entered 45 for **N**.

I entered 6 beside **I%**.

I entered 0 for **PV** because the first deposit does not go in until the end of the first year.

The annual RRSP deposit is $1000, so I entered −1000 for **PMT**. I entered 1 for both **P/Y** and **C/Y**.

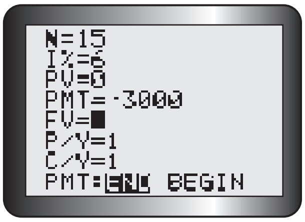
I solved for **FV**.

The future value of the RRSP deposits is $212 743.51.

Using the **ΣInt** function of the Finance Program, I calculated the interest over the given period.

The interest earned over 45 years is $167 743.51.

Option 2:



There are 15 annual deposits from age 50 to age 65, so I changed **N** to 15.

The annual RRSP deposit is $3000, so I changed **PMT** to −3000. I solved for **FV**.

The future value of the RRSP deposits is $69 827.91.

Using the **ΣInt** function of the Finance Program, I calculated the interest.

The interest earned over 15 years is $24 827.91.

EXAMPLE 2 Changing the payment frequency

Show how changing the payment frequency from semi-annual to weekly affects the amount of interest paid and the length of time needed to repay a loan of $5000.00 at 11%/a. For example, Joe makes semi-annual payments of $520 and interest is charged semi-annually, while Sarit makes weekly payments of $20 and interest is charged weekly.

Carmen's Solution

Compounding Period	TVM Solver Screen	Number of Payments	Time
Joe: semi-annual payments of $520		14.06	$\frac{14.06}{2}$ = 7.03 years
		The interest paid was $2308.32.	
Sarit: weekly payments of $20		356	$\frac{356}{52}$ = 6.85 years
		The interest paid was $2122.74	

Both Joe and Sarit are repaying the same amount each year.
$520 × 2 = $1040
$20 × 52 = $1040
I used the TVM Solver to compare the two options.

$2308.32 − $2122.74 = $185.58

The time needed to repay the loan decreases from 7.03 to 6.85 years under the weekly option, resulting in a savings of $185.58 in interest paid out.

EXAMPLE 3 Changing the payment

David has a credit card balance of $10 000 that is charged 23%/a interest, compounded monthly. He decides not to use his credit card again and to make monthly payments to pay off his debt.

a) How long does it take to reduce his credit card balance of $10 000 to 0 if he pays $1000 a month?

b) If he increases his monthly payment to $1600, how much sooner is the debt paid off?

c) How much will he save in finance charges if his payment is $1600 rather than $1000?

Mandy's Solution

a)

I entered 23 beside **I%** and 10 000 beside **PV**.

I entered −1000 for **PMT** because it is money David pays out.

The future value of the debt will be 0 when it is paid off, so I entered 0 for **FV**.

I entered 12 for **P/Y** and **C/Y**.

I placed the cursor beside **N** because I was solving for the number of monthly payments.

I solved for the number of compounding periods. I rounded up to 12 because David would have to make the monthly payment 11 times and a smaller payment the last time.

The credit card balance will be paid off after 12 months.

b)

I changed **PMT** to −1600 and placed the cursor beside **N**. I solved for the number of compounding periods, rounding up to 7.

The credit card balance will be paid off after 7 months.

$12 - 7 = 5$

I calculated the difference of the two numbers of monthly payments.

The debt will be paid off about 5 months earlier if David pays $1600 a month.

c) $(11.207\,658\,83 \times 1000) - (6.720\,813\,35 \times 1600) = 454.36$

I could have used the TVM ΣInt function to compare the interests paid, but I decided to calculate the total amount paid for payments of $1000 and subtracted the total amount paid for payments of $1600.

The amount saved if David pays $1600 a month is $454.36.

In Summary

Key Ideas

- Amortization tables show the amount of each loan payment or savings deposit, the interest portion, the principal portion, and the new balance after each payment. An amortization table can be created with spreadsheet software.
- Once an amortization table has been set up on a spreadsheet, it is easy to change the conditions of the situation and examine the impact the change has on quantities such as the duration of a loan, the amount of interest paid, and the total amount of money paid out.

