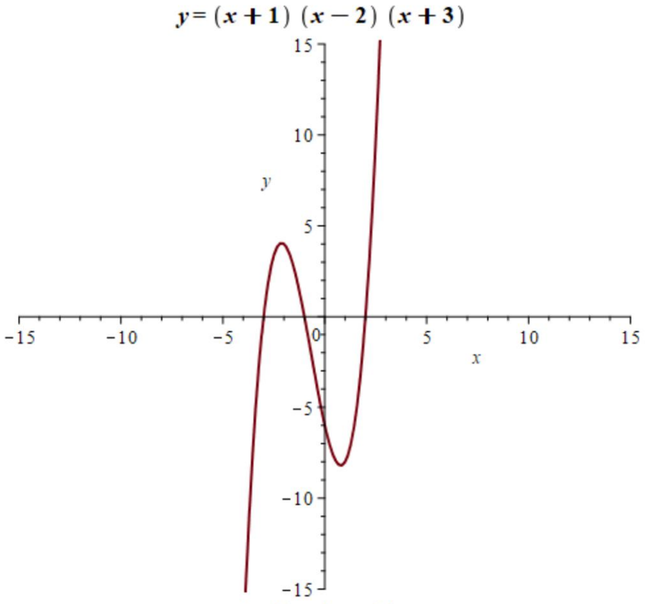
Advanced Functions

Course Notes

Chapter 2 – Polynomial Functions

***Learning Goals**: We are learning*

- *The algebraic and geometric structure of polynomial functions of degree three and higher*
- *Algebraic techniques for dividing one polynomial by another*
- *Techniques for using division to FACTOR polynomials*
- *To solve problems involving polynomial equations and inequalities*



A cubic polynomial

A∞Ω
Math@TD

Chapter 2 – Polynomial Functions

Contents with suggested problems from the Nelson Textbook (Chapter 3)

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2.1 Polynomial Functions: An Introduction

Learning Goal: We are learning to identify polynomial functions.

Definition 2.1.1

A **Polynomial Function** is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots a_2 x^2 + a_1 x^1 + a_0 x^0$$

where a_i , $i = 0, 1, 2, 3, \ldots n$, are coefficients

And the exponents are integers.

Examples of Polynomial Functions

a) $f(x) = \underset{a_4}{\boxed{8}}x^4 - 5x^3 + 2x^2 + 3x - 5$

$a_4 = 8$ $a_2 = 2$ $a_0 = -5$

b) $g(x) = 7x^6 - 4x^3 + 3x^2 + 2x$

$a_6 = 7$ $a^4 = 0$ $a_1 = 2$ $a_0 = 0$

Notes: The **TERM** $a_n x^n$ in any polynomial function (where *n* is the **highest power** we see) is

called the Leading term , and then we write all the following terms

in descending order .

The Leading term has two components:

1) Leading coefficient, a_n, is either positive or negative

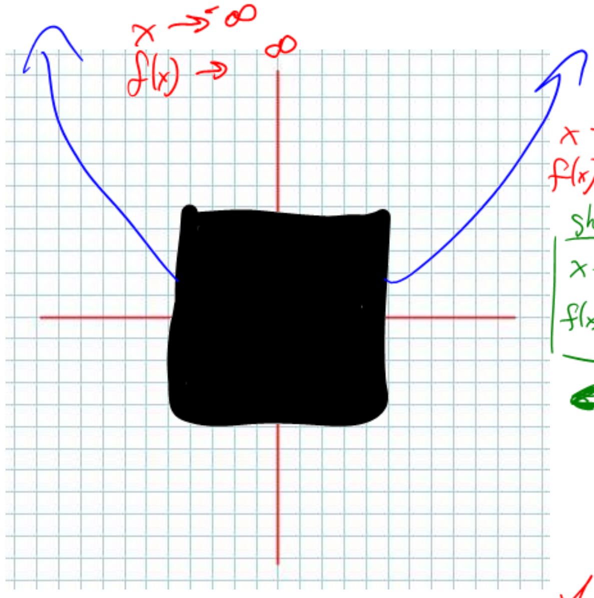
2) n, the highest power/degree, it can be even or odd

have nothing to do with symmetry

The Leading term tells us the **end behaviour** of the polynomial function.

✱ all polynomial functions have 4 possible end behaviors.

