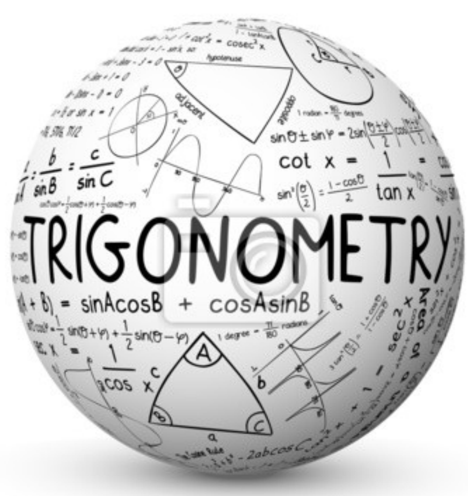
Advanced Functions

Course Notes

Unit 6 – Trigonometric Identities and Equations

We will learn

- *about Equivalent Trigonometric Relationships*
- *how to use compound angle formulas to determine exact values for trig ratios which DON'T involve the two special triangles*
- *techniques for proving trigonometric identities*
- *how to solve linear and quadratic trigonometric equations using a variety of strategies*



A∞Ω
MATH@TD

Chapter 6 – Trigonometric Identities and Equations

*Contents with suggested problems from the Nelson Textbook (**Chapter 7**)*

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

6.1 Basic Trigonometric Equivalencies
Pg. 392 – 393 #3cdef, 5cdef

6.2 Compound Angle Formulae
Pg. 400 – 401 #3 – 6, 8 – 10, 13

6.3 Double Angle Formulae
Pg. 407 – 408 Finish #2, 4, 12 – Do # 6, 7

6.4 Trigonometric Identities
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6.5 Linear Trigonometric Equations
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6.6 Quadratic Trigonometric Equations
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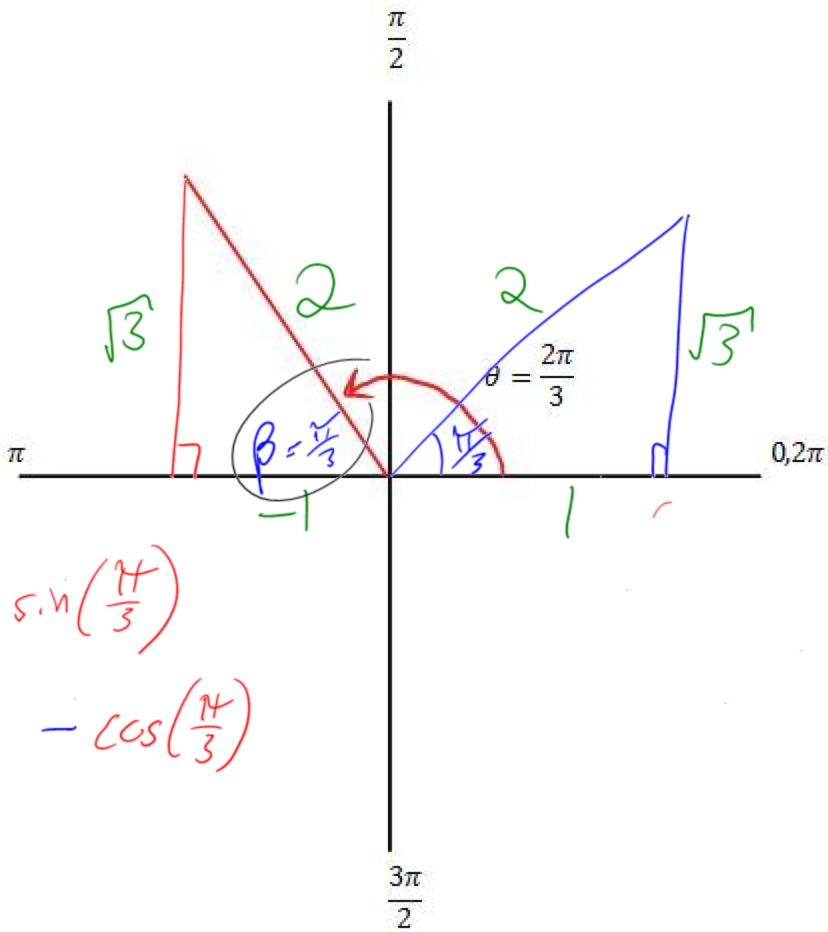
6.1 Basic Trigonometric Equivalencies

Learning Goal: We are learning to identify equivalent trigonometric relationships.

We have already seen a very basic trigonometric equivalency when we considered angles of rotation. For example, consider the angle of rotation for $\theta = \frac{2\pi}{3}$:

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \quad \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$

$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$

Periodic Equivalencies

Symmetry (odd and even)

Example 6.1.1

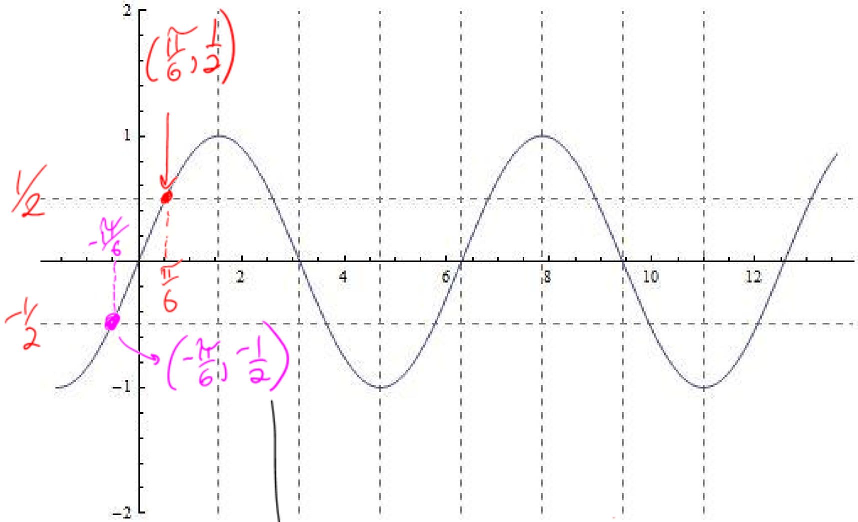
Consider the sketch of the function $f(\theta) = \sin(\theta)$

we say that $\sin\theta$ is 2π periodic

$\rightarrow \sin\theta = \sin(2\pi + \theta)$

Sine is an odd fn.

