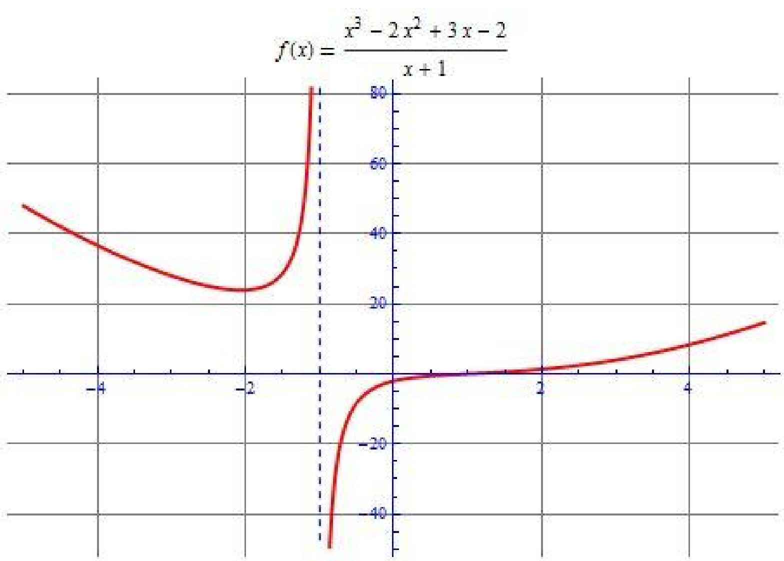
Advanced Functions

Course Notes

Unit 4 – Rational Functions, Equations and Inequalities

We are learning to

- *sketch the graphs of simple rational functions*
- *solve rational equations and inequalities with and without tech*
- *apply the techniques and concepts to solve problems involving rational models*



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MATH@TD

Unit 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

4.1 Introduction to Rational Functions and Asymptotes

Pg. 262 #1 - 3

4.2 Graphs of Rational Functions

Pg. 272 #1, 2 (Don't use any tables of values!), 4 – 6, 9, 10

4.4 Solving Rational Equations

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4.5 Solving Rational Inequalities

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4.1 Rational Functions, Domain and Asymptotes

Learning Goal: We are learning to identify the asymptotes of rational functions.

Definition 4.1.1

A **Rational Function** is of the form

$f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and both $p(x)$ and $q(x)$ are polynomial functions.

e.g. $f(x) = \dfrac{3x^2 - 5x + 1}{2x - 1}$ is a rational function

$g(x) = \dfrac{\sqrt{2x+5}}{3x-2}$ not because $\sqrt{2x+5}$ is not polynomial

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Domain

Definition 4.1.2

Given a rational function $f(x) = \dfrac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f = \{ x \in \mathbb{R} \mid q(x) \neq 0 \}$

zeros of $q(x)$

Example 4.1.1

Determine the natural domain of $f(x) = \dfrac{x^2 - 4}{x - 3}$.

$x \neq 3$

$D_f = \{ x \in \mathbb{R} \mid x \neq 3 \}$

$x \in (-\infty, 3) \cup (3, \infty)$

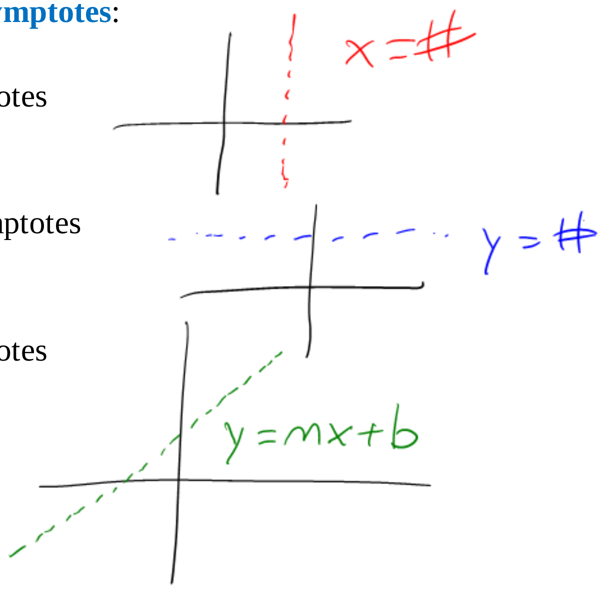
Asymptotes

There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes

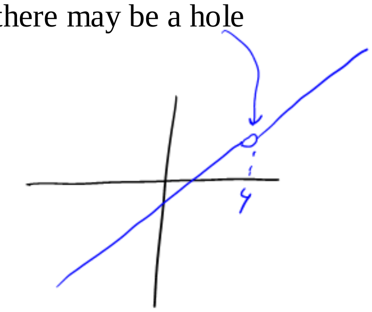
2) Horizontal Asymptotes

3) Oblique Asymptotes



Vertical Asymptotes

A rational function $f(x)=\frac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x)=0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.



Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x)=\frac{5x}{x^2-x-6}$

$$f(x)=\frac{5x}{(x-3)(x+2)}$$

$x=3$ $\quad$ $x=-2$

V.A. $\quad$ V.A

If the factor stay, they are V.A.

$$x\in(-\infty,-2)\cup(-2,3)\cup(3,\infty)$$

b) $h(x)=\dfrac{x+3}{x^2-9}$

$$h(x)=\frac{\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$$

$x=3$ $\quad$ $x=-3$

$x=3$ V.A.

$x=-3$ Hole because it disappeared!

$$h(x)=\frac{1}{x-3}$$

$x\in(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

c) $g(x)=\dfrac{x^2-4}{x+2}$

Factor everything

$$g(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}}$$

$x=-2$ Hole.

$g(x)=x-2$

$x\in(-\infty,-2)\cup(-2,\infty)$

Horizontal Asymptotes

Here we are concerned with the end behavior of the rational function

i.e. We are asking, given a rational function $f(x)=\frac{p(x)}{q(x)}$, how is $f(x)$ behaving as $x \to \pm\infty$.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $p(x)$ > Order of $q(x)$

   e.g. $f(x)=\frac{x^3-2}{x^2+1}$ $x^3-2 \to$ order 3, $x^2+1 \to$ order 2

When the order of p(x) is bigger than the order of q(x), there is $\boxed{\text{no}}$ horizontal asymptote.

2) Order of numerator = Order of denominator then the H.A. are the leading coefficients

   e.g. $f(x)=\frac{2x^2\boxed{-3x+1}}{3x^2\boxed{+4x-5}}$

If x is MASSIVE, the stuff behind the x's are inconsequential/irrelevant.

What's left is $\frac{2\cancel{x^2}}{3\cancel{x^2}} = \boxed{\frac{2}{3}=y}$ is the horizontal asymptote.

e.g. Determine the horizontal asymptote of $g(x)=\frac{3x-4x^5}{5x^5+2x-1}$

H.A: $y=\frac{-4}{5}$

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x) = \dfrac{x^2 - 5x + 6}{x^5 + 7}$

$$\frac{(100)^2 = 10{,}000}{(100)^5 = 10{,}000{,}000{,}000} = \frac{1}{\text{Really Big}} = \text{close to zero.}$$

H.A. is $y = 0$.

Oblique Asymptotes

These occur when the order of $p(x)$ is exactly one bigger than $q(x)$

e.g. $f(x) = \dfrac{x^2 - 2x + 3}{x + 1}$

With Oblique Asymptotes we are still dealing with end behaviors.

O.A. have the form $y = mx + b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: By polynomial division!
The O.A. is the quotient

$$f(x) = \frac{x^2 - 2x + 3}{x + 1}$$

$x = -1$

$$\begin{array}{r|rrr} -1 & 1 & -2 & 3 \\ & \downarrow & -1 & 3 \\ \hline & \boxed{1 \quad -3} & & 6 \end{array}$$

quotient remander

$y = 1x - 3$ is the line of the O.A.

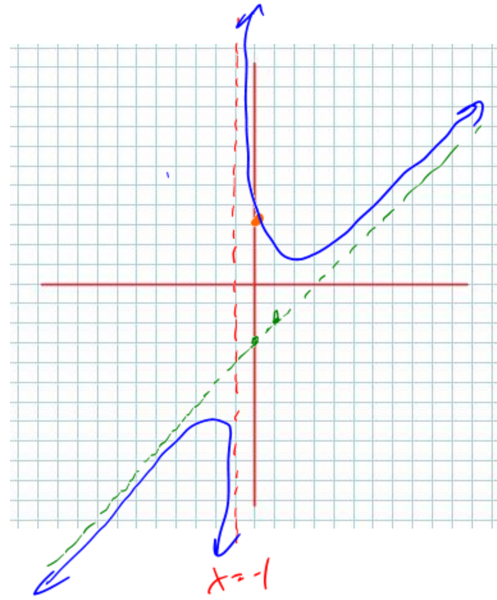
(Rough) Sketch of $f(x)=\dfrac{x^2-2x+3}{x+1}$

V.A. $x=-1$

H.A. None

O.A. $y=\frac{1}{1}x-3$

$f(0)=\frac{3}{1}=3$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x)=\dfrac{x+2}{x^2+3x+2}$ order 1 order 2

$f(x)=\dfrac{\cancel{(x+2)}}{(x+1)\cancel{(x+2)}}$

$f(x)=\dfrac{1}{x+1}$

V.A.	$x=-1$
Hole	$x=-2$
H.A.	$y=0$
O.A	None

b) $g(x)=\dfrac{\boxed{4}x^2-25}{\boxed{1}x^2-9}$

$g(x)=\dfrac{(2x-5)(2x+5)}{(x-3)(x+3)}$

V.A.	$x=-3,\ x=3$
Hole	None
H.A	$y=\frac{4}{1}=4$
O.A.	None

c) $h(x) = \dfrac{x^2 + 0x + 0}{x+3}$ order 2

order 1

-3	1	0	0
		-3	9
	1	-3	9

V.A	$x = -3$
Hole	None
H.A.	None
O.A	$y = x - 3$

Example 4.1.4

denominator

Determine an equation for a function with a vertical asymptote at $x = -3$, and a horizontal asymptote at $y = 0$.