Pictures



$x \to -\infty$

$f(x) \to \infty$

$x \to \infty$

$f(x) \to \infty$

shortcut

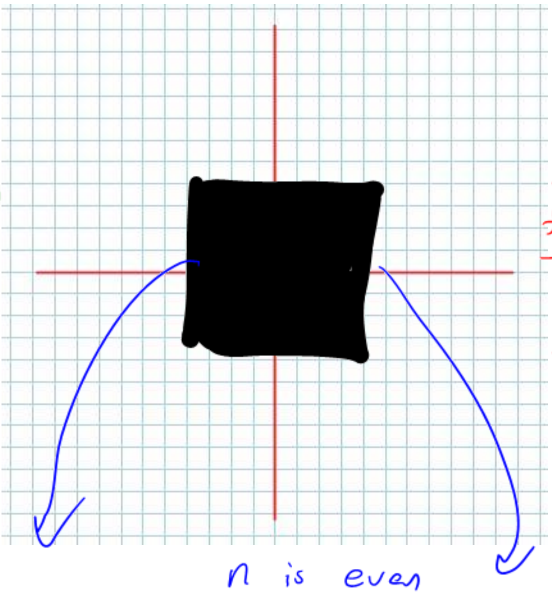
$x \to \pm \infty$

$f(x) \to \infty$

n is even

a_n is positive

✱ think of a parabola



$x \to -\infty$

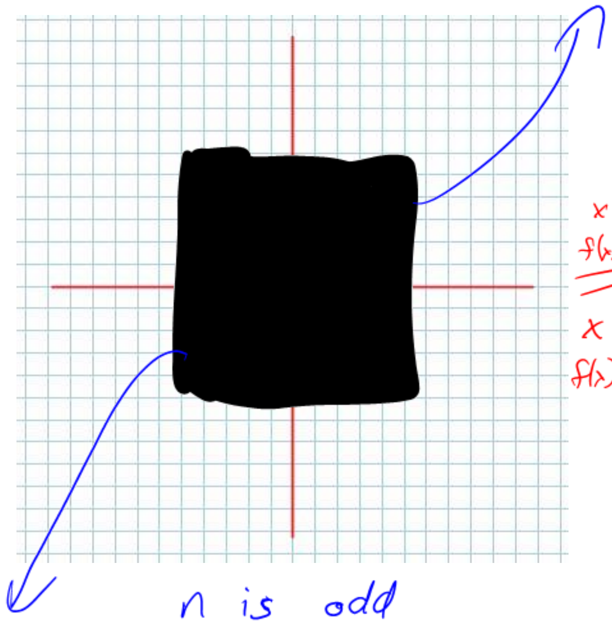
$f(x) \to -\infty$

$x \to \infty$

$f(x) \to -\infty$

n is even

a_n is negative



$x \to -\infty$

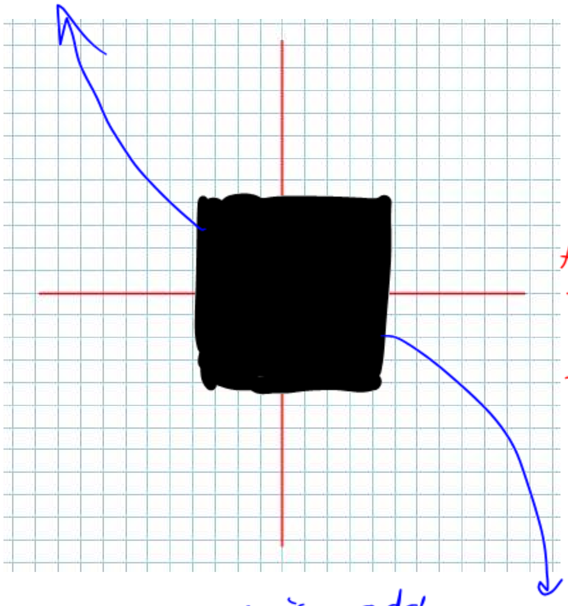
$f(x) \to -\infty$

$x \to \infty$

$f(x) \to \infty$

n is odd

a_n is positive



$x \to -\infty$

$f(x) \to \infty$

$x \to \infty$

$f(x) \to -\infty$

n is odd

a_n is negative

Same

Definition 2.1.2

The **order** of a polynomial function is the value of the highest power, or just the degree of the leading term.

ex: $g(x) = 2x^3 + 3x^2 \boxed{-8x^5} + 1$

The order of $g(x)$ is 5

Determine the end behavior of:

$$h(x) = 2(x-3)^2(2x+8)^3(4x+5)$$

$$2(x)^2(2x)^3(4x)$$

$$= 2(x^2)(8x^3)(4x)$$

$$= 64x^6$$

$\therefore$ $x \to -\infty$ $f(x) \to \infty$ $x \to \infty$ $f(x) \to \infty$

Success Criteria:

- I can justify whether a function is polynomial or not
- I can identify the degree of a polynomial function
- I can recognize that the domain of a polynomial is the set of all real numbers
- I can recognize that the range of a polynomial function may be the set of all real numbers, or it may have an upper/lower bound
- I can identify the shape of a polynomial function given its degree

2.2 Characteristics (Behaviours) of Polynomial Functions

Today we open, and look inside the black box of mystery

Learning Goal: We are learning to determine the turning points and end behaviours of polynomial functions.

Consider the sketch of the graph of some function, $f(x)$:

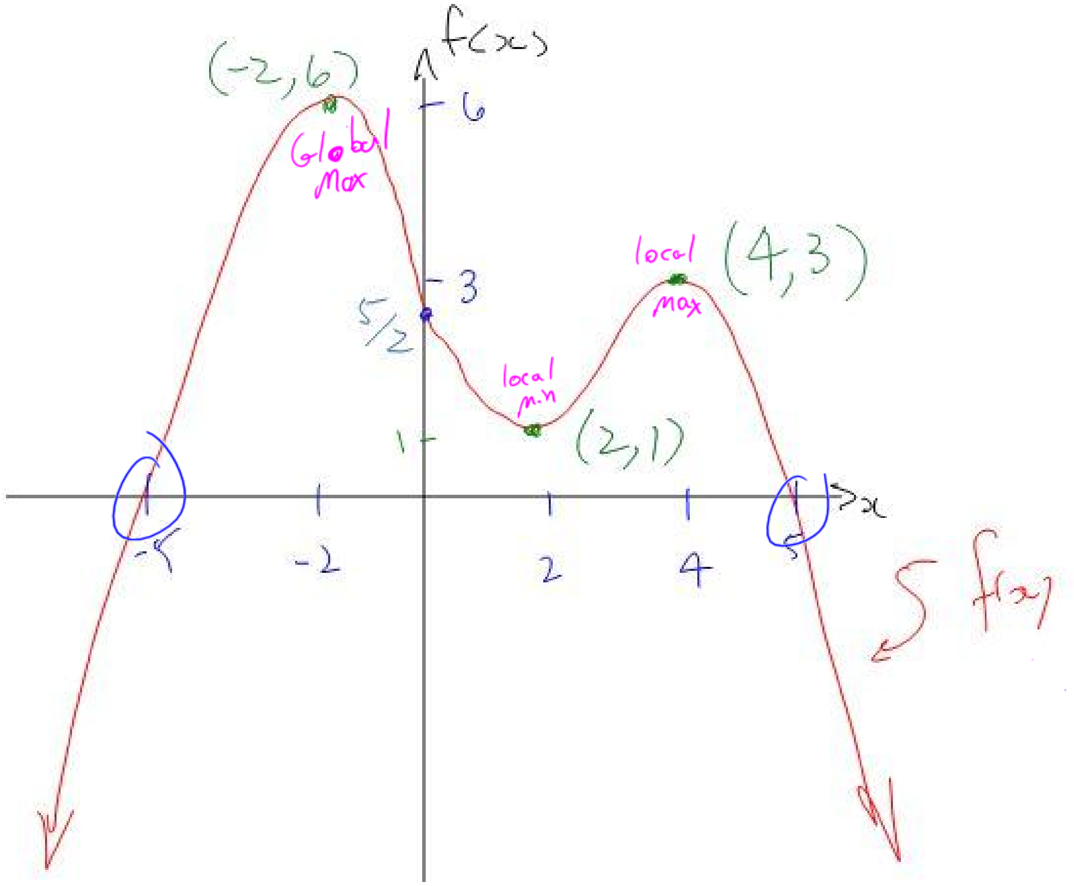


Figure 2.2.1

Observations about $f(x)$:

1) $f(x)$ is a polynomial of even order (degree). The end behaviors are the same

2) The leading coefficient is negative

3) $f(x)$ has 3 turning points (where the functional behaviour of INCREASING/DECREASING switches from one to the other.)

4) $f(x)$ has 2 zeros, $f(-5) = 0$ and $f(5), 0$

Zeros at $x = -5$, $x = 5$

5) $f(x)$ is increasing on $x \in (-\infty, -2) \cup (2, 4)$

$f(x)$ is decreasing on $x \in (-2, 2) \cup (4, \infty)$

6) $f(x)$ has a maximum $\overset{y}{\text{functional}}$ value, of 6.

This max is called the global maximum because it is the absolute highest value

☆ only even polynomial functions have a global max/min

7) $f(x)$ has a local minimum at $(2,1)$ [redacted]

and a local max at $(4,3)$.