$\therefore \sin\theta = -\sin(-\theta)$



Even: $f(x) = f(-x)$

Odd: $f(x) = -f(-x)$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

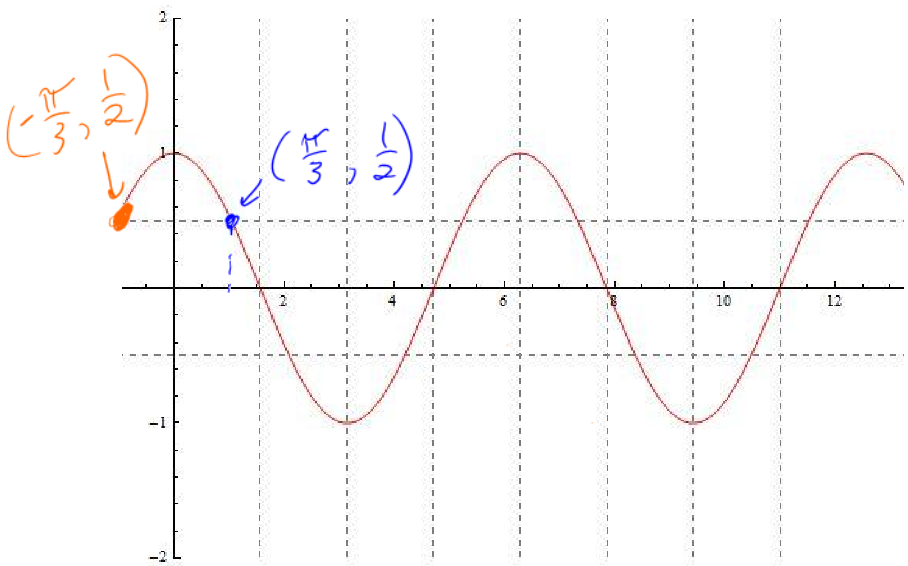
$-\sin\left(\frac{-\pi}{6}\right) = +\frac{1}{2}$

Example 6.1.2

Consider $g(x) = \cos(x)$

Cosine is also 2π periodic

$\longrightarrow \cos\theta = \cos(2\pi + \theta)$

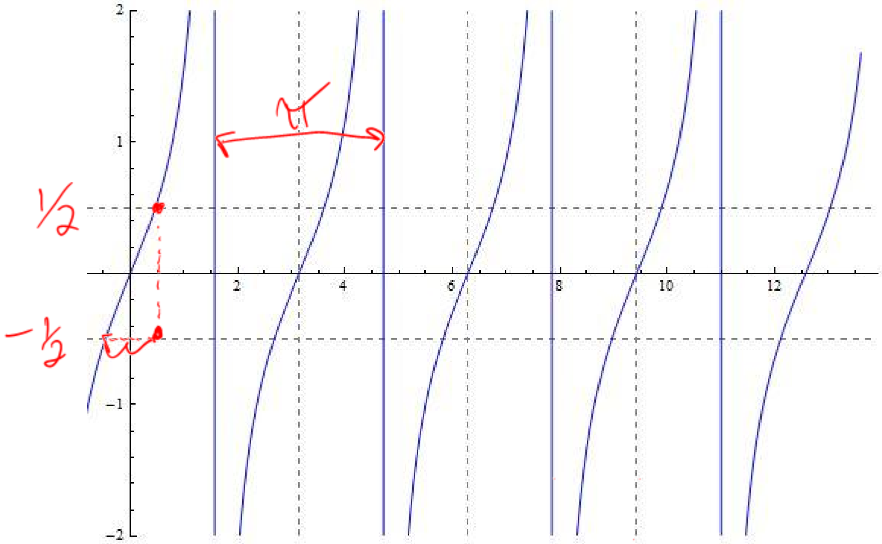


$\therefore$ Cosine is even

$\rightarrow \cos\theta = \cos(-\theta)$

Example 6.1.3

Consider $h(\theta) = \tan(\theta)$



tan is π periodic

$\Rightarrow \tan(\theta) = \tan(\pi + \theta)$

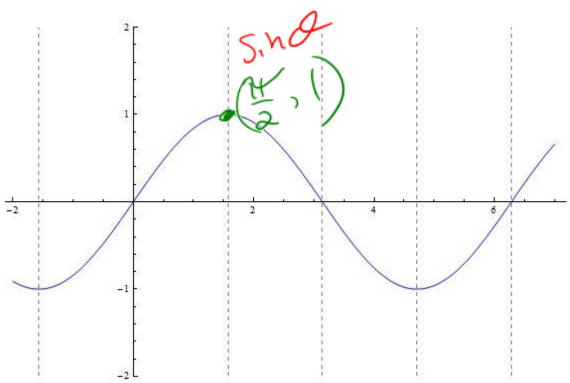
tan is an odd fn

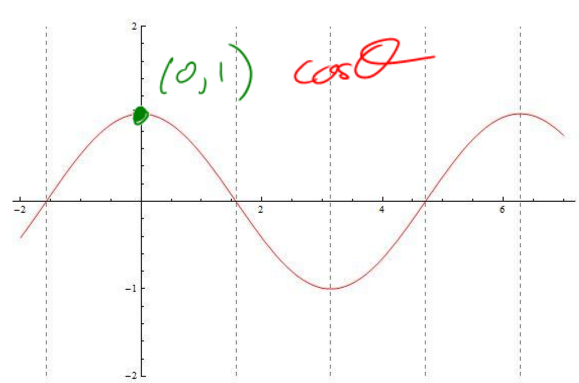
$\therefore \tan\theta = -\tan(-\theta)$

Shift Equivalencies

Example 6.1.4

Consider the sketches of the graphs for $f(\theta) = \sin(\theta)$ and $g(\theta) = \cos(\theta)$



