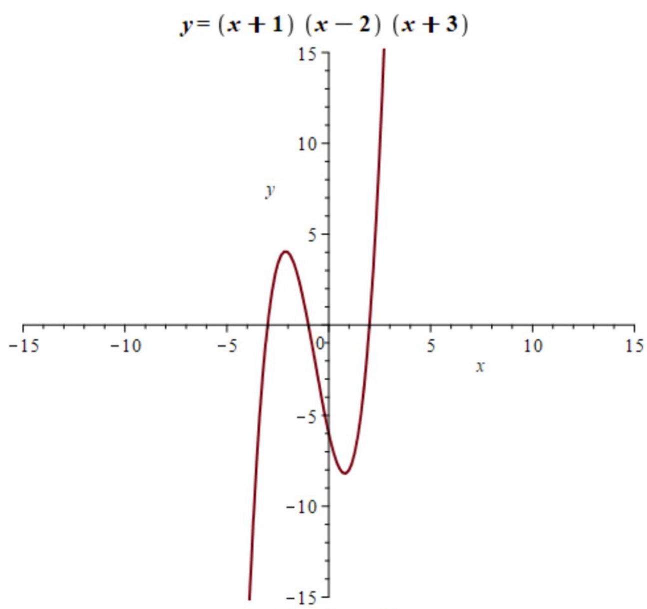
Advanced Functions

Course Notes

Chapter 2 – Polynomial Functions

***Learning Goals**: We are learning*

- *The algebraic and geometric structure of polynomial functions of degree three and higher*
- *Algebraic techniques for dividing one polynomial by another*
- *Techniques for using division to FACTOR polynomials*
- *To solve problems involving polynomial equations and inequalities*



A cubic polynomial

A∞Ω
MATH@TD

Chapter 2 – Polynomial Functions

Contents with suggested problems from the Nelson Textbook (Chapter 3)

2.1 Polynomial Functions: An Introduction

Learning Goal: We are learning to identify polynomial functions.

Definition 2.1.1

A **Polynomial Function** is of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x^1 + a_0 \boxed{x^0}_{=1}$$

where a_i, $i = 0, 1, 2, \ldots, n$, are coefficients.

The exponents are integers!

Examples of Polynomial Functions

a) $f(x) = 8x^4 - 5x^3 + 2x^2 + 3x \boxed{-5}$

$a_4 = 8 \qquad a_2 = 2 \qquad a_0 = -5$

b) $g(x) = 7x^6 - 4x^3 + 3x^2 + 2x$

$a_6 = 7 \qquad a_4 = 0 \qquad a_1 = 2 \qquad a_0 = 0$

Notes: The **TERM** $a_n x^n$ in any polynomial function (where n is the **highest power** we see) is

called the Leading term, and then we write all the following terms

in descending order.

The Leading Term has two components:

1) Leading coefficient, a_n, is positive or negative

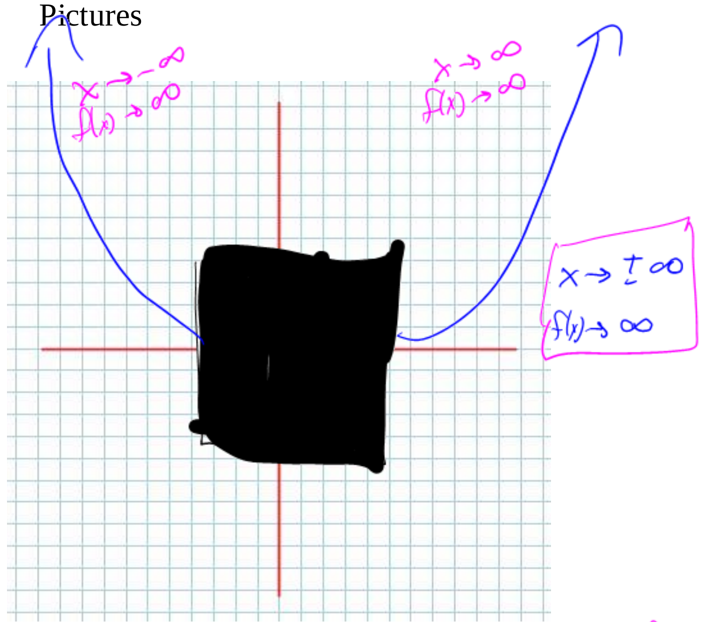
2) $n \to$ the highest power/degree, it can be odd or even

nothing to do with symmetry.