Consider the sketch of the graph of some function $g(x)$:

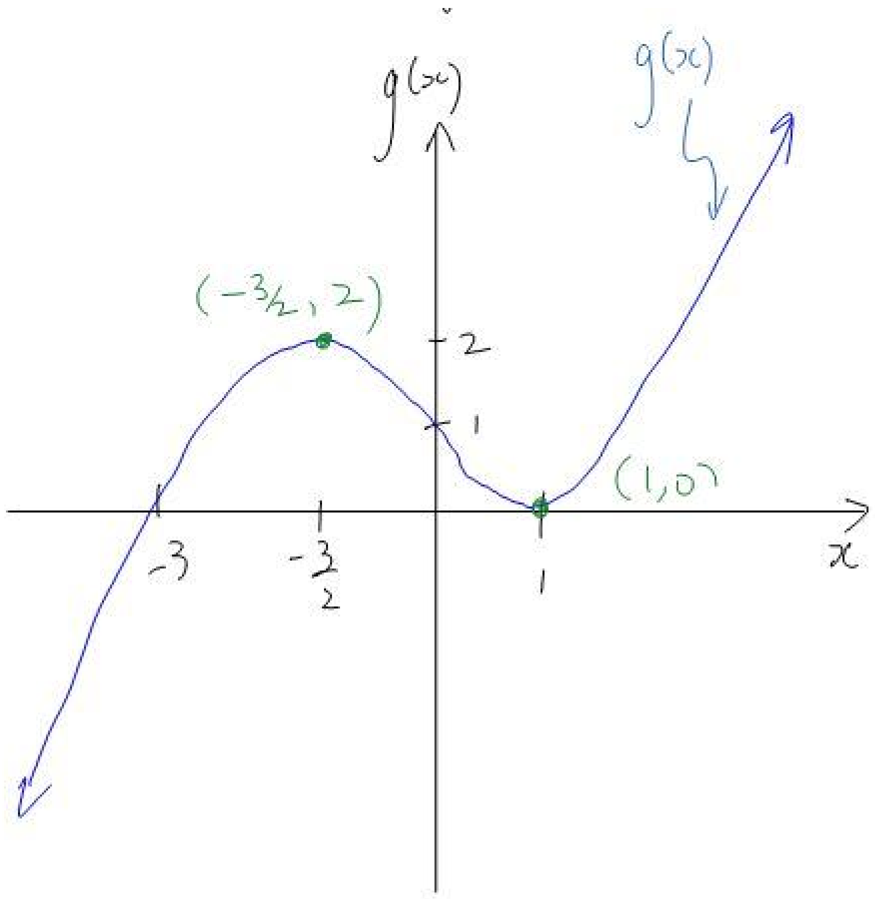


Figure 2.2.2

Observations about $g(x)$:

① Two turning points
local max at $(\frac{-3}{2}, 2)$ and a local min at $(1,0)$

② Increasing from $x \in (-\infty, -\frac{3}{2}) \cup (1, \infty)$
Decreasing from $x \in (\frac{-3}{2}, 1)$

③ $g(-3)=0$ and $g(1)=0$ ∴ 2 zeros

④ The Leading coefficient is positive

⑤ $g(x)$ is odd. End behaviors are different.

General Observations about the Behaviour of Polynomial Functions

1) The Domain of all Polynomial Functions is $x \in (-\infty, \infty)$

2) The Range of ODD ORDERED Polynomial Functions is

$f(x) \in (-\infty, \infty)$

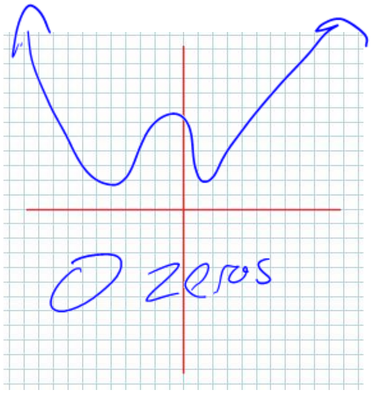
3) The Range of EVEN ORDERED Polynomial Functions

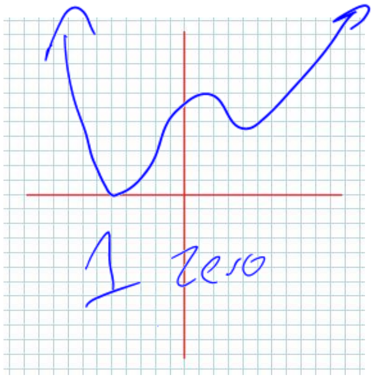
1. The sign of the leading coefficient: $+ \Rightarrow y \geq \#$
2. The value of the global max/min $- \Rightarrow y \leq \#$

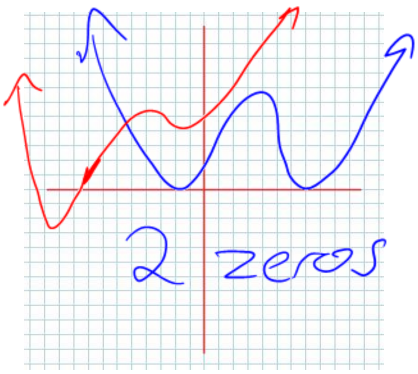
Even Ordered Polynomials

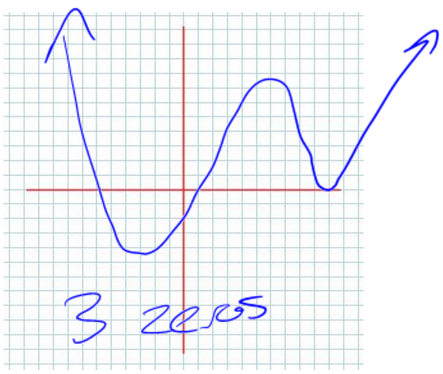
Zeros: A Polynomial Function, $f(x)$, with an even degree of "n" (i.e. $n = 2, 4, 6...$) can have 0 zeros, 1, 2, 3, ... , n zeros

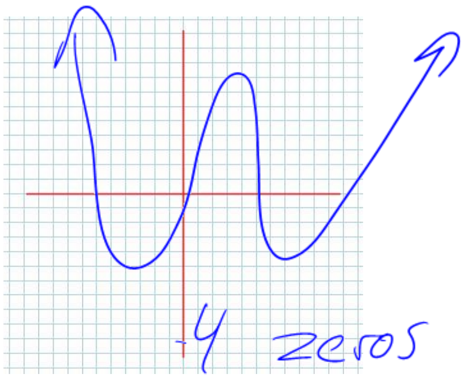
e.g. A degree 4 Polynomial Function (with a positive leading coefficient) can look like:







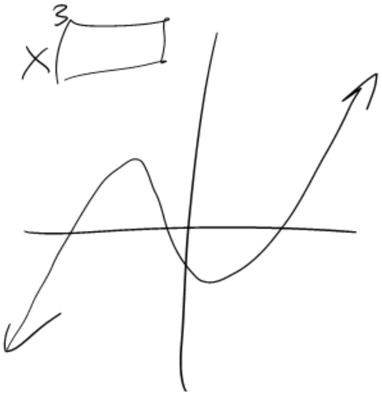


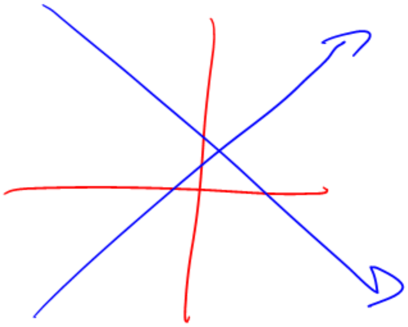


Turning Points:

The minimum number of turning points for an Even Ordered Polynomial Function is one. It must turn.