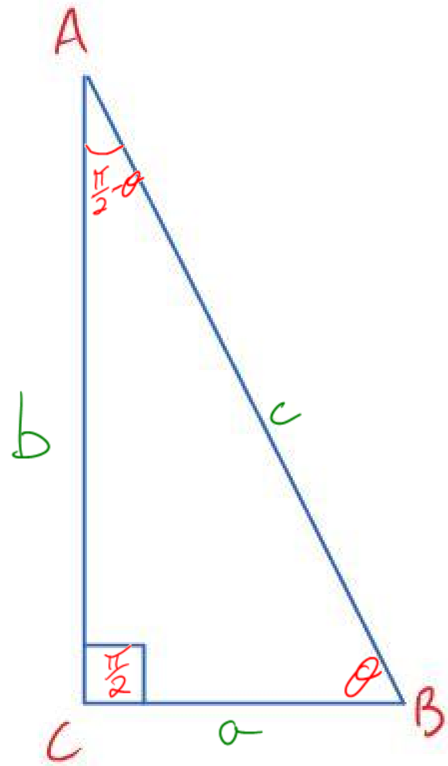


$\sin\theta = \cos(\theta - \frac{\pi}{2})$

$\cos\theta = \sin(\theta + \frac{\pi}{2})$

Cofunction Identities

Consider the right angle triangle



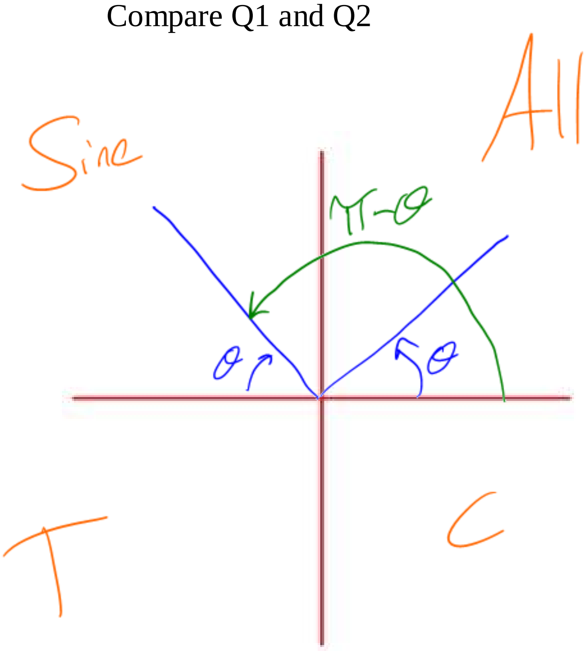
$$\sin\theta = \frac{b}{c} = \cos\left(\frac{\pi}{2}-\theta\right)$$

$$\cos\theta = \frac{a}{c} = \sin\left(\frac{\pi}{2}-\theta\right)$$

$$\tan\theta = \frac{b}{a} = \cot\left(\frac{\pi}{2}-\theta\right)$$

Using CAST, relating angles of rotation to π and 2π

Compare Q1 and Q2



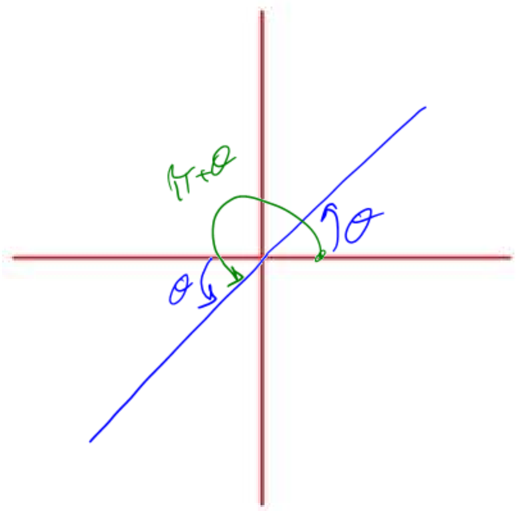
$$\underline{Q1} = \underline{Q2}$$

$$\sin\theta = \sin(\pi-\theta)$$

$$\cos\theta = -\cos(\pi-\theta)$$

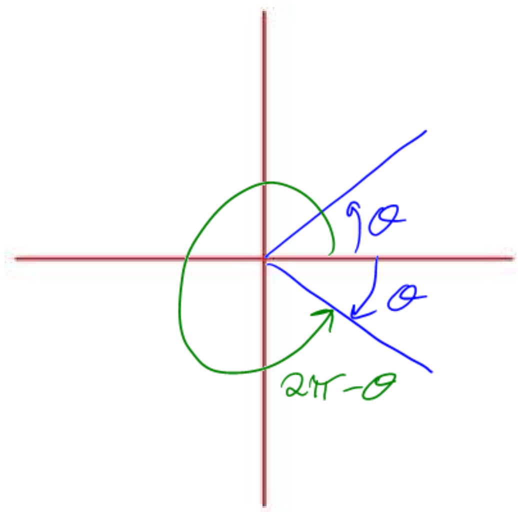
$$\tan\theta = -\tan(\pi-\theta)$$

Compare Q1 and Q3



$$\sin\theta = -\sin(\pi+\theta)$$
$$\cos\theta = -\cos(\pi+\theta)$$
$$\tan\theta = \tan(\pi+\theta)$$

Compare Q1 and Q4



$$\sin\theta = -\sin(2\pi-\theta)$$
$$\cos\theta = \cos(2\pi-\theta)$$
$$\tan\theta = -\tan(2\pi-\theta)$$

Example 6.1.5

From your text: Pg. 392 #3

Use a cofunction identity to find an equivalency:

a) $\sin\left(\underset{\theta}{\frac{\pi}{6}}\right) = \cos\left(\frac{\pi}{2} - \underset{\theta}{\frac{\pi}{6}}\right)$

$$= \cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

d) $\cos\left(\frac{5\pi}{16}\right) = \sin\left(\frac{\pi}{2} - \frac{5\pi}{16}\right)$

$$= \sin\left(\frac{8\pi}{16} - \frac{5\pi}{12}\right)$$

$$= \sin\left(\frac{3\pi}{16}\right)$$

Example 6.1.6

From your text: Pg. 393 #5

Using the related acute angle, find an equivalent expression:

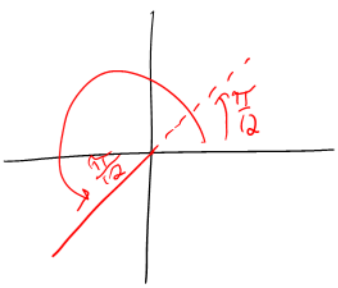
a) $\sin\left(\frac{7\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$

Q2

Q1

Q3

b) $\underbrace{\cos\left(\frac{13\pi}{12}\right)}_{-\#} = -\underbrace{\cos\left(\frac{\pi}{12}\right)}_{+\#}$



Success Criteria:

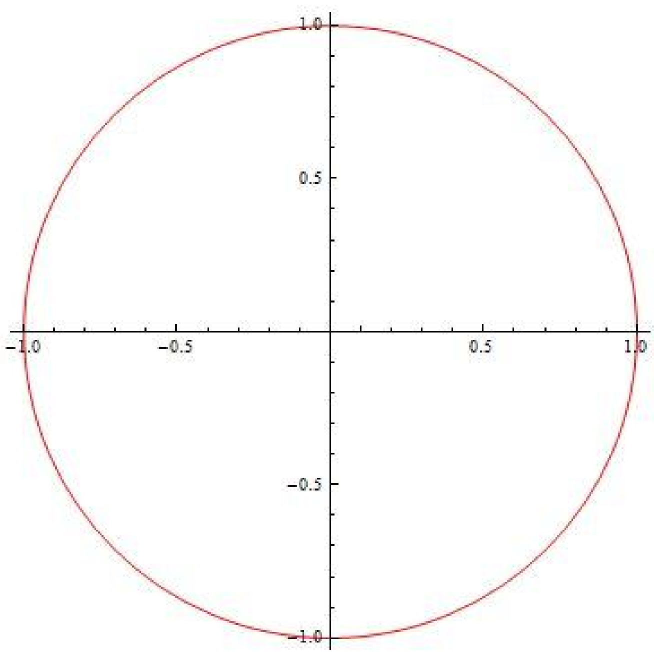
- I can recognize that there are many equivalent trigonometric expressions due to their periodic nature
- I can recognize several types of equivalencies
    - Shifting sin/cos by 2π OR $\pi/2$
    - Cos has even symmetry, Sin and Tan have odd symmetry
    - Cofunction equivalencies
    - Equivalencies based on the quadrant a principal angle is in

6.2 Compound Angle Formulae

Learning Goal: We are learning to use the compound angle formulas

Here we learn to find **exact trig ratios** for **non-special angles**!