Need to Know

- The TVM Solver can be used to investigate the effects of changing the conditions when borrowing or investing.
- **ΣInt** is a financial function that may be used after entering information in the TVM Solver. **ΣInt** is used to calculate the total interest paid from a starting payment number to an ending payment number.

CHECK Your Understanding

1. The formulas in the amortization table used to solve Example 1 (Option 2) are shown.

	A	B	C	D
1	Annual Rate	0.06	Start Year	50
2	Rate per Period	= B1/B3	End Year	65
3	Compounding Periods per Year	1	Final Year End	=D2−D1
4	Contribution	3000		
5	End of Year	Interest	Payment Made	New Balance
6	1			=B4
7	=A6+1	=D6*$B$2	=$B$4+B7	=D6+C7
	⋮			⋮
14	=A13+1	=D13*$B$2	=$B$4+B14	=D13+C14
15	=A14+1	=D14*$B$2	=$B$4+B15	=D14+C15

a) How is the interest calculated?

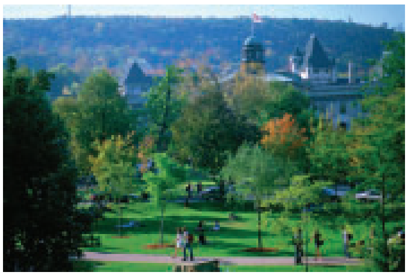
b) **Payment Made** is the sum of two numbers. What do these numbers represent?

2. An $18 000 car loan is charged 4%/a interest compounded quarterly.
   a) Determine the quarterly payments needed to pay the loan off in 5 years.
   b) How much faster would the loan be paid off with the same payments if the interest rate was lowered to 2.9%/a?
   c) What would be the total cost of the car, including interest, in parts (a) and (b)?

3. For each situation, show what would be entered into each TVM Solver variable. Do not solve the problem.
   a) A $10 000 loan is repaid with monthly payments of $350 for 13 years. Determine the annual interest rate, with monthly compounding.
   b) Martin took out a student loan for $15 000 at 7.5%/a compounded monthly. He is working now and wants to pay it off. When will it be paid off if he makes monthly payments of $500?

PRACTISING

4. Susan's parents would like to save $12 000 over the next four years to pay for her first year at McGill University in Montréal.
K



   a) How much should they deposit at the end of each month into an account that pays 7.25%/a, compounded monthly, to attain their goal?
   b) How much should they deposit at the end of every three months into an account that pays 7.25%/a, compounded quarterly, to attain their goal?
   c) Why is the payment in part (b) slightly over three times the payment in part (a)?

5. Joel and Katerina are each paying off loans of $5000. Joel makes monthly payments of $75 and interest is charged at 9%/a compounded monthly. Katerina pays the loan off in the same amount of time, but her monthly payments are only $65. Determine the annual interest rate that Katerina is charged if her interest is also compounded monthly.

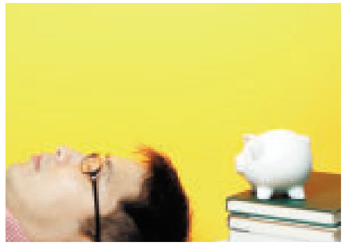
6. Bernice will repay a $30 000 loan with monthly payments. The term of the loan is 5 years. The interest rate is 7.25%/a compounded monthly.
   a) What is the monthly payment for this loan?
   b) What is the outstanding balance on the loan after each of the first 5 years?
   c) What is the interest and principal that she has paid at the end of the 5-year term?

7. If $1000 is deposited at the end of each year in an account that pays 13.5%/a compounded annually, about how many years will it take to accumulate to $20 000?

8. Jack's life savings total $320 000. He wants to use the money to buy an annuity earning interest at 10%/a compounded semi-annually so that he will receive equal semi-annual payments for 20 years. How much is each payment if the first is 6 months from the date of purchase?

9. T An account pays 9.2%/a compounded annually. What deposit on January 1 of this year will allow you to make 10 annual withdrawals of $5000, beginning January two years from now?