The maximum number of turning points for a Polynomial Function of (even) order *n* is $n-1$





Odd Ordered Polynomials

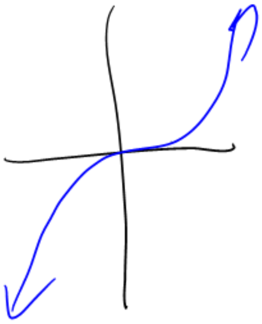
Zeros: min is one.

max is n

Turning Points:

min # of T.P. is zero

max # is $n-1$



Example 2.2.1 (*#2, for #1b, from Pg. 136*)

Determine the minimum and maximum number of zeros and turning points the given function may have: $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

g(x) is odd

L.C. is positive

Zeros: Min = 1, max = 5 (n)

Turning Points: Min = 0, max = 4 ($n-1$)

End behaviors $x \to -\infty$, $g(x) \to -\infty$

$x \to \infty$, $g(x) \to \infty$

Example 2.2.2 (*#4d from Pg. 136*)

Describe the end behaviour of the polynomial function using the order and the sign on the leading coefficient for the given function: $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$

↳ even and negative

End Behaviors

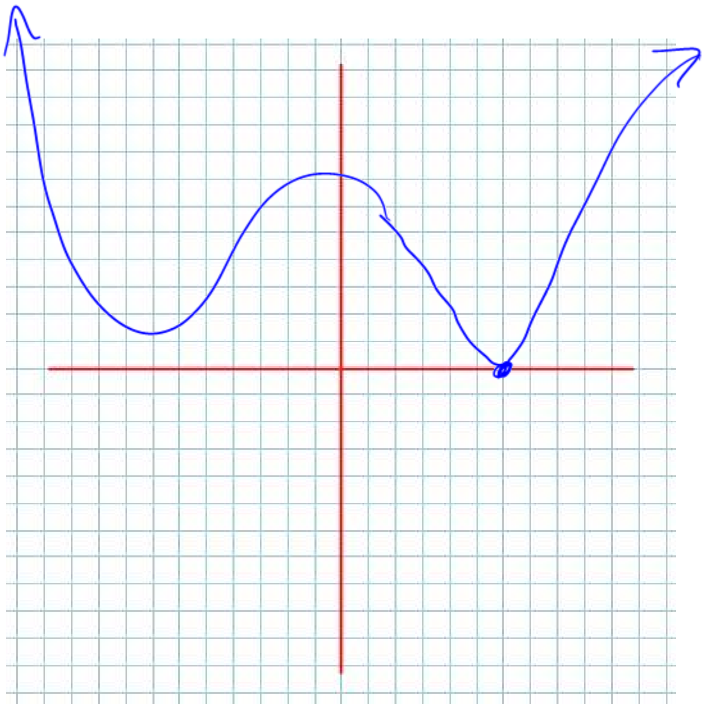
$x \to -\infty, f(x) \to -\infty$

$x \to \infty, f(x) \to -\infty$

Example 2.2.3 (*#7c from Pg. 137*)

Sketch a graph of a polynomial function that satisfies the given set of conditions:

Degree 4 - positive leading coefficient - 1 zero - 3 turning points.



Success Criteria:

- I can differentiate between an even and odd degree polynomial
- I can identify the number of turning points given the degree of a polynomial function
- I can identify the number of zeros given the degree of a polynomial function
- I can determine the symmetry (if present) in polynomial functions

2.3 Zeros of Polynomial Functions

(Polynomial Functions in Factored Form)

*Today we take a deeper look inside the Box of Mystery, **carefully examining Zeros of Polynomial Functions***

Learning Goal: We are learning to determine the equation of a polynomial function that describes a particular situation or graph and vice-versa.

We'll begin with an **Algebraic Perspective**:

Consider the polynomial function in factored form:

$$f(x) = (2x-3)(x-1)(x+2)(x+3)$$

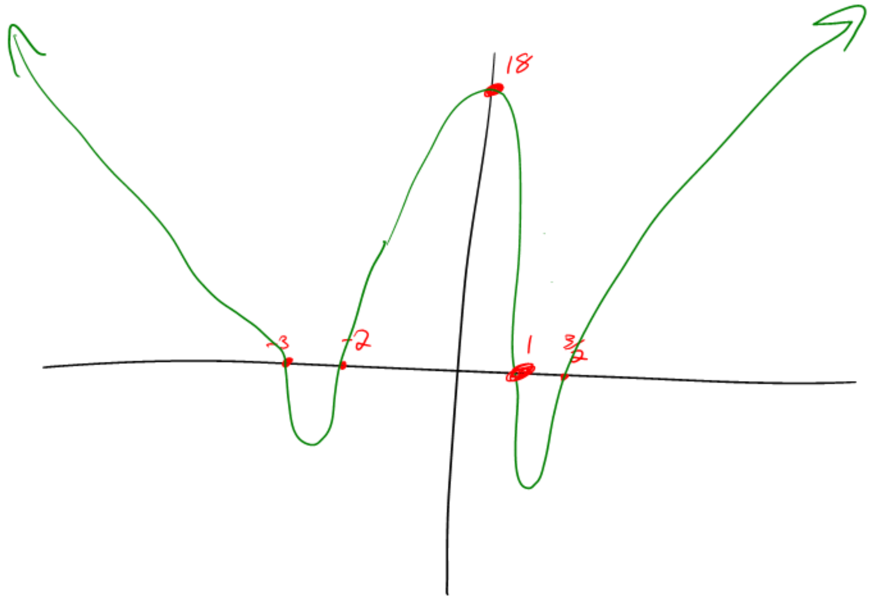
Observations: Leading Term is $2x^4$

1. $f(x)$ is even and positive, therefore $x \to \pm\infty$, $f(x) \to \infty$

2. Order/degree is 4.

3. 4 zeros at $x = \frac{3}{2}, 1, -2, -3$

4. y-int is: $f(0) = (-3)(-1)(2)(3) = 18$



Now, consider the polynomial function $g(x) = (x-3)^2(x-1)(x+2)$

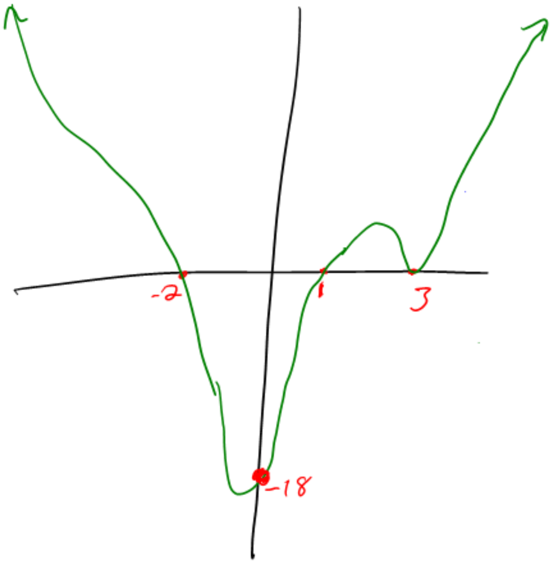
Observations: Leading terms is: $(x)^2(x)(x) = x^4$