Consider the picture:



Can we find $\sin(A+B)$ if we know $\sin(A)$ and $\sin(B)$?

or $\cos(A+B)$?

or $\tan(A+B)$?

Your text has a nice proof of one of the six compound angle formulas (there are six of them!...see Pg. 394)

Namely your text proves that $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

Using some trig equivalencies (from 6.1) we will find the other 5 compound angle formulae.

We now know $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Example 6.2.1

Find a formula for $\cos(A \mp B)$

$$\cos(A + (\underset{\text{"B"}}{-B})) = \cos A \underbrace{\boxed{\cos(-B)}}_{\text{even}} - \sin A \underbrace{\boxed{\sin(-B)}}_{\text{odd}}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Recall from 6.1

cosine is even

$\therefore \cos(-\theta) = \cos(\theta)$

Sine is odd

$\therefore \sin(-\theta) = -\sin(\theta)$

$\sin(-30) = -\sin(30)$

$$\rightarrow \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Example 6.2.2

Determine a compound angle formula for $\sin(A+B)$ using **a cofunction identity** and a **cosine compound angle formula**.

We know $\sin(A+B) = \sin A \cos B + \sin B \cos A$

We want $\sin(A-B)$

$$= \sin(A+(-B)) = \sin A \underbrace{\cos(-B)}_{\text{even}} + \underbrace{\sin(-B)}_{\text{odd}} \cos A$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A$$

$$\rightarrow \quad \sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

Example 6.2.3

Using the fact that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ determine a compound angle formula for $\tan(A+B)$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B}$$

Divide each term by $\underline{\cos A\cos B}$

$$= \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\sin B\cancel{\cos A}}{\cancel{\cos A}\cos B}}{\underbrace{\dfrac{\cancel{\cos A\cos B}}{\cancel{\cos A\cos B}}}_{1} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

tangent is odd
$\therefore \tan(-B) = -\tan B$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Example 6.2.4

From your text: Pg. 400 #3acd

Express each given angle as a compound angle using a pair of special triangle angles

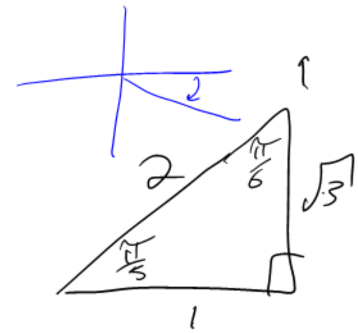
$\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$

$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

a) $75^\circ = 30 + 45$

c) $-\frac{\pi}{6} = \frac{\pi}{6} - \frac{\pi}{3}$ or $2\pi - \frac{\pi}{6}$

$-30 = 30 - 60$



$\frac{\pi}{3} = \frac{4\pi}{12}$

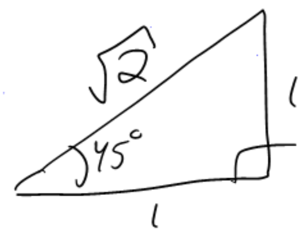
$\frac{\pi}{4} = \frac{3\pi}{12}$

$\frac{\pi}{6} = \frac{2\pi}{12}$

d) $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12}$ OR $\frac{\pi}{4} - \frac{\pi}{6}$

15°

$= \frac{\pi}{3} - \frac{\pi}{4}$



$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

Example 6.2.5

From your text: Pg. 400 #4ac, 8bd

Determine the **EXACT** value of the trig ratio

a) $\sin(75^\circ) = \sin(\overset{A}{30} + \overset{B}{45})$

$= (\sin 30)\cos(45) + (\sin 45)(\cos 30)$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

$(3+\sqrt{3})(3+\sqrt{3})$

$9 + \cancel{3}6\sqrt{3} + 3$

$12 + 6\sqrt{3}$

b) $\tan\left(\frac{5\pi}{12}\right) = \tan\left(\overset{A}{\frac{\pi}{4}} + \overset{B}{\frac{\pi}{6}}\right)$

$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$

$= \frac{\frac{1\times 3}{1\times 3} + \frac{\sqrt{3}}{3}}{\frac{1\times 3}{1\times 3} - (1)\left(\frac{\sqrt{3}}{3}\right)}$

$= \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}} \div$

$= \frac{3+\sqrt{3}}{\cancel{3}} \times \frac{\cancel{3}}{3-\sqrt{3}}$

$= \frac{3+\sqrt{3}}{3-\sqrt{3}}$

you may stop....

$= \frac{3+\sqrt{3}}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}}$

$= \frac{\overset{2}{\cancel{12}} + \overset{1}{\cancel{6}}\sqrt{3}}{\cancel{9-3}\ \cancel{6}\ 1}$

$= 2 + \sqrt{3}$

$-\tan(15) = -\tan(45-30)$

45 - 60

#8b) $\tan\left(-15^\circ\right) = \tan(30-45)$

$$= \frac{\tan 30 - \tan 45}{1 + \tan 30 \tan 45}$$

$$= \frac{\frac{\sqrt{3}}{3} - \cancel{1}\frac{3}{3}}{\frac{3}{3}\cancel{1} + \frac{\sqrt{3}}{3}(1)}$$

$$= \frac{\frac{\sqrt{3}-3}{\cancel{3}}}{\cancel{\frac{3+\sqrt{3}}{3}}} \times \frac{\cancel{3}}{3+\sqrt{3}} \div$$

$3\sqrt{3} - 3 - 9 + 3\sqrt{3}$

$6\sqrt{3} - 12$

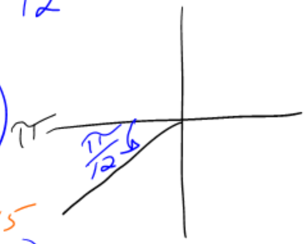
$$= \frac{\sqrt{3}-3}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$= \frac{6\sqrt{3}-12}{6}$$

$$= \sqrt{3} - 2$$

$\frac{4\pi}{12}, \frac{3\pi}{12}, \frac{2\pi}{12}$

d) $\sin\left(\frac{13\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right)$ π



60 − 45

$$= -\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= -\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right)$$

$$= -\left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}\right)$$

$$= -\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)$$

$$= -\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)$$

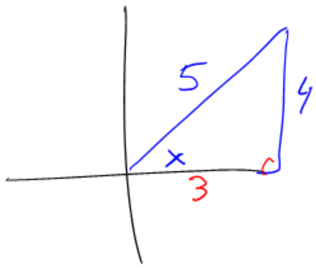
$$= \frac{-\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

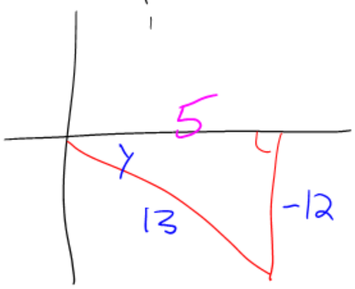
Example 6.2.6

From your text: Pg. 401 #9a

Q1 Q4

If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, where $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le y \le 2\pi$ evaluate $\cos(x+y)$.