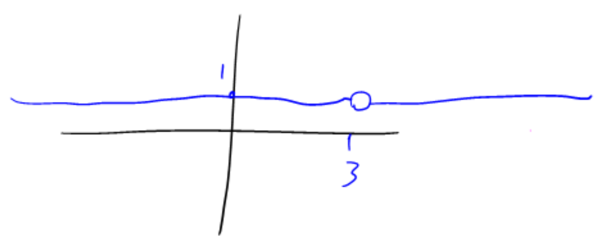
$(x+3)$

$$f(x) = \frac{10}{(x+3)}$$

Example 4.1.5

Determine an equation for a function with a hole discontinuity at $x = 3$.

$$f(x) = \frac{(x-3)}{(x-3)}$$



Success Criteria:

- I can identify a hole when there is a common factor between *p(x)* and *q(x)*
- I can identify a vertical asymptote as the zeros of *q(x)*
- I can identify a horizontal asymptote by studying the degrees of *p(x)* and *q(x)*
- I can identify an oblique asymptote when the degree of *p(x)* is exactly 1 greater than *q(x)*

4.2 Graphs of Rational Functions

Learning Goal: We are learning to sketch the graphs of rational functions.

Note: In Advanced Functions we will only consider rational functions of the form

$$f(x) = \frac{ax+b}{cx+d}$$

Rational Functions of the form $f(x) = \frac{ax+b}{cx+d_{\neq 0}}$ will have:

1) One Vertical Asymptote

$cx + d = 0$ The zero of the denominator.

$x = \frac{-d}{c}$

2) One Zero (unless $a=0$)

↳ x-intercept

Ther zero of the numerator $ax + b = 0$

$x = \frac{-b}{a}$

3) Functional Intercept

Y

$f(0) = \frac{\cancel{a(0)} + b}{\cancel{c(0)} + d} = \frac{b}{d}$ $\left(0, \frac{b}{d}\right)$

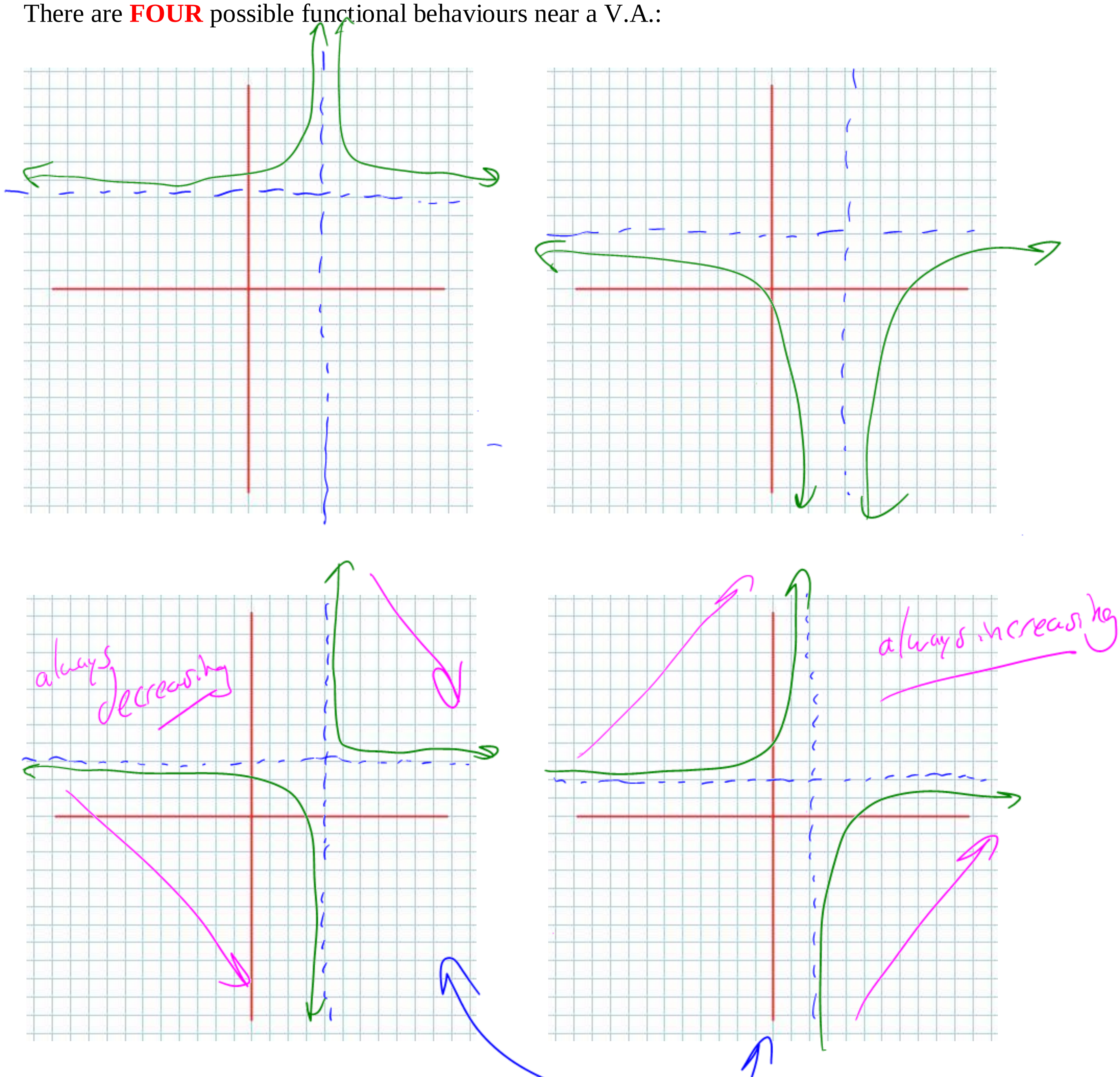
4) A Horizontal Asymptote because the orders are both one.