The Leading term tells us the **end behaviour** of the polynomial function.

☆ all poly fns have $\underline{4}$ possible end behaviours.

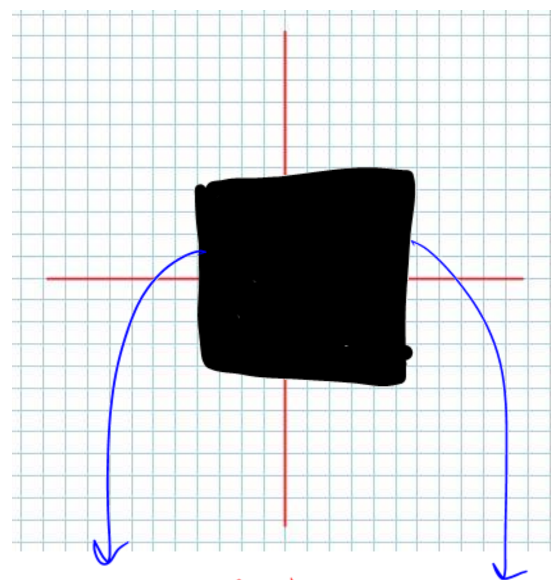
Pictures



n is even

$a_n > 0$

☆ think of a parabola!

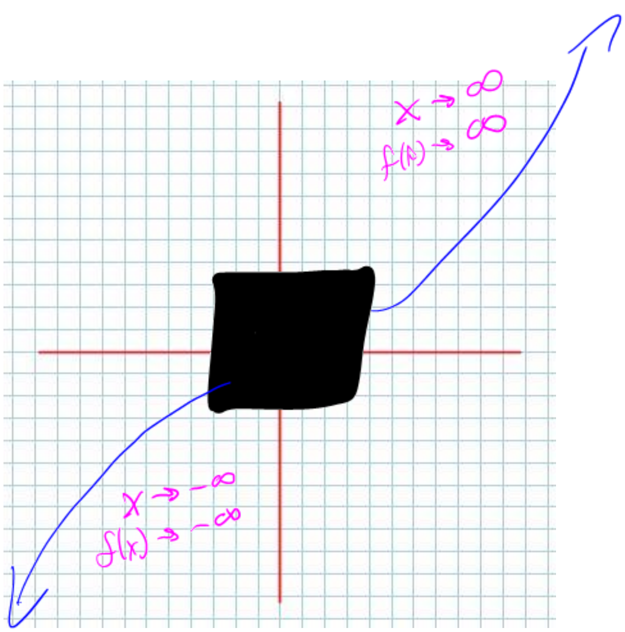


x → ±∞

f(x) → −∞

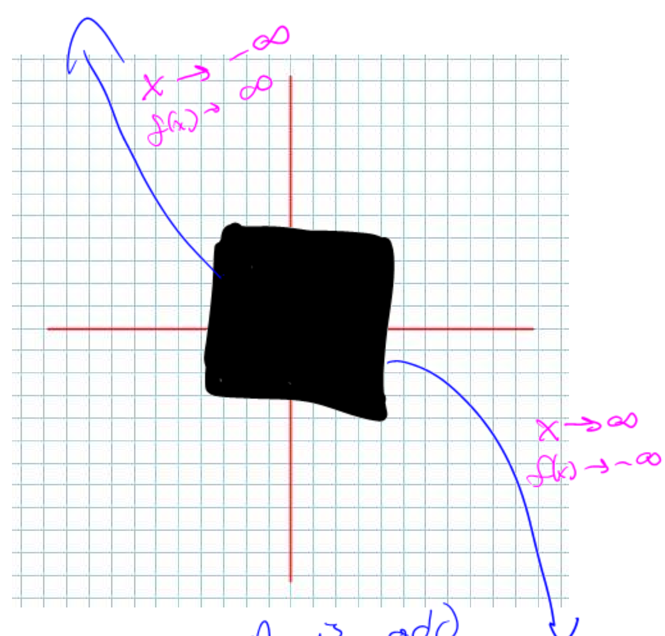
n is even

$a_n < 0$



n is odd

$a_n > 0$



n is odd

$a_n < 0$

Definition 2.1.2

The **order** of a polynomial fn is the value of the highest power, or just the degree of the leading term

ex: $g(x) = 2x^3 + 3x^2 - 8x^5 + 1$

The order of $g(x)$ is 5.