$169 - 144 = 25$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$$

$$= \frac{15}{65} + \frac{48}{65}$$

$$= \frac{63}{65}$$

Success Criteria:

- I can identify SIX compound angle formulas
- I can use compound angle formulas to obtain exact values for trigonometric ratios
- I can use compound angle formulas to show that some trigonometric expressions are equivalent

6.3 Double Angle Formulae

This is a nice extension of the compound angle formulae from section 6.2.

Learning Goal: We are learning to use double angle formulas

Recall the compound Angle Formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

1) $\sin(2A) = \sin(\overset{\text{"A"}}{A} + \overset{\text{"B"}}{A})$

$= \sin A \cos A + \sin A \cos A$

$= 2\sin A \cos A$

Recall: $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

2) $\cos(2A) = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

$\boxed{① \cos(2A) = \cos^2 A - \sin^2 A}$ $\Rightarrow (\cos A - \sin A)(\cos A + \sin A)$

② $\cos(2A) = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1$ ②

③ $\cos(2A) = 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

3) $\tan(2A) = \tan(A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

Example 6.3.1

From your text: Pg. 407 #2ae

Express as a single trig ratio and evaluate:

a) $2\sin(\underset{A}{45^\circ})\cos(\underset{A}{45^\circ}) = \sin(2 \times 45)$

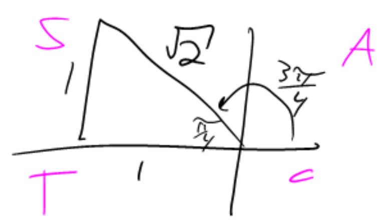
$$= \sin(90)$$

$$= 1$$

e) $1 - 2\sin^2\left(\overset{A}{\frac{3\pi}{8}}\right) = \cos\left(\cancel{2}^{1} \times \frac{3\pi}{\cancel{8}_{4}}\right)$

$$= \cos\left(\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} \Rightarrow \frac{-\sqrt{2}}{2}$$

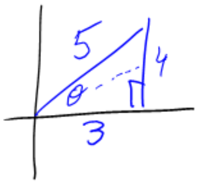


Example 6.3.2

From your text: Pg. 407 #4

Q1

Determine the values of $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$ given $\cos(\theta)=\frac{3}{5}$, $0\le\theta\le\frac{\pi}{2}$



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{3}{5}\right)^2 - 1 = 2\left(\frac{9}{25}\right) - 1 = \frac{18}{25} - \frac{25}{25} = \frac{-7}{25}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{\frac{9}{9}-\frac{16}{9}} = \frac{\frac{8}{3}}{\frac{-7}{9}} = \frac{8}{\cancel{3}_1} \times \frac{-\cancel{9}\,3}{7} = \frac{-24}{7}$$

Example 6.3.3

From your text: Pg. 408 #12

Use the appropriate angle and double angle formulae to determine a formula for:

a) $\sin(3\theta) = \sin(\underset{A}{2\theta} + \underset{B}{\theta})$

the formula can only use sine

$\sin A\cos B + \sin B\cos A$

$$= \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$$

$$= (2\sin\theta\cos\theta)\cos\theta + \sin\theta(1-2\sin^2\theta)$$

$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$

$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$

$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

Example 6.3.4

From your text: Pg. 407 #8

Determine the value of a in the following:

$$2\tan(x)-\tan(2x)+2a=1-\tan(2x)\cdot\tan^2(x)$$

$$2a = 1 - \tan(2x)\cdot\tan^2(x) - 2\tan(x) + \tan(2x)$$

$$2a = \tan(2x) - \tan(2x)\tan^2(x) - 2\tan(x) + 1$$

$$2a = \tan(2x)(1-\tan^2(x)) - 2\tan(x) + 1$$

$$2a = \left(\frac{\cancel{2\tan(x)}}{\cancel{1-\tan^2(x)}}\right)\cancel{(1-\tan^2(x))} - \cancel{2\tan(x)} + 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

$$3 - 6x$$

$$= 3(1-2x)$$

Success Criteria:

- I can identify FIVE double angle formulas
- I can use the double angle formulas to simplify expressions and to calculate exact values
- I can use the double angle formulas to develop other equivalent formulas

6.4 Trigonometric Identities

Learning Goal: We are learning to prove trigonometric identities.

Proving Trigonometric Identities is so much fun, it's plainly ridiculous. I should be paid extra for letting you play with these proofs! We will be using **ALGEBRA** (remember the rules?). Inside our algebra we will be using the following tools:

Reciprocal Identities

e.g. $\csc(\theta) = \dfrac{1}{\sin(\theta)}$ $\quad \sin\theta = \dfrac{1}{\csc\theta}$

Quotient Identities

e.g. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, or $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

The Pythagorean Trig Identities

$\sin^2(\theta) + \cos^2(\theta) = 1$ $\qquad 1 + \tan^2(\theta) = \sec^2(\theta)$ $\qquad 1 + \cot^2(\theta) = \csc^2(\theta)$

$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$ $\qquad \Rightarrow 1 = \sec^2\theta - \tan^2\theta$ $\qquad \Rightarrow \csc^2\theta - 1 = \cot^2\theta$

or $\sin^2\theta = 1 - \cos^2\theta$ $\qquad$ or $\tan^2\theta = \sec^2\theta - 1$ $\qquad$ or $1 = \csc^2\theta - \cot^2\theta$

The Compound Angle Formulae

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \begin{cases} 1 - 2\sin^2\theta \\ 2\cos^2\theta - 1 \end{cases}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

(And let's not forget our friends, "The Trig Equivalencies" such as the Cofunction Identities!)

A General Rule of Thumb

Convert everything to sine and cosine

Example 6.4.1

Prove $1 + \tan^2(x) = \sec^2(x)$

L.S. $1 + \tan^2(x)$

$$= \frac{1}{1}\cdot\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \quad 1$$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x) \quad \therefore LS = RS$$

Example 6.4.2

Prove $\sin(x+y)\cdot\sin(x-y)=\cos^2(y)-\cos^2(x)$

L.S. $(\sin(x+y))\cdot(\sin(x-y))$

$= (\sin x\cos y + \sin y\cos x)(\sin x\cos y - \sin y\cos x)$ Difference of Squares.