H.A: $y = \frac{a}{c}$ If $a = 0$, H.A. is $y = 0$

5) These functions will always be either increasing or decreasing, meaning no turning points.

Functional Behaviour Near A Vertical Asymptote

There are **FOUR** possible functional behaviours near a V.A.:



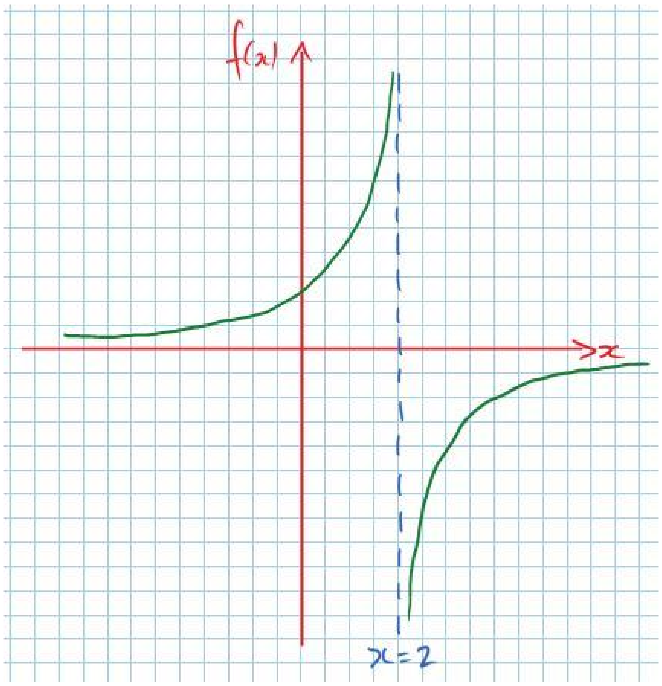
For functions of the form $f(x)=\frac{ax+b}{cx+d}$ we will see behaviours

The questions is, **how do we know which**?

We need to **analyze** the function **near the V.A.**

We need to become familiar with some **Notation.** Skipping Just graph it.

Consider some rational function with a sketch of its graph which looks like:



Example 4.2.1

Determine the functional behaviour of $f(x)=\dfrac{2x+1}{x-3}$ near its V.A.

We now have the tools to sketch some graphs!