10. A Michael has a student loan of $15 000 at 8.5%/a compounded monthly. He will pay off the loan over the next 5 years. The first payment will be made 1 month from now.



a) What is his monthly loan payment?
b) What is the interest that Michael will pay over the term of the loan?
c) At the end of one year, Michael decides to pay off the rest of the loan. How much interest did he save by repaying the loan in 1 year?

11. C Describe three different ways to save money when taking out a loan. Why do they work?

Extending

12. The repayment schedule for a loan lists each payment, and shows how much of each payment is interest and how much goes to reduce the principal. It also shows the outstanding balance after each payment.
a) Identify an item that you would like to buy for which you might need a loan.
b) The prime interest rate is the interest rate that banks charge their preferred customers. Determine the current prime lending rate of a major bank.
c) Create a repayment schedule for the loan, showing the amount of the loan, annual interest rate, issue date, number of payments, and monthly payment. The loan is repaid with equal monthly payments over 2 years.

13. Sumiko buys a car with a $22 000 loan at 9.25%/a compounded monthly. She will repay the loan with 60 equal monthly payments.
a) Determine the monthly payment.
b) After a year, Sumiko decides to increase her payments by $150 a month. How many more payments are required to pay off her loan?
c) How much interest does she save by making the greater monthly payment?

Curious | Math

Buy Now, No Payments for a Year: Is It Always a Good Deal?

A local home-furnishing store offers "no payments, no interest" for a year if you charge your purchase to the store's credit card. However, the store will charge a $35 administration fee. This fee, with 6% GST and 8% PST, must be paid within 30 days. If the balance of the purchase is not paid at the end of the year, interest on the full amount is added to the bill. The interest rate is 24%/a compounded annually. A customer makes a purchase of $3495, not including tax.

1. What is the cost of borrowing if the bill is paid at the end of the year? Explain.

2. What is the cost of borrowing if the bill is not paid at the end of the year? Explain.

3. If the customer is unable to pay after 1 year, the store usually arranges to have the customer pay the bill with 12 equal monthly payments. As the payments are made, interest is charged on any unpaid balance at 24%/a compounded monthly. Calculate the monthly payment and the total paid in cash by the customer.

4. Suppose that, after 1 year, the customer is able to transfer the unpaid balance to another credit card that charges 9%/a compounded monthly. The customer intends to make the same monthly payment. How much sooner will the debt be paid off? How much interest will the customer save by transferring the debt to the credit card company offering the lesser interest rate?

8 Chapter Review

FREQUENTLY ASKED Questions

Q: How can a timeline help you better visualize how an annuity works?

A: An annuity is a series of payments or deposits made at regular intervals. Annuities earn interest on each regular deposit from the time the money is deposited to the end of the term of the annuity. A timeline is a tool to help you visualize how the factor $(1 + i)$ affects the present value of withdrawals or future value of deposits. The future values of deposits are multiplied by $(1 + i)$ for each compounding period. The present values of withdrawals are divided by $(1 + i)$ for each compounding period.

Study *Aid*

- See Lessons 8.5 and 8.6, Example 1.
- Try Chapter Review Question 7.

Q: How can you calculate the future value or the present value of an annuity?

A: You can use

- the formula $FV = \frac{R[(1 + i)^n - 1]}{i}$ or $PV = \frac{R[1 - (1 + i)^{-n}]}{i}$

  where
  - FV is the future value (amount), in dollars
  - PV is the present value (principal), in dollars
  - R is the regular payment, in dollars
  - i is the interest rate per compounding period, expressed as a decimal
  - n is the total number of payments
- the TVM Solver
- a spreadsheet

Study *Aid*

- See Lessons 8.5 and 8.6, Examples 2 and 3.
- Try Chapter Review Questions 8 to 15.

Q: What is an amortization table?

A: For each payment made during the term of the amortization, an amortization table shows the payment number, the payment amount, how much of each payment is interest, how much of each payment goes to reduce the principal, and the outstanding principal. An amortization table can be created with spreadsheet software. It is designed to allow you to change the parameters of an annuity problem and analyze the effects of the changes.

Study *Aid*

- See Lesson 8.7, Examples 1, 2, and 3.
- Try Chapter Review Question 16.