$= \boxed{\sin^2 x}\cos^2 y - \boxed{\sin^2 y}\cos^2 x$

$= (1-\cos^2 x)\cos^2 y \boxed{-} (1-\cos^2 y)\cos^2 x$

$= \cos^2 y - \cancel{\cos^2 x\cos^2 y} - \cos^2 x + \cancel{\cos^2 x\cos^2 y}$

$= \cos^2 y - \cos^2 x \qquad \therefore L.S. = R.S$

Example 6.4.3

Prove $\sin(\theta)\cdot\tan(\theta)=\sec(\theta)-\cos(\theta)$

R.S. $\sec\theta - \cos\theta$

$= \dfrac{1}{\cos\theta} - \dfrac{\cos\theta\,(\cos\theta)}{1\;\cos\theta}$

$= \dfrac{1-\cos^2\theta}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta}$

$= \boxed{\dfrac{\sin\theta\sin\theta}{\cancel{\cos\theta}}}$

$\dfrac{\sin\theta}{1}\cdot\dfrac{\sin\theta}{\cos\theta}$

$= \sin\theta\tan\theta$ $\square$

$\therefore L.S. = R.S$

Example 6.4.4

Prove $\tan(x)\cdot\tan(y) = \dfrac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}$

R.S. $\dfrac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}$

$$= \frac{\tan(x)+\tan(y)}{\frac{1}{\tan(x)}\frac{\tan(y)}{\tan(y)} + \frac{1}{\tan(y)}\frac{\tan(x)}{\tan(x)}}$$

$$= \frac{\tan(x)+\tan(y)}{\frac{\tan(y)+\tan(x)}{\tan(x)\tan(y)}} \div$$

$$= \frac{\cancel{\tan(x)+\tan(y)}}{1} \times \frac{\tan(x)\tan(y)}{\cancel{\tan(y)+\tan(x)}}$$

$$= \tan(x)\tan(y) \quad \square \qquad \therefore L.S = R.S.$$

Example 6.4.5

From your text: Pg. 417 #9a

Prove $\dfrac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin(\theta)\cos(\theta)}=1-\tan(\theta)$

a^2-b^2

$=(a+b)(a-b)$

L.S.

$$\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin\theta\cos\theta}\Bigg\}\text{ factor}$$

$$=\frac{\cancel{(\cos\theta+\sin\theta)}(\cos\theta-\sin\theta)}{\cos\theta\cancel{(\cos\theta+\sin\theta)}}$$

$$=\frac{\cos\theta-\sin\theta}{\cos\theta}$$

$$=\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=1-\tan\theta \quad \square$$

$\therefore$ L.S. = R.S.

Success Criteria:

- I can prove trigonometric identities using a variety of strategies
- I can recognize the proper form to proving trigonometric identities

6.5 Linear Trigonometric Equations

Learning Goal: We are learning to solve linear trigonometric equations.

By this time, asking you to solve a "linear equation" is almost an insult to your intelligence. BUT it is never an insult to ask you to solve problems with math. Instead it is a special treat to be able to spend time thinking mathematically. And so, **you're very welcome**.

e.g. Solve the linear equation

$$3x - 4 = 9$$

$+4 \quad +4$

$$\frac{3x}{3} = \frac{13}{3}$$

$$x = \frac{13}{3}$$

You undo the operations acting on "x"!

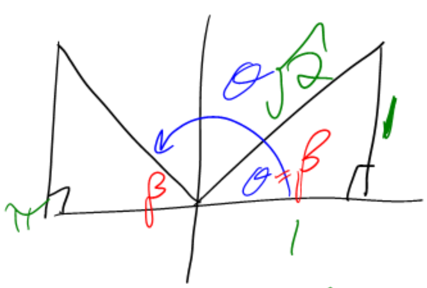
Example 6.5.1

From your text: Pg. 427 #6

For $\theta \in [0, 2\pi]$, solve the linear trigonometric equation

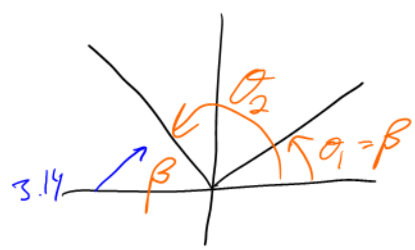
2 answers.

a) $\sin(\theta) = \frac{1}{\sqrt{2}}$ exactly, and using a calculator.



$$\theta_1 = \frac{\pi}{4}$$

$$\theta_2 = \frac{3\pi}{4}$$

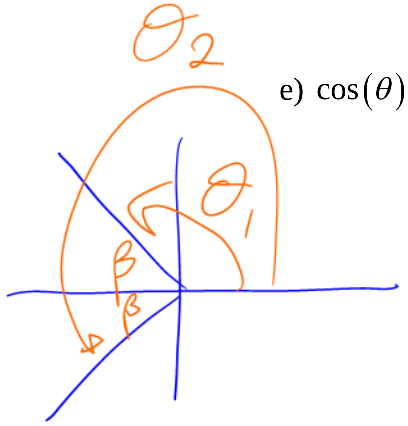


$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta_1 = 0.79 \text{ radians}$$

$$\theta_2 = 3.14 - 0.79$$
$$= 2.35 \text{ rad}$$



e) $\cos(\theta) = -\frac{1}{\sqrt{2}}$ exactly and using a calculator.

Example 6.5.2

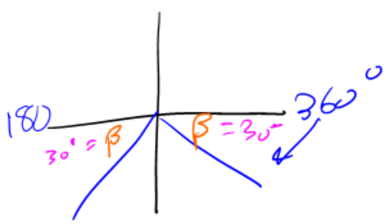
From your text: Pg. 427 #7

Using a calculator, determine solutions for $0^\circ \le \theta \le 360^\circ$

a) $2\sin(\theta) = -1$

Note: Our Domain is in Degrees!!