Example 4.2.2

Sketch the graph of the given function. State the domain, range, intervals of increase/decrease and where the function is positive and negative.

a) $f(x) = \dfrac{2x+1}{x-1}$

H.A. $y = \frac{2}{1} = 2$

V.A. $x = 1$

x-int $x = \frac{-1}{2}$

y-int $y = \frac{1}{-1} = -1$

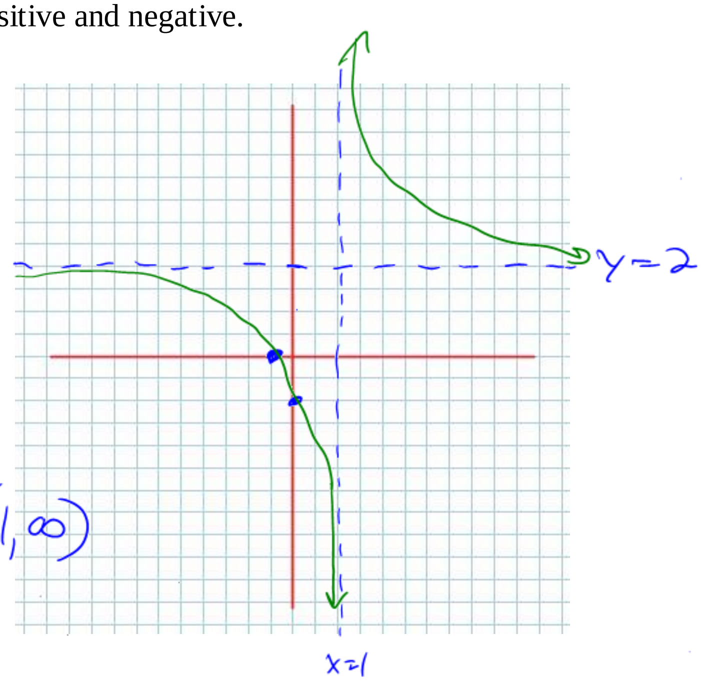
$D_f: x \in (-\infty, 1) \cup (1, \infty)$

$R_f: f(x) \in (-\infty, 2) \cup (2, \infty)$

$f(x)$ is decreasing

$f(x) > 0$ when $x \in (-\infty, \frac{-1}{2}) \cup (1, \infty)$

$f(x) < 0$ $x \in (\frac{-1}{2}, 1)$



b) $g(x) = \dfrac{3x-2}{2x+5}$

H.A. $y = \frac{3}{2}$

V.A. $x = \frac{-5}{2}$

x-int: $x = \frac{2}{3}$

y-int: $y = \frac{-2}{5}$

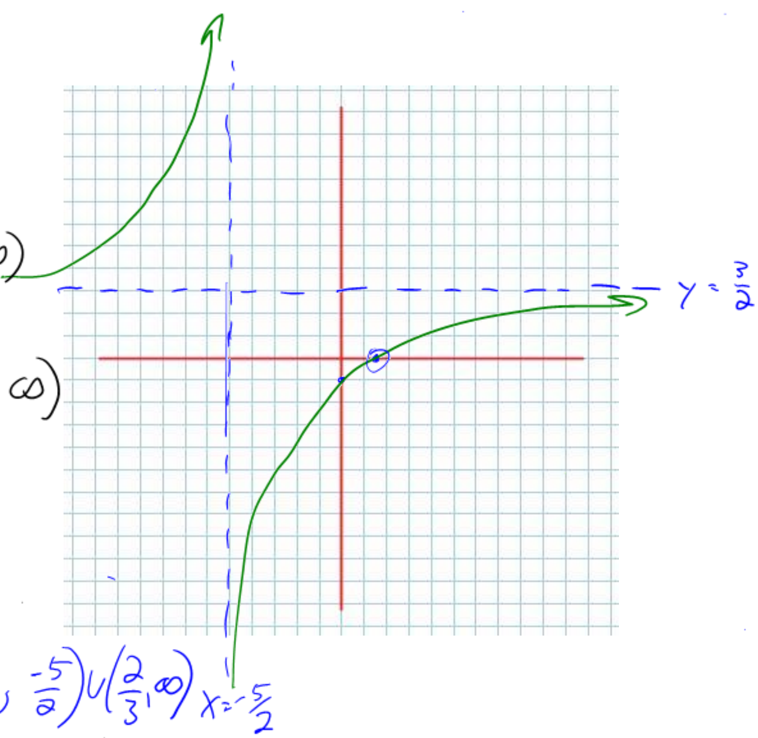
$D_g: x \in (-\infty, \frac{-5}{2}) \cup (\frac{-5}{2}, \infty)$