Determine the end behavior of $h(x) = 2(x-3)^2(2x+8)^3(4x+5)$

Leading term is $2(x^2)((2x)^3)(4x)$

$= 2(x^2)(8x^3)(4x)$

$= 64x^6$

$\therefore\ x \to \pm\infty,\ h(x) \to \infty$

Success Criteria:

- I can justify whether a function is polynomial or not
- I can identify the degree of a polynomial function
- I can recognize that the domain of a polynomial is the set of all real numbers
- I can recognize that the range of a polynomial function may be the set of all real numbers, or it may have an upper/lower bound
- I can identify the shape of a polynomial function given its ~~degree~~ leading term.

2.2 Characteristics (Behaviours) of Polynomial Functions

Today we open, and look inside the black box of mystery

Learning Goal: We are learning to determine the turning points and ~~end behaviours~~ number of zeros of polynomial functions.

Consider the sketch of the graph of some function, $f(x)$:

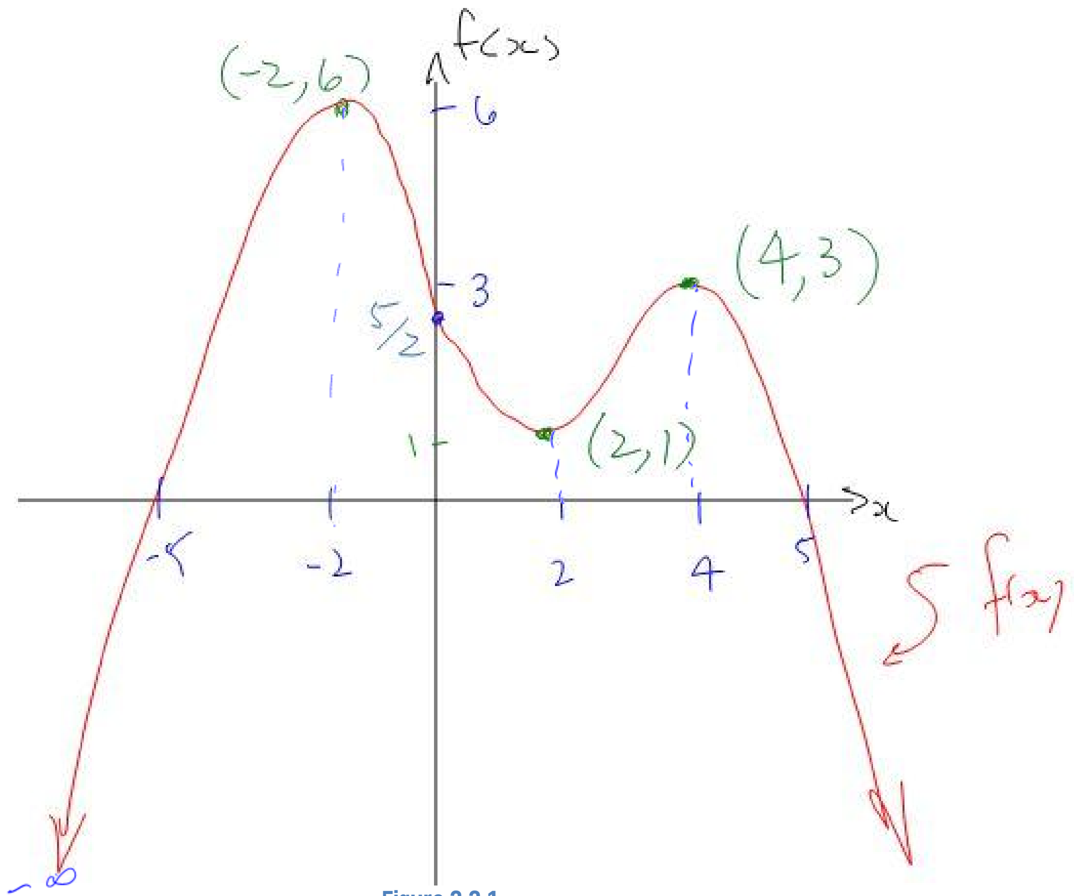


Figure 2.2.1

Observations about $f(x)$:

1) $f(x)$ is a polynomial of even order (degree). The end behaviors are the same.

2) The leading coefficient is negative

3) $f(x)$ has 3 turning points (where the functional behaviour of INCREASING/DECREASING switches from one to the other.)

4) $f(x)$ has 2 zeros, $f(-5)=0$ and $f(5)=0$

Zeros at $x=-5$ and $x=5$

5) $f(x)$ is increasing on $x \in (-\infty,-2) \cup (2,4)$

$f(x)$ is decreasing on $x \in (-2,2) \cup (4,\infty)$

6) $f(x)$ has a maximum functional -y value. of 6.

This max is called the global maximum because it is the absolute highest value.

☆ only even polynomial fn's have global max/min

7) $f(x)$ has a local min at $(2,1)$

and a local max at $(4,3)$

Consider the sketch of the graph of some function $g(x)$:

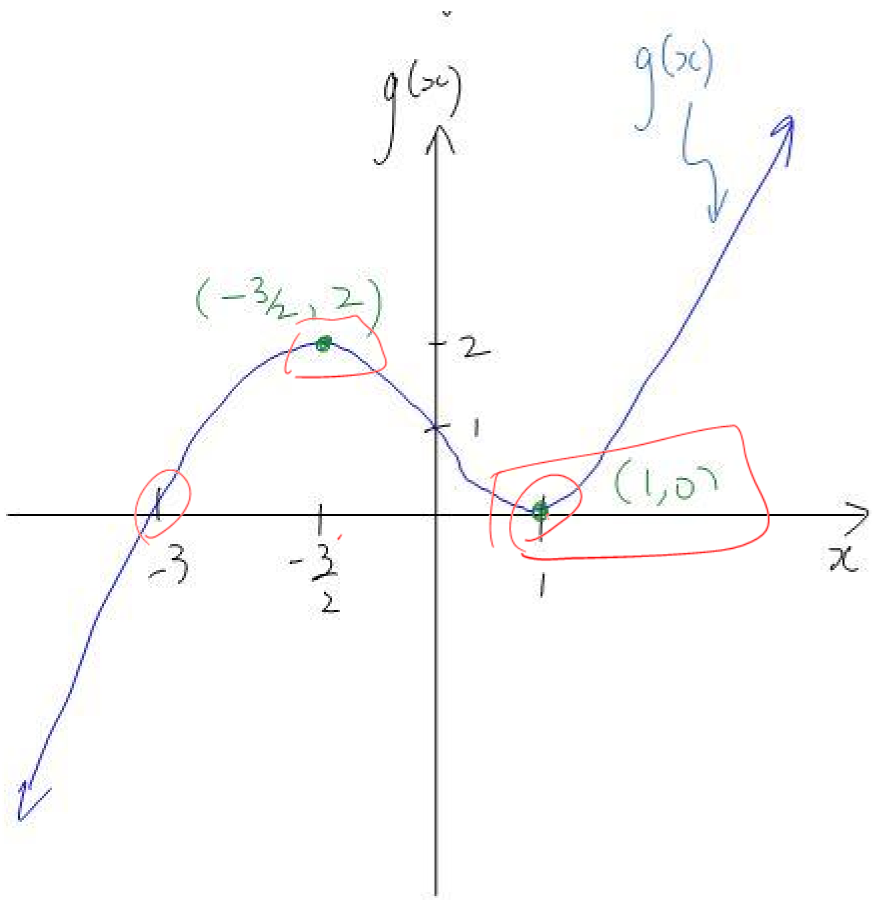


Figure 2.2.2

Observations about $g(x)$:

① g(x) is odd   End behaviors are different.