$\sin\theta = -\frac{1}{2}$



$\beta = \sin^{-1}\left(\frac{1}{2}\right)$

$\beta = 30^\circ$

$\theta_1 = 180 + 30$

$\theta_1 = 210^\circ$

$\theta_2 = 360 - 30$

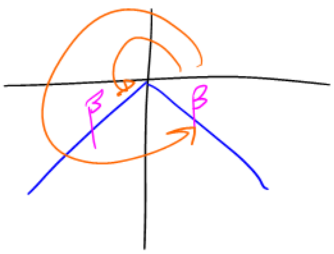
$\theta_2 = 330^\circ$

d) $-3\sin(\theta)-1=1$ (correct to one decimal place)

$-3\sin\theta = 2$

$\overline{-3} \quad \overline{-3}$

$\sin\theta = -\frac{2}{3}$



$\sin\beta = \frac{2}{3}$

$\beta = \sin^{-1}\left(\frac{2}{3}\right)$

$\beta = 41.8°$

$\theta_1 = 180 + 41.8$
$= 221.8°$

$\theta_2 = 360 - 41.8$
$= 318.2°$

Example 6.5.3

From your text: Pg. 427 #8

Determine solutions to the equations for $0 \le x \le 2\pi$.

a) $3\sin(x) = \sin(x) + 1$

$-\sin(x) \quad -\sin(x)$

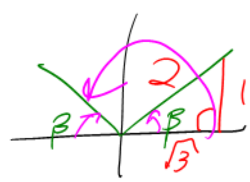
$3x^2 = x^2 + 1$

$2\sin(x) = 1$

$\sin(x) = \frac{1}{2}$

$x = \frac{\pi}{6}$

$x = \frac{5\pi}{6}$



Example 6.5.4

From your text: Pg. 427 #9f

Solve for $x \in [0, 2\pi]$

$$\overset{-8}{8} + 4\cot(x) = \overset{-8}{10}$$

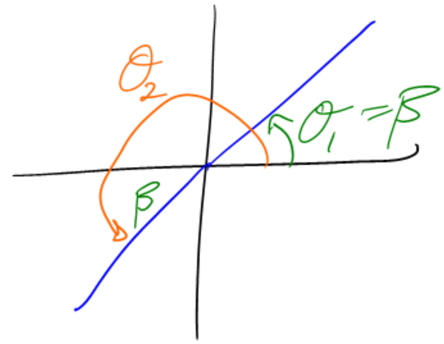
$$\frac{4\cot(x)}{4} = \frac{2}{4}$$

$$\cot(x) = \frac{1}{2}$$

$$\frac{1}{\tan(x)} = \frac{1}{2}$$

FLIP!!

$$\tan(x) = \frac{2}{1}$$



$$x = \tan^{-1}(2)$$

$$x = 1.11 \quad \text{and} \quad x = 3.14 + 1.11$$
$$= 4.25$$

Success Criteria:

- I can solve a linear trigonometric equation using: special triangles, a calculator, a sketch of the graph, and/or the CAST rule
- I can recognize that because of their periodic nature, there are infinite solutions. We normally want solutions within a specified interval.

6.6 Quadratic Trigonometric Equations

Learning Goal: We are learning to solve quadratic trigonometric equations.

Before moving on to Quadratic Trigonometric Equations, we need to consider a mind stretching problem, because it's good stretch from time to time *(in Baseball parlance, this would be the Lesson 6 Stretch)*.

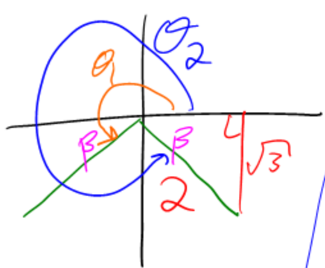
Don't be afraid of the **3**! (though it does give one some concern…)

Example 6.6.1

Solve $\sin(3x) = -\frac{\sqrt{3}}{2}$ exactly on $x \in [0, 2\pi]$

$f(x) = \sin(3x)$

Period $= \frac{2\pi}{3}$



$\beta = \frac{\pi}{3}$

$\theta_1 = \pi + \frac{\pi}{3}$ | $\theta_2 = 2\pi - \frac{\pi}{3}$

$\theta_1 = \frac{4\pi}{3}$ | $\theta_2 = \frac{5\pi}{3}$

$3x = \frac{4\pi}{3}$ | $3x = \frac{5\pi}{3}$

$x_1 = \frac{4\pi}{9}$ | $x_2 = \frac{5\pi}{9}$

These are in $\left[0, \frac{2\pi}{3}\right]$

Add the period $\frac{2\pi}{3} \Rightarrow \frac{6\pi}{9}$

$x_3 = \frac{10\pi}{9}$ | $x_4 = \frac{11\pi}{9}$ $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

$x_5 = \frac{16\pi}{9}$ | $x_6 = \frac{17\pi}{9}$ $\left[\frac{4\pi}{3}, \frac{6\pi}{3}\right]$ 2π

6 total answers

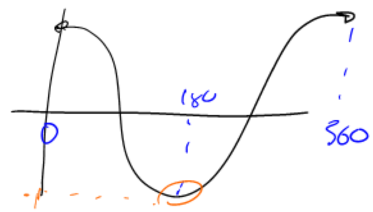
In Quadratic Trigonometric Functions the highest power on the trig 'factor' will be 2.

Example 6.6.2

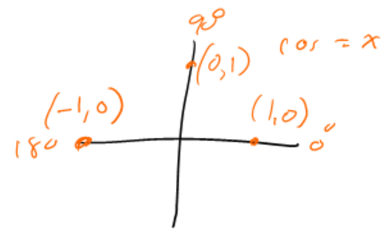
From your text: Pg. 436 #4: Solve, to the nearest degree, for $0^\circ \le \theta \le 360^\circ$

b) $\sqrt{\cos^2(\theta)} = \sqrt{1}$

$\cos\theta = \pm 1$



$\cos\theta = 1$	$\cos\theta = -1$
$\theta = 0, 360$	$\theta = 180^\circ$

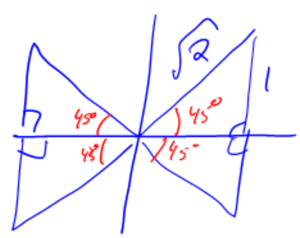


$\therefore \theta = 0^\circ, 180^\circ, 360^\circ$

f) $\dfrac{2\sin^2(\theta)}{2} = \dfrac{1}{2}$

$\sqrt{\sin^2\theta} = \pm\sqrt{\dfrac{1}{2}}$

$\sin\theta = \pm\dfrac{1}{\sqrt{2}}$



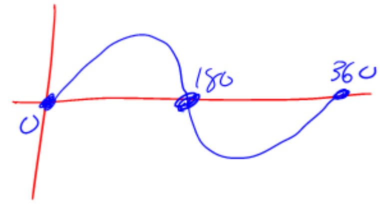
$\theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

$\dfrac{\pi}{4}$ $\quad \dfrac{3\pi}{4}$ $\quad \dfrac{5\pi}{4}$ $\quad \dfrac{7\pi}{4}$

Example 6.6.3

From your text: Pg. 436 #5: Solve for $0^\circ \le x \le 360^\circ$

b) $\sin(x)\left(\cos(x)-1\right)=0$



$\sin(x)=0$

$x=0^\circ, 180^\circ, 360^\circ$

$\cos(x)-1=0$

$\cos(x)=1$

$x=0^\circ, 360^\circ$

$\therefore x=0^\circ, 180^\circ, 360^\circ$

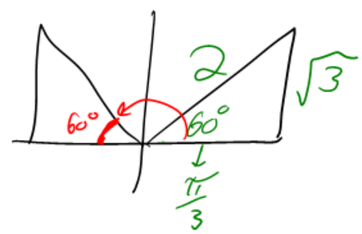
d) $\cos(x)\left(2\sin(x)-\sqrt{3}\right)=0$