$R_g: g(x) \in (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

$g(x)$ is increasing

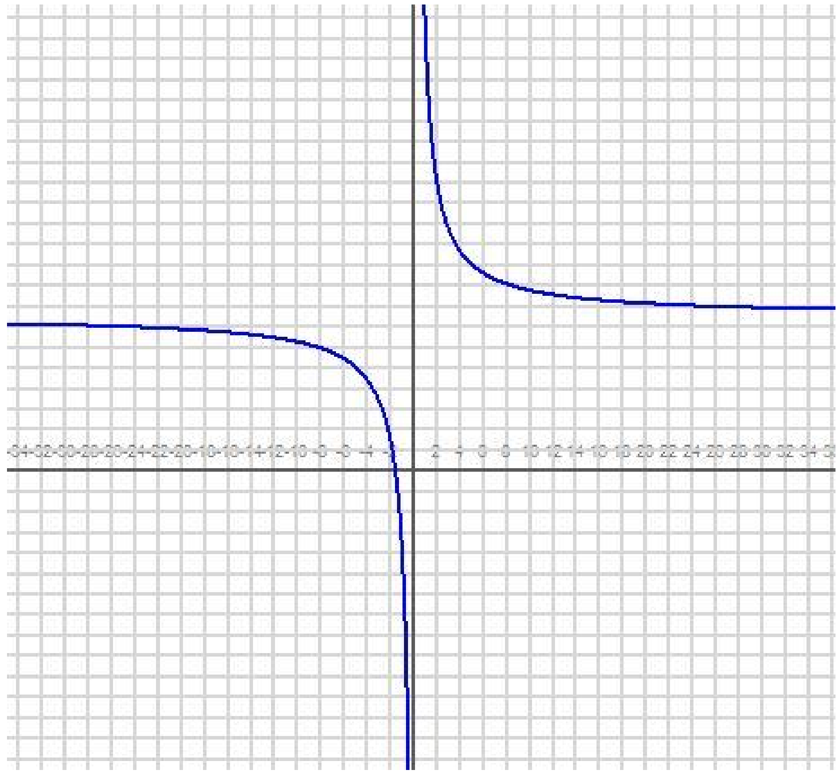
$g(x) > 0$ when $x \in (-\infty, \frac{-5}{2}) \cup (\frac{2}{3}, \infty)$

$g(x) < 0$ when $x \in (\frac{-5}{2}, \frac{2}{3})$



Example 4.2.3

Consider question #9 on page 274: $I(t) = \dfrac{15t + 25}{t}$



Success Criteria:

- I can identify the horizontal asymptote as $\frac{a}{c}$
- I can identify the vertical asymptote as $-\frac{d}{c}$
- I can identify the y-intercept as $\frac{b}{d}$
- I can identify the x-intercept as $-\frac{b}{a}$

Know it.

4.4 Solving Rational Equations

Learning Goal: We are learning to solve rational equations. Think rationally!

Solving a Rational Equation is **VERY MUCH** like solving a Polynomial Equation. Thus, this stuff is so much fun it should be illegal. But it isn't illegal unless you break a rule of algebra. Math Safe!

KEY (*this is a major key for you music buffs*)

Multiplying by the Multiplicative Inverse of the Common Denominator is wonderful to use **WHEN YOU HAVE** something like:

$$\text{RATIONAL}_1 + \text{RATIONAL}_2 = \text{RATIONAL}_3$$

(2)(x)(x-2) Common Denominator

2(x-2)(x)

e.g.

$$\frac{3}{x-2} + \frac{3}{2} = \frac{4(x+5)}{x}$$

$$(2)(x)\cancel{(x-2)}\left(\frac{3}{\cancel{x-2}}\right) + \cancel{2}(x)(x-2)\left(\frac{3}{\cancel{2}}\right) = (2)\cancel{(x)}(x-2)\left(\frac{4(x+5)}{\cancel{x}}\right)$$

$$6x + 3x(x-2) = 8(x-2)(x+5)$$

Expand, "stuff = 0", then solve.

Make Sure To Keep **RESTRICTIONS ON X** In Mind

This means that restrictions cannot be solutions.

Example 4.4.1

a) Solve $\frac{x}{5} = \frac{9}{18}$

No restrictions

$$\frac{18x}{18} = \frac{45}{18}$$

$$x = \frac{5}{2}$$

b) Solve $\frac{1}{x} - \frac{5x}{3} = \frac{2}{5}$ $(5)(3)(x)$

RESTRICTIONS

$x \neq 0$

CD: $15x$ or $(5)(3)(x)$

$$15 - 25x^2 = 6x$$

$$0 = 25x^2 + 6x - 15$$

D.N.F.

Use technology or Quadratic Formula

$x = -0.9$ and $x = 0.66$

c) Solve $\left(\frac{3}{x}+\frac{4}{x+1}=\frac{2}{1}\right)(x)(x+1)$

RESTRICTIONS

$x \neq 0$

$x \neq -1$

C.D.

$(x)(x+1)$

$3(x+1)+4x=2x(x+1)$

$3x+3+4x=2x^2+2x$

$-3x \quad -3 \quad -4x \quad -7x \quad -3$

$0=2x^2-5x-3$

M: -6 A: -5 $\boxed{-6,1}$

$0=\underbrace{2x^2+1x}_{x}\underbrace{-6x-3}_{-3}$

$0=(x-3)(2x+1)$

$x=3$ ✓

$x=\frac{-1}{2}$ ✓

d) Solve $\left(\frac{10}{\underbrace{x^2-2x}_{x(x-2)}}+\frac{4}{x}=\frac{5}{x-2}\right)(x)(x-2)$

RESTRICTIONS

$x \neq 0$

$x \neq 2$

C.D.