② g(x) is positive

③ Two zeros at $x=-3$ and $x=1$

④ Two turning points.

⑤ Local max at $x=-\frac{3}{2}$ of 2

⑥ Local min at $x=1$ of 0.

⑦ Increasing on $x \in (-\infty, -\frac{3}{2}) \cup (1, \infty)$

Decreasing on $x \in (-\frac{3}{2}, 1)$

General Observations about the Behaviour of Polynomial Functions

1) The Domain of all Polynomial Functions is $x \in (-\infty, \infty)$

2) The Range of ODD ORDERED Polynomial Functions is

$$f(x) \in (-\infty, \infty)$$

3) The Range of EVEN ORDERED Polynomial Functions depends on:

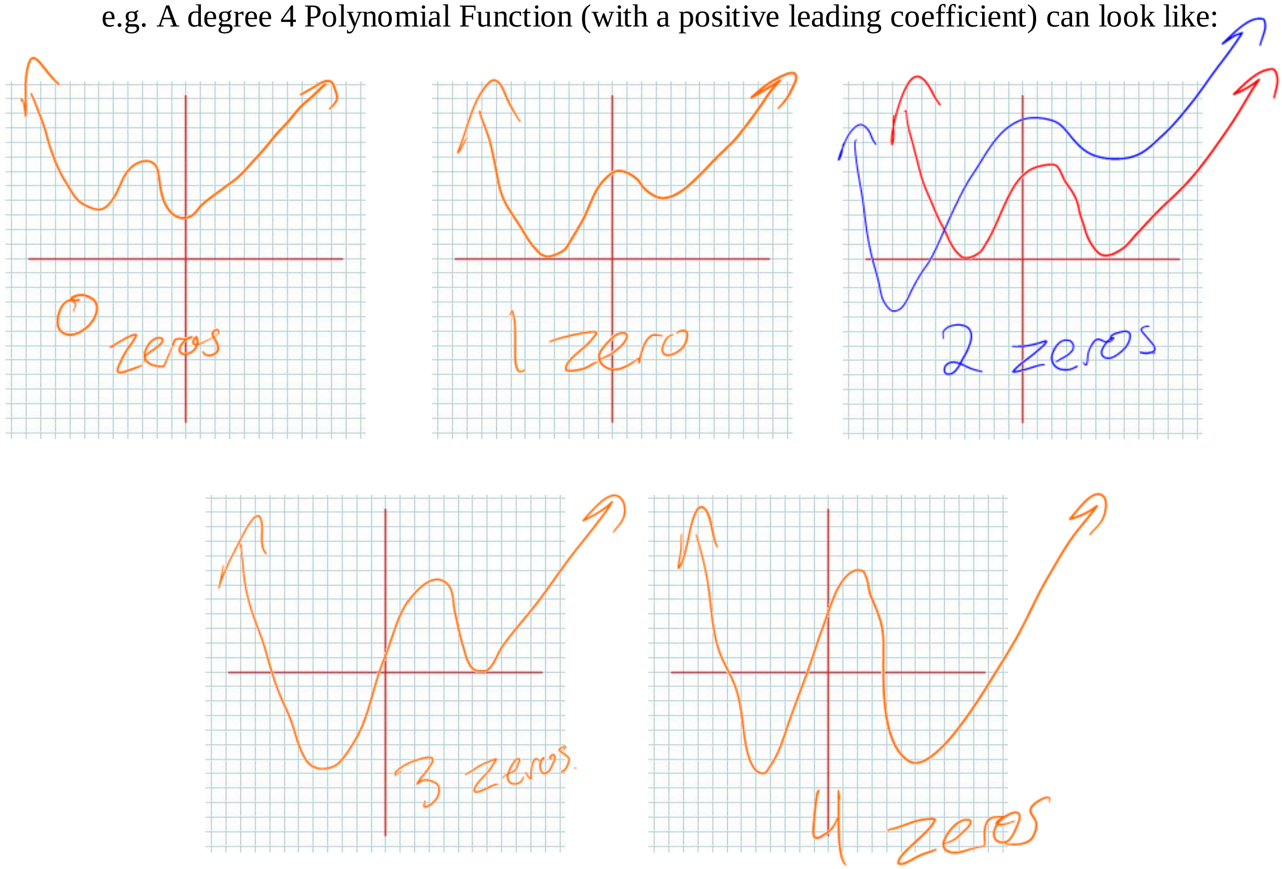
1. The sign of the leading coefficient → if positive, $\geq$
   if negative, $\leq$
2. The value of the global max/min ↳ positive ↳ negative