$\cos(x)=0$

$x=90^\circ, 270^\circ$

$2\sin(x)-\sqrt{3}=0$

$\sin(x)=\dfrac{\sqrt{3}}{2}$



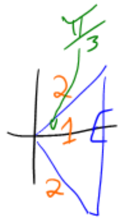
$x=60^\circ, 120^\circ$

$\therefore x=60^\circ, 90^\circ, 120^\circ, 270^\circ$

Example 6.6.4

From your text: Pg. 436 #6: Solve for $0 \le x \le 2\pi$

d) $\left(2\cos(x)-1\right)\left(2\sin(x)+\sqrt{3}\right)=0$



$2\cos(x)-1=0$

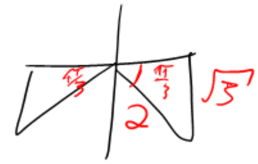
$\cos(x)=\frac{1}{2}$

$x=\frac{\pi}{3}, \frac{5\pi}{3}$

$2\sin(x)+\sqrt{3}=0$

$\sin(x)=-\frac{\sqrt{3}}{2}$

$x=\frac{4\pi}{3}, \frac{5\pi}{3}$



$\therefore x=\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Example 6.6.5

From your text: Pg. 436 #7: Solve for $0 \le \theta \le 2\pi$ to the nearest hundredth (if necessary).

a) $2\cos^2(\theta)+\cos(\theta)-1=0$ FACTOR!!

$(2\cos\theta-1)(\cos\theta+1)=0$

$2\cos\theta-1=0$

$\cos\theta=\frac{1}{2}$ $\quad \theta=\frac{\pi}{3}, \frac{5\pi}{3}$

$\cos\theta+1=0$

$\cos\theta=-1$

$\theta=\pi$

$2x^2+x-1$

$\frac{(2x-1)(2x+2)}{2}$

$(2x-1)(x+1)$

$\therefore \theta=\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

e) $3\tan^2(\theta)-2\tan(\theta)=1$

$3\tan^2\theta-2\tan\theta-1=0$

$(\tan\theta-1)(3\tan\theta+1)=0$

$3x^2-2x-1$

$\frac{(3x-3)(3x+1)}{3}$

$(x-1)(3x+1)$

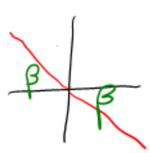
$\tan\theta-1=0$

$\tan\theta=1$

$\theta=\frac{\pi}{4}, \frac{5\pi}{4}$

$3\tan\theta+1=0$

$\tan\theta=-\frac{1}{3}$



$\beta=\tan^{-1}\left(\frac{1}{3}\right)$

$\beta=0.32$

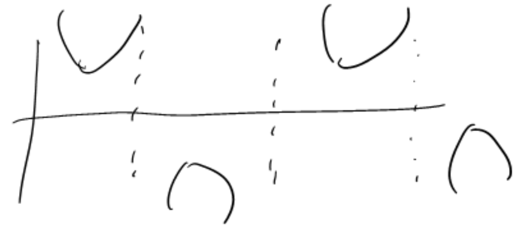
$\theta_1=3.14-0.32=2.82$

$\theta_2=6.28-0.32=5.96$

$\therefore \theta=\frac{\pi}{4}, 2.82, \frac{5\pi}{4}, 5.96$

Example 6.6.6 (decimals are between the sixes!)

From your text: Pg. 436 #8: Solve for $x \in [0, 2\pi]$



a) $\underbrace{\sec(x)\cdot\csc(x)}\underbrace{-2\csc(x)} = 0$

$\csc(x)\,(\sec(x) - 2) = 0$

$\csc(x) = 0$

no solutions

$\sec(x) - 2 = 0$

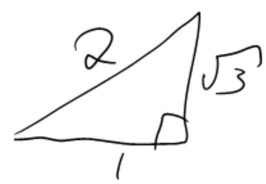
$\sec(x) = 2$

$\cos(x) = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

c) $\underbrace{2\sin(x)\cdot\sec(x)}\underbrace{-2\sqrt{3}\sin(x)} = 0$

$2\sin(x)\,(\sec(x) - \sqrt{3}) = 0$



$2\sin(x) = 0$

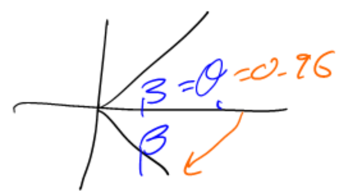
$\sin(x) = 0$

$x = 0, \pi, 2\pi$

$\sec(x) - \sqrt{3} = 0$

$\sec(x) = \sqrt{3}$

$\cos(x) = \frac{1}{\sqrt{3}}$



$6.28 - 0.96$

$x = 0.96, 5.35$

$\therefore x = 0, 0.96, \pi, 5.35, 2\pi$

Example 6.6.7

From your text: Pg. 437 #9: Solve for $x \in [0, 2\pi]$. Round to two decimal places.

a) $5\cos(2x) - \cos(x) + 3 = 0$

$$5(2\cos^2 x - 1) - \cos(x) + 3 = 0$$

$$10\cos^2 x - 5 - \cos(x) + 3 = 0$$

$$10\cos^2(x) - \cos(x) - 2 = 0$$

$$(2\cos(x) - 1)(5\cos(x) + 2) = 0$$

$$10x^2 - x - 2$$

$$\frac{(10x - 5)}{5}\frac{(10x + 4)}{2}$$

$$(2x - 1)(5x + 2)$$

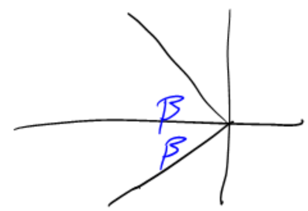
$$2\cos(x) - 1 = 0$$

$$\cos(x) = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$5\cos(x) + 2 = 0$$

$$\cos(x) = \frac{-2}{5}$$



$$\beta = \cos^{-1}\left(\frac{2}{5}\right)$$

$$\beta = 1.16$$

$$x = 3.14 - 1.16$$

$$= 1.98$$

$$x = 3.14 + 1.16$$

$$x = 4.30$$

$$x = \frac{\pi}{3}, 1.98, 4.30, \frac{5\pi}{3}$$

Success Criteria:

- I can solve quadratic trigonometric equations by factoring, or using the quadratic formula
- I can recognize when I must use other trigonometric identities to create a quadratic equation with only a single trigonometric function
- I can recognize when I need to use special triangles VS a calculator to solve quadratic trigonometric equations