1. $g(x)$ is positive and even

$x \to \pm\infty$, $g(x) \to \infty$

2. Zeros at $x = 3, 1, -2$

3. y-int $g(0) = (-3)^2(-1)(2) = -18$



Geometric Perspective on Repeated Roots (zeros) of order **2**

Consider the quadratic in factored form: $f(x) = (x-1)^2$

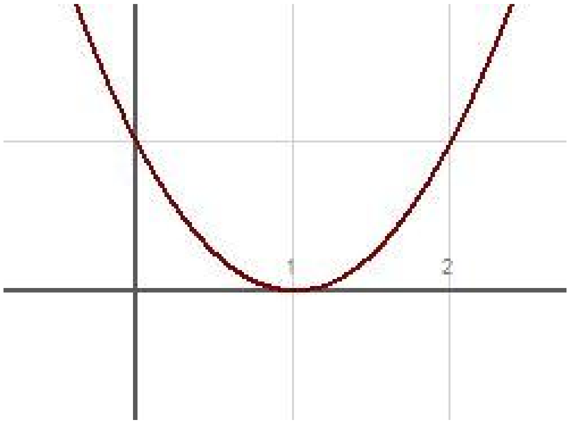


Figure 2.3.1

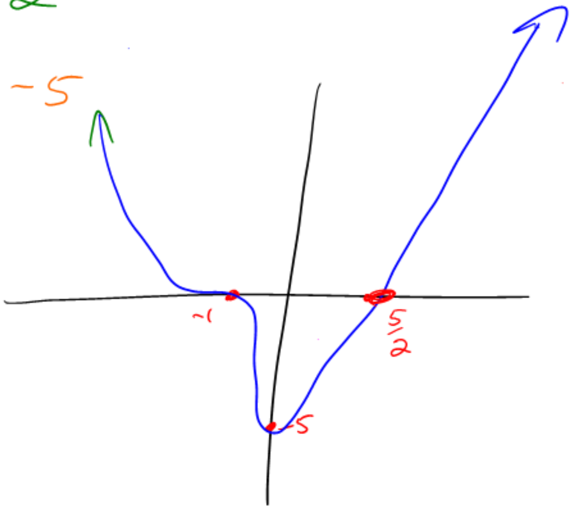
Consider the polynomial function in factored form: $h(t) = \boxed{(t+1)^3}(2t-5)$

Observations: Leading term $(t)^3(2t) = 2t^4$

1. $h(t)$ is even and positive, as $t \to \pm\infty$, $h(t) \to \infty$

2. Zeros at $\overset{\text{order } 3}{\boxed{t=-1}}$, $\frac{5}{2}$

3. y-int: $h(0) = (1)^3(-5) = -5$



Geometric Perspective on Repeated Roots (zeros) of order **3**

Consider the function $f(x) = (x-1)^3$

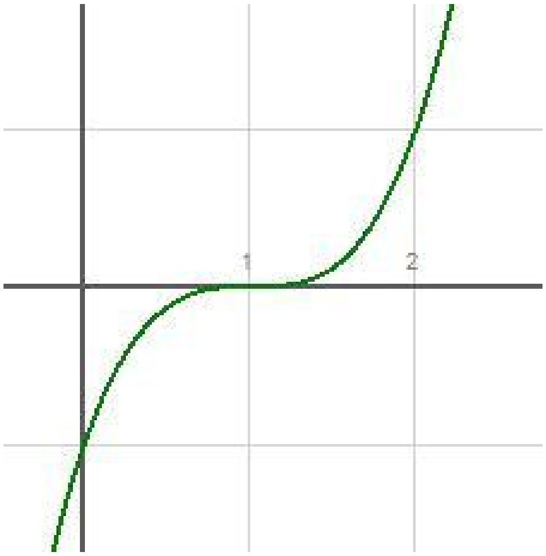
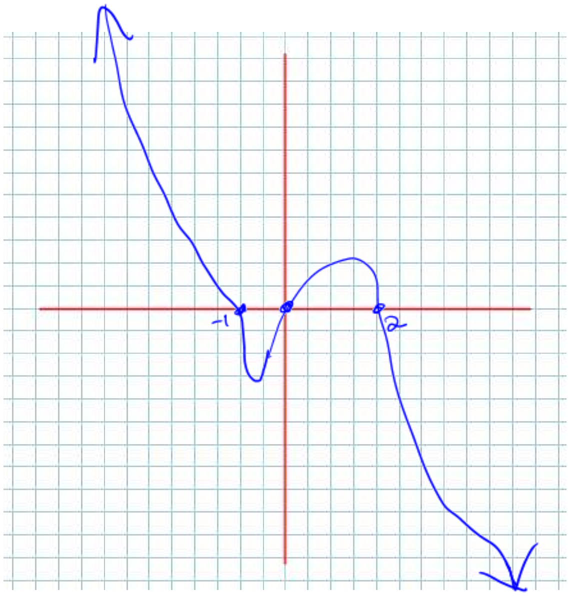


Figure 2.3.2

Example 2.3.1

Sketch a (**possible**) graph of $f(x) = -2\overset{x=0}{\overset{\downarrow}{x}}(x+1)(x-2)$



Leading term is $-2x^3$

$f(x)$ is negative and odd

$x \to -\infty,\ f(x) \to \infty$

$x \to \infty,\ f(x) \to -\infty$

Zeros at $x = \underbrace{-1, 2, 0}_{\text{all order one}}$

y-int is $f(0) = 0$

Families of Functions

Polynomial functions which share the same order are "broadly related" (e.g. **all** quadratics are in the "order 2 family").

Polynomial Functions which share the same order and zeros are more tightly related.

Polynomial Functions which share the same order, zeros, end behaviours are like siblings.