Even Ordered Polynomials

Zeros: A Polynomial Function, $f(x)$, with an even degree of "n" (i.e. $n = 2, 4, 6...$) can have

0 zeros, 1, 2, ..., n zeros

e.g. A degree 4 Polynomial Function (with a positive leading coefficient) can look like:

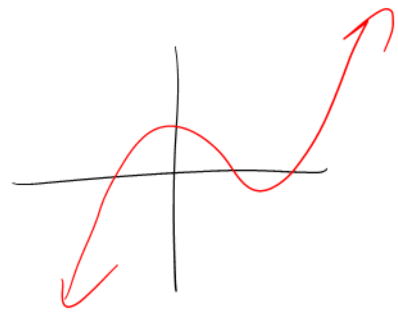


Turning Points:

The minimum number of turning points for an Even Ordered Polynomial Function is one You must turn!

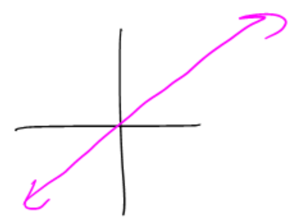
The maximum number of turning points for a Polynomial Function of (even) order *n* is $n-1$

Odd Ordered Polynomials



Zeros: min # of zeros is one.

max # of zeros is $\boxed{n}$



Turning Points:

min # of T.P. is 0.

max # of T.P. is $n-1$

Example 2.2.1 (*#2, for #1b, from Pg. 136*)

Determine the minimum and maximum number of zeros and turning points the given function may have: $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

leading term.

order/degree is 5 ∴ odd

L. coefficient is 2 ∴ positive.

Zeros: min = 1 max = 5 (n)

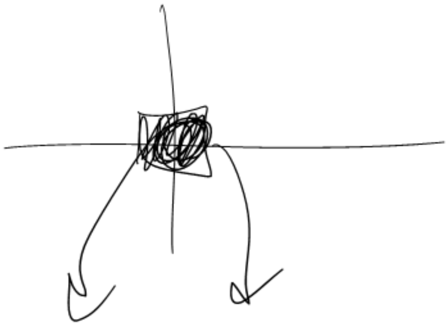
T.P. min = 0 max = 4 $(n-1)$

End behaviors: $x \to -\infty,\ f(x) \to -\infty$

$x \to \infty,\ f(x) \to \infty$

Example 2.2.2 (*#4d from Pg. 136*)

Describe the end behaviour of the polynomial function using the order and the sign on the leading coefficient for the given function: $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$