PRACTICE Questions

Lesson 8.1

1. Copy and complete the table.

	Principal ($)	Annual Interest Rate (%)	Years	Compounding Frequency	Amount ($)	Interest Earned ($)
a)	400.00	5%	15	semi-annually		
b)	450.00	4.5%	10	monthly		
c)		3.4%	10	weekly	875.00	
d)	508.75		3	semi-annually	568.24	
e)	10 000.00	2.34%		quarterly		1000.00

2. Suppose you were to graph the accumulated simple interest on an investment and the accumulated compound interest on the same investment on the same graph. What would be the similarities and differences between the two graphs? Be as detailed as possible.

Lesson 8.2

3. Determine the future value of a $5000 compound-interest Canada Savings Bond at $8\frac{1}{2}$%/a compounded annually after each amount of time.

a) 4 years **b)** 8 years

Lesson 8.3

4. Your mother wants to give you $25 000 in 15 years' time. How much should she invest now at 8%/a interest compounded monthly to meet this goal?

Lesson 8.4

5. An investment of $1500 grows to $3312.06 in 10 years. What is the interest rate of the investment if interest is compounded quarterly?

6. Kadie invested $3000 at 6%/a compounded quarterly. How long will it take for the investment to be worth $8500?

Lesson 8.5

7. For 10 years, Sheila deposits $750 at the end of every 3 months in a savings account that pays 8%/a compounded quarterly.

a) Draw a partial timeline to represent the first 3 months and the last 3 months of the annuity.

b) Calculate the amount of the annuity and the total interest earned.

8. At the end of every 6 months, Parvati deposited $200 into a savings account that paid 3.5%/a compounded semi-annually. She made the first deposit when her son was 6 months old and the last deposit on his 18th birthday. The money remained in the account until he turned 21. How much did Parvati's son receive?

9. David is 8 years old when his parents start an education fund. They deposit $450 at the end of every 3 months in a fund that pays 8%/a compounded quarterly.

a) How old is David when the fund is worth $20 000?

b) How much less time would it take to build the fund to $20 000 if the regular deposit were $550?

10. The Huang family borrowed $30 000 at 9%/a compounded monthly to buy a motor home. The Huangs will make payments at the end of each month. They have two choices for the term: 5 years or 8 years.

a) Determine the monthly payment for each term.

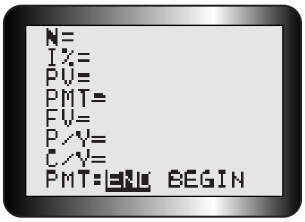
b) How much interest would they save by selecting the shorter term?

11. Raymond has $53 400 in his savings account, and he withdraws $250 at the end of every 3 months. If the account earns 5%/a compounded quarterly, what will his bank balance be at the end of 4 years?

Lesson 8.6

12. Adrianna wants to buy a used car. She can afford payments of $300 each month and wants to pay off the debt in 3 years. The bank offers a rate of 9.8%/a compounded monthly. What is the most Adrianna can spend on a vehicle?

13. For each situation described, indicate the values you would enter beside each variable for the TVM Solver.



a) Cecilia deposits $1500 at the end of each year in a savings account that pays 4.5%/a compounded annually. What is her balance after 5 years?

b) Farouk would like to have $200 000 in his account in 15 years. How much should he deposit at the end of each month in an account that pays 3.75%/a compounded monthly?

c) A $10 000 loan is repaid with monthly payments of $334.54 for 3 years. Determine the annual interest rate, compounded monthly.

d) Determine the total amount of interest earned on an annuity consisting of quarterly deposits of $2000 for 8 years if the annuity earns 9%/a interest compounded quarterly.

14. How much must be in a fund paying 6%/a compounded semi-annually if you wish to withdraw $1000 every 6 months, starting 6 months from now, for the next 5 years?

15. Karsten is preparing his will. He wants to leave the same amount of money to his two daughters. His elder daughter is careful with money, but the younger daughter spends it carelessly, so he decides to give them the money in different ways. How much must his estate pay his younger daughter each month over 20 years so that the accumulated present value will be equal to the $50 000 cash his elder daughter will receive upon his death? Assume that the younger daughter's inheritance earns 6%/a compounded monthly over the 20 years.