$f(x) = -2(x-3)^2(x+1)$

$g(x) = -5(x-3)^2(x+1)$

Example 2.3.2

The family of functions of order 4, with zeros $x = -1, 0, 3, 5$ can be expressed as:

$f(x) = \boxed{a}(x+1)(x+0)(x-3)(x-5)$

$f(x) = x(x+1)(x-3)(x-5)$

this is what distinguishes from family members.

Example 2.3.3

Sketch a graph of $g(x) = \overset{L.T.}{\boxed{4x^4}} - 16x^2$

g(x) is posit~~ive and~~ even $x \to \pm\infty, \ g(x) \to \infty$

Factor to get zeros:

$g(x) = 4x^2(x^2 - 4)$

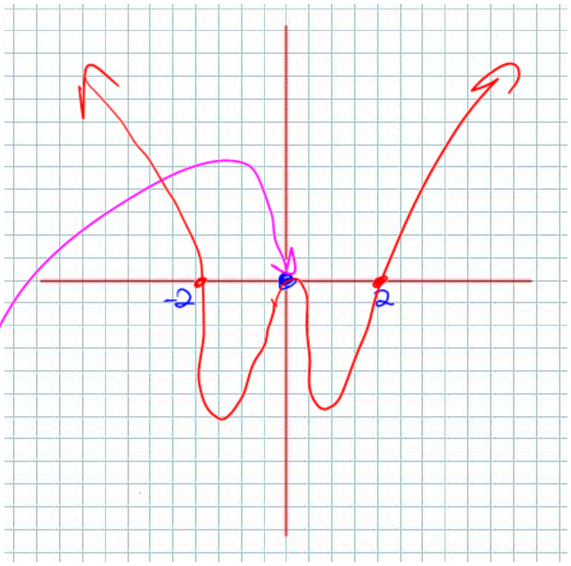
$g(x) = 4x^2(x-2)(x+2)$

$x = 0$ order 2

$x = -2$

$x = 2$

y-int is 0



Example 2.3.4

Sketch a (possible) graph of $h(t) = (t-1)^3(t+2)^2$

Leading term is t^5

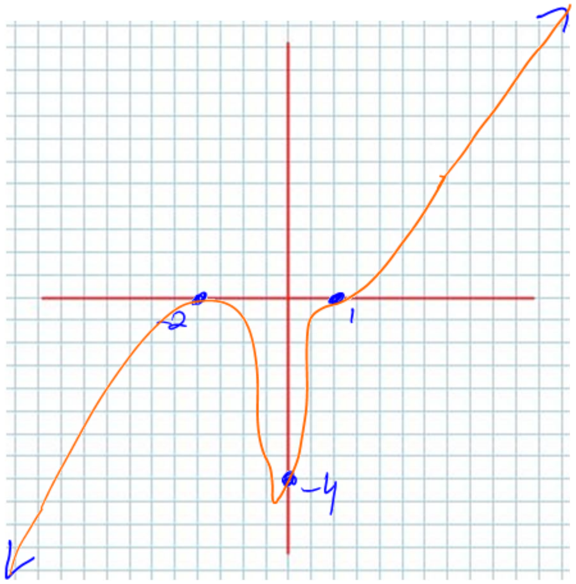
Odd, positive, $t \to -\infty, \ h(t) \to -\infty$

$t \to \infty, \ h(t) \to \infty$

Zeros at $t = 1$ order 3

$t = -2$ order 2 ∩ ∪

y-int: $h(0) = (-1)^3(2)^2 = -4$



Example 2.3.5

order 4

Determine **the** quartic function, $f(x)$, with zeros at $x = -2, 0, 1, 3$, if $f(-1) = -2$.

$$f(x) = a(x+2)(x+0)(x-1)(x-3)$$

$$-2 = a(-1+2)(-1+0)(-1-1)(-1-3)$$

$$-2 = a(1)(-1)(-2)(-4)$$

$$\frac{-2}{-8} = \frac{a(\cancel{-8})}{\cancel{-8}}$$

$$\frac{1}{4} = a$$

$$\therefore f(x) = \frac{1}{4}x(x+2)(x-1)(x-3)$$

Success Criteria:

- I can determine the equation of a polynomial function in factored form
- I can determine the behaviour of a zero based on the order/exponent of that factor

2.4a Dividing a Polynomial by a Polynomial

(The Hunt for Factors)

Learning Goal: We are learning to divide a polynomial by a polynomial using long division

> *Note: In this course we will almost always be dividing a polynomial by a ~~monomial~~* linear divisor

Before embarking, we should consider some "basic" terms (and notation):

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

the thing you are dividing

the thing you are dividing by

the "answer"

any left overs

The division statement

$$\boxed{\text{dividend} = (\text{quotient})(\text{divisor}) + \text{remainder}}$$

$$25 = (12)(2) + 1$$

Note: The Divisor and the Quotient will both be

FACTORS

IF

The remainder is zero

Example 2.4.1

Use **LONG DIVISION** for the following division problem:

$$\frac{5x^4+3x^3-2x^2+6x-7}{x-2}$$

Please read Example 1 (Part A) on Pgs. 162 – 163 in your textbook.

$$5x^3+13x^2+24x+54$$

$$(\boxed{x}-2)\overline{)\,5x^4+3x^3-2x^2+6x-7}$$

$$-(\boxed{5x^4}-10x^3)$$

$$13x^3-2x^2$$

$$-(13x^3-26x^2)$$

$$24x^2+6x$$

$$-(24x^2-48x)$$

$$54x-7$$

$$-(54x-108)$$

$$101$$

$$(x)(\underline{5x^3}) = 5x^4$$

$$(x)(\underline{13x^2}) = 13x^3$$

$$(x)(\underline{24x}) = 24x^2$$

$$(x)(\underline{54}) = 54x$$

Eliminate the first term every step

$$\therefore\ 5x^4+3x^3-2x^2+6x-7 = (5x^3+13x^2+24x+54)(x-2) + 101$$

KEY OBSERVATION:

(x-2) is not a factor.

Example 2.4.2

$0x^4 \quad 0x^2$

Using Long Division, divide $\frac{2x^5+3x^3-4x-1}{x-1}$.

$$2x^4+2x^3+5x^2+5x+1$$

$$(x-1)\overline{)\,2x^5+0x^4+3x^3+0x^2-4x-1}$$

$$-(2x^5-2x^4)$$

$$2x^4+3x^3$$

$$-(2x^4-2x^3)$$

$$5x^3+0x^2$$

$$-(5x^3-5x^2)$$

$$5x^2-4x$$

$$-(5x^2-5x)$$

$$x-1$$

$$-(x-1)$$

$$0$$

$$\therefore 2x^5+3x^3-4x-1=(x-1)(2x^4+2x^3+5x^2+5x+1)$$

KEY OBSERVATION:

$(x-1)$ is a factor.