$x(x-2)$

$10+4(x-2)=5x$

$10+4x-8=5x$

$2=x$

However, $x \neq 2$ $\therefore$ there are no solutions to this equation!

e) Solve $\left(\frac{16x}{1} - \frac{5}{x+2} = \frac{15}{x-2} - \frac{60}{(x-2)(x+2)}\right)(x-2)(x+2)$

Restrictions: $x \neq -2, 2$

CD: $(x-2)(x+2)$

$$16x\cancel{(x-2)(x+2)} - 5(x-2) = 15(x+2) - 60$$

$$(x^2-4)$$

$$16x^3 - 64x - 5x + 10 = \boxed{15x + 30 - 60}$$

$$\frac{16x^3 - 84x + 40 = 0}{4}$$

$$4x^3 - 21x + 10 = 0$$

$\pm 1, \pm 2, \pm 5, \pm 10$

Try $f(2) = 4(2)^3 - 21(2) + 10$

$= 32 - 42 + 10$

$= 0!$ $\therefore (x-2)$ is a factor.

2	4	0	-21	10
		8	16	-10
	4	8	-5	0

$= 4x^2 + 8x - 5$

M: -20
A: 8

$-2, 10$

$$(x-2)(4x^2 + 8x - 5) = 0$$

$$(x-2)(\underbrace{4x^2 - 2x}_{2x} \underbrace{+ 10x - 5}_{+5})$$

$$(x-2)(2x+5)(2x-1) = 0$$

$\therefore$ ~~$x = 2$~~ inadmissable

$x = \frac{-5}{2}$

$x = \frac{1}{2}$

Example 4.4.2

From your Text: Pg. 285 #10

The Turtledove Chocolate factory has two chocolate machines. Machine A takes *m* minutes to fill a case with chocolates, and machine B takes *m + 10* minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

Work or rate problem

$$\frac{\text{a job}}{\text{time}} = \frac{\text{filling } \boxed{a} \text{ case}}{\text{minutes}}$$

A: $\frac{1}{m}$ B: $\frac{1}{m+10}$ Together: $\frac{1}{15}$

$$\left(\frac{1}{m} + \frac{1}{m+10} = \frac{1}{15}\right)^{(15)(m)(m+10)}$$

Restrictions: $m \neq 0$, $m \neq -10$

CD: $(15)(m)(m+10)$

$$15(m+10) + 15m = m(m+10)$$

$$15m + 150 + 15m = m^2 + 10m$$

$-15m$ -150 $-15m$ $-30m$ -150

$$0 = m^2 - 20m - 150 \quad \text{D.N.F.}$$

by Q.F. or technology

$$m = 25.8$$

∴ Machine A takes 25.8 minutes and Machine B takes 35.8 minutes

Success Criteria:

- I can recognize that the zeros of a rational function are the zeros of the numerator
- I can solve rational equations by multiplying each term by the lowest common denominator, then solving the resulting polynomial equation
- I can identify inadmissible solutions based on the context of the problem

4.5 Solving Rational Inequalities

Learning Goal: We are learning to solve rational inequalities using algebraic and graphical approaches.

The joy, wonder and peace these bring is really quite amazing

e.g. Solve $\left(\frac{x-2}{7} \geq 0\right)7$

$x-2 \geq 0$

$x \geq 2$

Note: For Rational Inequalities, with a variable in the denominator, you **CANNOT** multiply by the multiplicative inverse of the common denominator!!!!

Why? If the factor, $x+3$, is negative the inequality would need to flip.

Example 4.5.1

Solve $\frac{x-2}{x+3} \geq 0$

We solve by using an Interval Chart

For the intervals, we split $(-\infty, \infty)$ at all zeros (**where the numerator is zero**), and all restrictions (**where the denominator is zero**) of the (SINGLE) rational expression. Keep in mind that it may take a good deal of algebraic manipulation to get a SINGLE rational expression…

zeros at $x=2$, restriction at $\boxed{x \neq -3}$

Intervals	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
Test Values	-4	0	3
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
Ratio	$+$	$-$	$+$

$\therefore \frac{x-2}{x+3} \geq 0$ when $x \in (-\infty, -3) \cup [2, \infty)$

Example 4.5.2

Solve $\frac{1}{x+5} < 5$ $\quad -5 \quad -5$

DO NOT CROSS MULTIPLY (or else)