Lesson 8.7

16. Create an amortization table showing the amortization of a loan that will be repaid with equal monthly payments over 5 years. The loan of $10 000 has an interest rate of 8%/a compounded monthly.

17. Jerzy borrowed $4831 at 7.5%/a compounded monthly. He has made 30 monthly payments of $130 each. He is now in a position to pay off the balance. What is his remaining balance?

8

Chapter Self-Test

1. Explain why more money accumulates if the interest is compounded than if the interest is simple.
2. Determine the amount and interest earned when \$10 500 is invested for 4 years at 4.8%/a compounded monthly.
3. Determine the present value and the interest earned on a loan of \$21 500 due in 6 years. The interest rate is 8%/a compounded quarterly.
4. Carter deposits \$12 000 in an account that pays 6%/a compounded semi-annually. After 5 years, the interest rate changes to 6%/a compounded monthly. Calculate the value of the money 8 months after the change in the interest rate.
5. Janet and her fiancé plan to buy a new house 3 years from now. They intend to make a down payment of \$20 000. They can invest money in an account offering 9.25%/a compounded monthly. How much money must they invest today to reach their goal?
6. A principal of \$500 grew to \$620 in 11 years. Determine the annual interest rate, compounded quarterly.
7. Raj invested \$1000 in a GIC that paid 4%/a compounded weekly. He received \$1350 at the end of the term. For how long was the money invested?
8. Draw a timeline showing the future value of \$350 deposited semi-annually for 1.5 years at 3.75%/a compounded semi-annually.

9. Kay Chung wants to travel to China in 20 months. The trip will cost \$3200. How much should she deposit at the end of each month in an account that pays 9%/a compounded monthly to save \$3200?
10. Since the birth of their daughter, the Tranters have deposited \$450 every 3 months in an education savings plan. The interest rate is 7.5%/a compounded quarterly. What is the plan's value when their daughter turns 17?
11. What are the components of an amortization table and what is its purpose?

8 Chapter Task

Investigating RRSP Investments

Teresa began to contribute to her RRSP at age 20. She made monthly contributions of \$50, starting 1 month after her birthday. Her RRSP earned interest at an average rate of 7.5%/a compounded monthly, until her 60th birthday, when she retired. One month later, she started to withdraw a monthly amount.

A. Determine the amount of her RRSP on her 60th birthday.

B. Suppose Teresa transferred her RRSP into a Registered Retirement Income Fund (RRIF)—an annuity that uses the RRSP savings to provide the holder with a regular retirement income over a term of 25 years. What monthly pension can Teresa withdraw for the next 25 years?

C. Choose different contribution amounts, times for investments, and interest rates. Determine what effects they have on monthly retirement pensions. Prepare a report of your findings that compares these different scenarios to Teresa's situation.

Task *Checklist*

In your report,

- ✔ Did you show the necessary calculations to support your answers for parts A and B?
- ✔ Did you vary
  - – the contribution amount?
  - – the length of time?
  - – the interest rates?
- ✔ Did you discuss how these changes affect the amount of monthly retirement payment?
- ✔ Did you compare your finding to Teresa's situation?

Chapters 7–8 Cumulative Review

Multiple Choice

1. Which expression has a value of 64?

a) $16^{\frac{3}{2}}$ c) $16^{\frac{1}{2}}$
b) $-4^{\frac{3}{2}}$ d) $\sqrt[3]{4}$

2. If the value of the variable is 3, which of the following is true?

a) $(p \times p^3)^3 = 3^9$ c) $(n^2)^3 \div n = 243$
b) $(t^2)^2 \times t^0 = 10\ 000$ d) $c^3 = 512$

3. Which number is equivalent to $16^{\frac{3}{4}}$?

a) 4 c) -4
b) 8 d) $\frac{1}{2}$

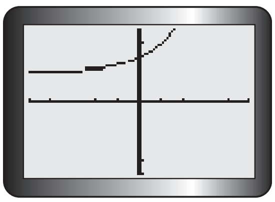
4. Identify the expression that is false.