Classwork: Pg. 169 #5 (Yep, that's it for today)

Success Criteria:

- I can use long division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after long division, there is no remainder

$$2x^3 + 4x^2 - 3x + 24$$

$$x^2 + 2x - 4 \overline{\smash{)}\, 2x^5 + 8x^4 - 3x^3 + 2x^2 - 10x + 8}$$

$$-(2x^5 + 4x^4 - 8x^3)$$

$$4x^4 + 5x^3 + 2x^2$$

$$-(4x^4 + 8x^3 - 16x^2)$$

$$-3x^3 + 18x^2 - 10x$$

$$-(-3x^3 - 6x^2 + 12x)$$

$$24x^2 - 22x + 8$$

$$-(24x^2 + 48x - 96)$$

$$-70x + 104$$

$$\square \div x^2 + 5$$

$$\uparrow$$

$$0x$$

$$x^2 + 0x + 5 \overline{\smash{)}\,}$$

2.4b Dividing a Polynomial by a Polynomial

(*The Hunt for Factors – Part 2*)

Learning Goal: We are learning to divide a polynomial by a polynomial using synthetic division

Here we will examine an alternative form of polynomial division called **Synthetic Division**. Don't be fooled! This is not "fake division". You're thinking with the wrong meaning for "synthetic". (Do a search online and see if you can come up with the meaning I am taking!)

In Synthetic Division we concern ourselves with the coefficients of the dividend and the zero of the divisor.

↳ we are using linear divisors ex: $2x+1$

$x-3$

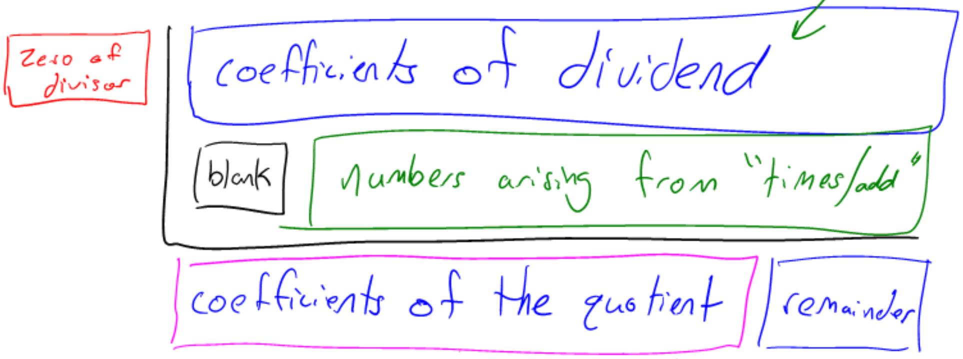
Synthetic Division uses

- only numbers
- these steps: ① Bring down ② times ③ add

Note: only linear divisors.

The Set-up

all powers must be present.



Example 2.4.3

Divide using synthetic division:

divisor

$$(4x^3-5x^2+2x-1)\div(x-2)$$

① Bring Down

② times

③ add

$$\begin{array}{r|rrrr} 2 & 4 & -5 & 2 & -1 \\ + & & 8 & 6 & 16 \\ \hline & 4 & 3 & 8 & 15 \end{array}$$

$$\underbrace{4 \quad 3 \quad 8}_{x^2 \quad x^1 \quad x^0} \qquad \underbrace{15}_{\text{remainder}}$$

$$\therefore 4x^3-5x^2+2x-1=(x-2)(4x^2+3x+8)+15$$

Example 2.4.4

$0x^3$

Divide using synthetic division:

$$\frac{4x^4+3x^2-2x+1}{\boxed{x+1}} \qquad x=-1$$

$$\begin{array}{r|rrrrr} -1 & 4 & 0 & 3 & -2 & 1 \\ & \downarrow & -4 & 4 & -7 & 9 \\ \hline & 4 & -4 & 7 & -9 & \textcircled{10} \end{array}$$

remainder

$$\therefore 4x^4+3x^2-2x+1=(x+1)(4x^3-4x^2+7x-9)+10$$

Example 2.4.5

Divide using your choice of method *(and you choose synthetic division…amen)*

$$(2x^3-9x^2+x+12)\div(2x-3) \rightarrow \text{is a factor!!}$$

$x=\frac{3}{2}$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -9 & 1 & 12 \\ & \downarrow & 3 & -9 & -12 \\ \hline & 2 & -6 & -8 & 0 \\ \div 2 & \boxed{1} & \boxed{-3} & \boxed{-4} & \end{array}$$

no remainder!!

when you have a fraction, divide the results by the denominator

$$\therefore\ 2x^3-9x^2+x+12=(2x-3)(x^2-3x-4)$$
$$=(2x-3)(x-4)(x+1)$$

$$\left(x-\frac{3}{2}\right)(2x^2-6x-8)$$

Example 2.4.6

Is $3x-1$ a factor of the function $f(x)=6x-x^3+2+3x^4$? $\Rightarrow 3x^4-x^3+0x^2+6x+2$

$$\begin{array}{r|rrrrr} \frac{1}{3} & 3 & -1 & 0 & 6 & 2 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & 3 & 0 & 0 & 6 & \boxed{4} \end{array}$$

$\therefore$ $3x-1$ is not a factor.

Example 2.4.7 (OK…this is a lot of examples!)

Consider again (from **Example 2.4.6**) $f(x)=3x^4-x^3+6x+2$, and calculate $f\left(\frac{1}{3}\right)$.

$$f\left(\frac{1}{3}\right)=3\left(\frac{1}{3}\right)^4-\left(\frac{1}{3}\right)^3+6\left(\frac{1}{3}\right)+2$$

$$=\cancel{3}\left(\frac{1}{\underset{27}{\cancel{81}}}\right)-\left(\frac{1}{27}\right)+2+2$$

$$=4$$

WAIT!!! This is the same remainder when dividing by $3x-1$.

Example 2.4.8

Consider **Example 2.4.5**. Let $g(x)=2x^3-9x^2+x+12$, and calculate $g\left(\frac{3}{2}\right)$.

$$g\left(\frac{3}{2}\right)=2\left(\frac{3}{2}\right)^3-9\left(\frac{3}{2}\right)^2+\frac{3}{2}+12$$

$$=\cancel{2}\left(\frac{27}{4\cancel{8}}\right)-9\left(\frac{9}{4}\right)+\frac{3\times 2}{2\times 2}+\frac{12\times 4}{1\times 4}$$

$$=\frac{27}{4}-\frac{81}{4}+\frac{6}{4}+\frac{48}{4}=\frac{0}{4}=0!$$

The Remainder Theorem

Given a polynomial function, $f(x)$, divided by a linear binomial, $x-k$, then the remainder of the division is the value $f(k)$.

Proof of the Remainder Theorem

Consider $f(x) \div (x-k)$

Then $f(x) = (x-k)(\overset{\text{quotient}}{q(x)}) + r$

$$f(k) = \cancel{(\underbrace{k-k}_{=0})(q(k))} + r$$

$$f(k) = r \ \square$$

Example 2.4.9

Determine the remainder of $\dfrac{\boxed{5x^4-3x^3-50} = f(x)}{x-2}$.