- Get everything on one side ✓
- Simplify into a single Rational Expression using a common denominator
- Interval Chart it up

hard part

$$\frac{1}{x+5} - \frac{5}{1}\frac{(x+5)}{(x+5)} < 0$$

$$\frac{1-5(x+5)}{x+5} < 0$$

$$\frac{1-5x-25}{x+5} < 0$$

$$\frac{-5x-24}{x+5} < 0$$

$-5x-24=0$

$-5x=24$

$x=\frac{24}{-5}$

zero at $x=\frac{-24}{5}=-4.8$

restriction at $x=-5$

Interval	$(-\infty, -5)$	$(-5, -4.8)$	$(-4.8, \infty)$
T.V.	-6	-4.9	0
$-5x-24$	$+$	$+$	$-$
$x+5$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$

$\therefore \frac{1}{x+5} < 5$ when $x \in (-\infty, -5) \cup (-4.8, \infty)$

Example 4.5.3

Solve $\dfrac{x^2+3x+2}{x^2-16} \geq 0$

FACTORED FORM IS YOUR FRIEND

$$\frac{(x+1)(x+2)}{(x-4)(x+4)} \geq 0$$

zeros at $x = -1, -2$

restrictions $x \neq -4, 4$

Intervals	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
T.V.	-5	-3	-1.5	0	5
$x+1$	$-$	$-$	$-$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$	$+$	$+$
Ratio	$+$	$-$	$+$	$-$	$+$

$\therefore \dfrac{x^2+3x+2}{x^2-16} \geq 0$ when $x \in (-\infty, -4) \cup [-2, -1] \cup (4, \infty)$

$$3x + 5 \geq 4x - 8$$
$$\quad -3x \quad +8 \qquad -3x \quad +8$$
$$13 \geq x$$

this is bigger, so move things to the right side

Example 4.5.4

Solve $\dfrac{3}{x+2} \leq x$

$$0 \leq \frac{x}{1}\cdot\frac{(x+2)}{(x+2)} - \frac{3}{x+2}$$

$$0 \leq \frac{x^2 + 2x - 3}{x+2}$$

$$\boxed{0 \leq} \frac{(x+3)(x-1)}{x+2}$$

Zeros: $x = -3, 1$

Restrictions: $x \neq -2$

Intervals	$(-\infty, -3)$	$(-3, -2)$	$(-2, 1)$	$(1, \infty)$
T.V.	-4	-2.5	0	2
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$	$+$

$$\therefore \frac{3}{x+2} \leq x \text{ when } x \in [-3, -2) \cup [1, \infty)$$

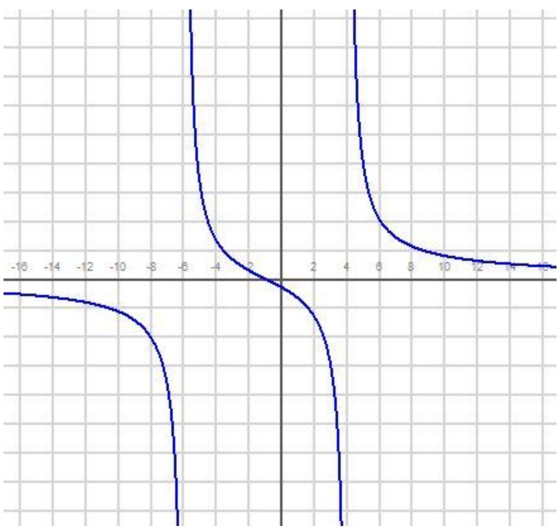
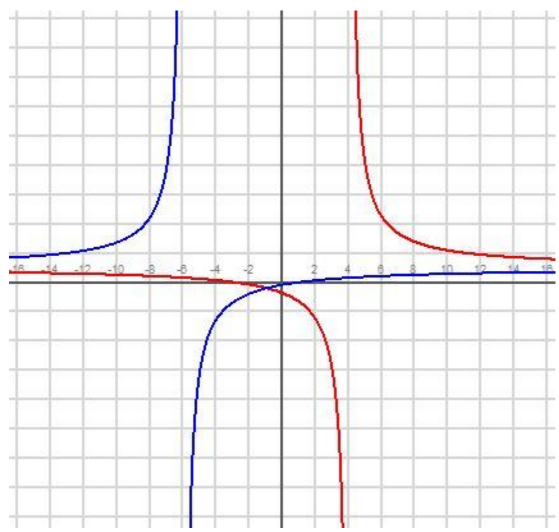
Example 4.5.5

From your Text: Pg. 296 #6a

Using **Graphing Tech**

Solve $\dfrac{x+3}{x-4} \geq \dfrac{x-1}{x+6}$

Note: There are **TWO** methods, both of which require a **FUNCTION** (let $f(x) = \ldots$ returns)

1) Get a Single Function (on one side of the inequality)	2) Use Two Functions (one for each side)
	

Success Criteria:

- I can recognize that an inequality has many possible intervals of solutions
- I can solve an inequality algebraically, using an interval chart
- I can solve an inequality graphically