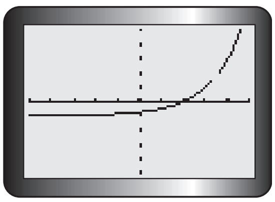
a) $(9^{\frac{1}{2}})(4^{\frac{1}{2}}) = (9 \times 4)^{\frac{1}{2}}$
b) $\left(\frac{1}{9} \times \frac{1}{4}\right)^{-1} = (9)(4)$
c) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 + 4)^{\frac{1}{2}}$
d) $[(9^{\frac{1}{3}})(4^{\frac{1}{3}})]^6 = 9^2 4^2$

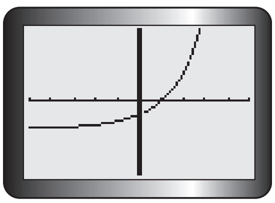
5. Which expression does not have a value of 9 when $a = 1$, $b = 3$, and $c = 2$?

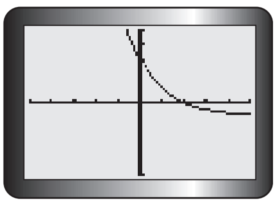
a) $(-a \div b)^{-c}$ c) $(ab)^{-c}$
b) $a^c b^c$ d) $(a^b b^a)^c$

6. Identify the exponential function whose equation of the asymptote is $y = 2$.

a)


c)


b)


d)


7. A bacteria culture doubles in size every 15 min. Given the formula $P(n) = 20(2)^{\frac{n}{15}}$, how long will it take for a culture of 20 bacteria to grow to a population of 163 840?

a) 2048 min c) 65 min
b) 12 h d) 195 min

8. Thorium-227 has a half-life of 18.4 days. Given the formula $M(t) = 50\left(\frac{1}{2}\right)^{\frac{t}{18.4}}$, how many days, t, will a 50 mg sample take to decompose to 10 mg?

a) 73.6 c) 42.72
b) 21.09 d) 7.36

9. Four years ago, Sam invested a sum of money at 5%/a compounded semi-annually. Today there is $921.35 in Sam's account. How much did she invest?

a) $756.19 c) $920.00
b) $46.06 d) 875.29

10. How much will $7500 be worth if it is invested now for 10 years at 6%/a compounded annually?

a) $12 000 c) $13 431.36
b) $16 637.84 d) $4500

11. Phong wants to purchase a motorcycle. He can borrow $6500 at 10%/a compounded quarterly, if he agrees to repay the loan by making equal quarterly payments for 4 years. Determine a reasonable quarterly payment.

a) $500 c) $650
b) $300 d) $65

12. In order to repay a loan in less time, you could

a) increase the periodic payment and increase the interest rate
b) increase the periodic payment and decrease the interest rate
c) decrease the periodic payment and increase the interest rate
d) none of the above

Investigations

13. Ball Bounce

Marisa drops a small rubber ball from a height of 6 m onto a hard surface. After each bounce, the ball rebounds to 60% of the maximum height of the previous bounce.

a) Create a table to show the height of the ball after each bounce for the first 5 bounces.

b) Graph the height versus bounce number.

c) Create a function that models the height of the ball as a function of bounce number.

d) Estimate the height of the ball after 12 bounces from your graph. Verify this result with your function.

e) Use your graph to estimate when the ball's maximum height will be 28 cm. Verify this result using your function.

14. Retirement Plans

Sara and Ritu have just finished school and have started their first full-time jobs. Sara had decided that she is going to start putting away $100/month in a Retirement Plan. Ritu thinks that Sara is crazy because Sara is only 22 and should enjoy her money now and worry about saving for retirement later. Ritu is not going to worry about saving money until she is 45.

Suppose both young women retire at age 60 and that they can invest their money at 9%/a compound monthly.

How much money will Ritu have to contribute monthly to retire with the same savings as Sara?

15. Buying a Car

Matt is going to buy his first car for $17 500. He will make a down payment of $3500 and finance the rest at 9%/a compounded monthly. He will make regular monthly payments for 4 years. He estimates that his car will depreciate in value at a rate of 18% per year. Provide a complete analysis of this situation by determining the amount of Matt's monthly payments, the total interest that he will be paying, and the value of his car when he has completed his payments. Include graphs, charts, and tables with your analysis.