WAIT!!!! We MUST have a FUNCTION

$$f(2) = 5(2)^4 - 3(2)^3 - 50$$

$$= 80 - 24 - 50$$

$$= 6.$$

Success Criteria:

- I can appreciate that synthetic division is “da bomb”
- I can use synthetic division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after synthetic division, there is no remainder (The Remainder Theorem)

2.5 The Factor Theorem

(Factors have been FOUND)

Learning Goal: We are learning the connections between a polynomial function and its remainder when divided by a binomial

The Factor Theorem

Given a polynomial function, $f(x)$, then $x-a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

Example 2.5.1

Use the Factor Theorem to factor x^3+2x^2-5x-6.

WAIT!!!! We need a FUNCTION

$$f(x) = (x-a)(x-b)(x-c)$$

$$(a)(b)(c) = -6$$

∴ the factors must divide -6

$$f(x) = x^3+2x^2-5x-6$$

Test the possible factors of -6

$\pm 1, \pm 2, \pm 3, \pm 6$

Test $x=1$ or $(x-1)$

$$f(1) = 1^3+2(1)^2-5(1)-6 \neq 0$$

Test $x -1$ or $(x+1)$

$$f(-1) = (-1)^3+2(-1)^2-5(-1)-6$$

$$= -1+2+5-6 = 0$$

∴ $(x+1)$ is a factor

now divide

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \\ & x^2 & x & \# & \end{array}$$

$$\therefore x^3+2x^2-5x-6 = (x+1)(x^2+x-6)$$

$$= (x+1)(x+3)(x-2)$$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Example 2.5.2

Factor **fully** $x^4 - x^3 - 16x^2 + 4x \boxed{+48}$

Test $x = 1$, $(x-1)$

$f(1) = 1^4 - 1^3 - 16(1)^2 + 4(1) + 48$

$\neq 0$

Test $x = -2$

$f(-2) = (-2)^4 - (-2)^3 - 16(-2)^2 + 4(-2) + 48$

$= 16 + 8 - 64 - 8 + 48$

$= 0$ $\therefore$ $(x+2)$ is a factor

☺

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -16 & 4 & 48 \\ & \downarrow & -2 & 6 & 20 & -48 \\ \hline & 1 & -3 & -10 & 24 & 0 \end{array}$$

$\therefore (x+2)\underbrace{(x^3 - 3x^2 - 10x + 24)}_{\text{factor this}}$

Test factors of 24 Simon says try $x = 2$ $(x-2)$

$g(2) = 2^3 - 3(2)^2 - 10(2) + 24$

$= 8 - 12 - 20 + 24$

$= 0!$

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & \downarrow & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$\therefore x^4 - x^3 - 16x^2 + 4x + 48 = (x+2)(x-2)(x^2 - x - 12)$

$= (x+2)(x-2)(x-4)(x+3)$

$\pm 1, \pm 2 \ldots\ldots \pm 48$

Example 2.5.3 (*Pg 177 #6c in your text*)

Factor fully $x^4 + 8x^3 + 4x^2 - 48x$

$$= x(\underbrace{x^3 + 8x^2 + 4x - 48}_{g(x)})$$

Test $x=2$ $(x-2)$

$$g(2) = 2^3 + 8(2)^2 + 4(2) - 48$$
$$= 8 + 32 + 8 - 48$$
$$= 0$$

$$\begin{array}{r|rrrr} 2 & 1 & 8 & 4 & -48 \\ & & 2 & 20 & 48 \\ \hline & 1 & 10 & 24 & 0 \end{array}$$

$$\therefore x^4 + 8x^3 + 4x^2 - 48x = x(x-2)(x^2 + 10x + 24)$$
$$= x(x-2)(x+4)(x+6)$$

Example 2.5.4 (*Pg 177 #10*)

$f(1) = 10$

When $\overset{f(x)}{ax^3} - x^2 + 2x + b$ is divided by $x-1$ the remainder is 10. When it is divided by $x-2$ the remainder is 51. Find *a* and *b*.

This problem is very instructive.

$$f(1) = a(1)^3 - (1)^2 + 2(1) + b = 10$$
$$a - 1 + 2 + b = 10$$
$$a + b = 9$$

$$f(2) = a(2)^3 - (2)^2 + 2(2) + b = 51$$
$$8a - 4 + 4 + b = 51$$
$$8a + b = 51$$
$$-(a + b = 9)$$
$$7a = 42$$
$$a = 6$$

$$\boxed{\therefore a = 6,\ b = 3}$$

Success Criteria:

- I can use test values to find the factors of a polynomial function
- I can factor a polynomial of degree three or greater by using the factor theorem
- I can recognize when a polynomial function is not factorable

2.6 Factoring Sums and Differences of Cubes

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Knowing how to factor a sum or difference of cubes is a simple matter of remembering patterns.

Learning Goal: We are learning to factor a sum or difference of cubes.

Example 2.6.1 (*Recalling the pattern for factoring a Difference of Squares*)

Factor $4x^2 - 25$

$(2x - 5)(2x + 5)$

Note: Sums of Squares DO NOT factor!!

e.g. Simplify $x^2 + 4$

$8x^3 \pm 27$

Differences of Cubes

Same as the question

opposite the question

Always Positive

Pattern

$(cube_1 - cube_2) = (cuberoot_1 - cuberoot_2)(cuberoot_1^{\ 2} + cuberoot_1 \times cuberoot_2 + cuberoot_2^{\ 2})$

$8x^3 - 27$ $(2x - 3)(4x^2 + 6x + 9$

TWO POSITIVES and ONE NEGATIVE

Sums of Cubes (These DO factor!!)

Pattern

$(cube_1 + cube_2) = (cuberoot_1 + cuberoot_2)(cuberoot_1^{\ 2} - cuberoot_1 \times cuberoot_2 + cuberoot_2^{\ 2})$

$(8x^3 + 27)$ $(2x + 3)(4x^2 - 6x + 9)$

SOAP

Example 2.6.2

Factor $x^3 - 8$ $= (x-2)(x^2+2x+4)$

Example 2.6.3

Factor $27x^3 + 125y^3$ $= (3x+5y)(9x^2 - 15xy + 25y^2)$

Example 2.6.4

Factor $1 - 64z^3$ $= (1-4z)(1+4z+16z^2)$

Example 2.6.5

Factor $1000x^3 + 27$

$$= (\underline{10x} + \underline{3})(100x^2 - 30x + 9)$$

Example 2.6.6

Factor $x^6 - 729$

$(x^2)^3$

$$= \underline{(x^2 - 9)}(x^4 + 9x^2 + 81)$$

$$= (x-3)(x+3)(x^4 + 9x^2 + 81)$$

$$x^{64} - 1 = (x^{32}+1)(x^{32}-1)$$

↓

↓

Success Criteria:

- I can use patterns to factor a sum of cubes
- I can use patterns to factor a difference of cubes