even and negative

$x \to -\infty, \ f(x) \to -\infty$

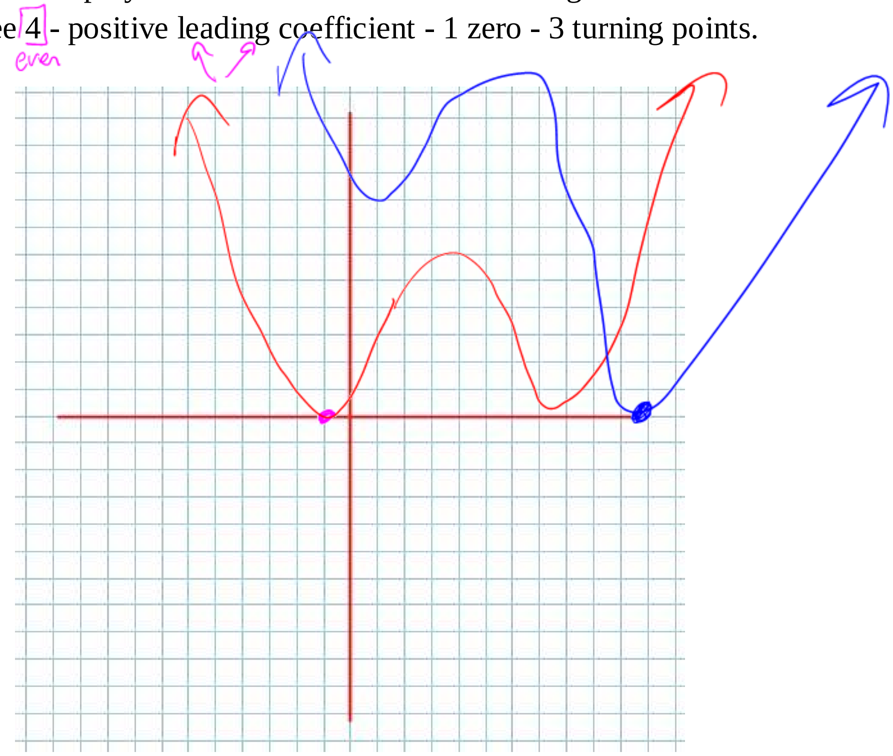
$x \to \infty, \ f(x) \to -\infty$

Example 2.2.3 (*#7c from Pg. 137*)

Sketch a graph of a polynomial function that satisfies the given set of conditions:

Degree $\boxed{4}$ - positive leading coefficient - 1 zero - 3 turning points.

even



Success Criteria:

- I can differentiate between an even and odd degree polynomial
- I can identify the number of turning points given the degree of a polynomial function
- I can identify the number of zeros given the degree of a polynomial function
- I can determine the symmetry (if present) in polynomial functions

2.3 Zeros of Polynomial Functions

(*Polynomial Functions in Factored Form*)

Today we take a deeper look inside the Box of Mystery, ***carefully examining Zeros of Polynomial Functions***

Learning Goal: We are learning to determine the equation of a polynomial function that describes a particular situation or graph and vice-versa.

We'll begin with an **Algebraic Perspective**:

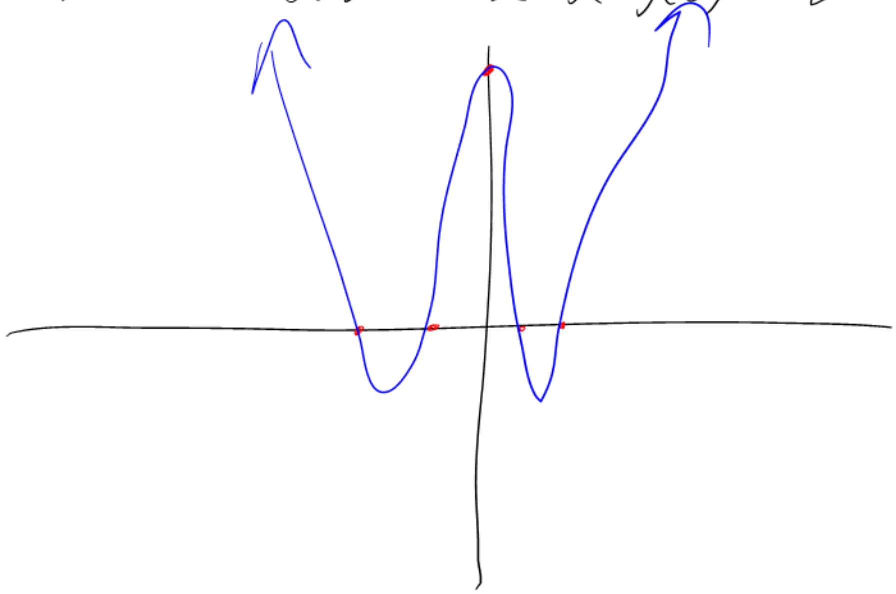
Consider the polynomial function in factored form:

$$f(x) = (2x-3)(x-1)(x+2)(x+3)$$

Observations: Leading term $(2x)(x)(x)(x) = 2x^4$

$x \to -\infty$, $f(x) \to \infty$
$x \to \infty$, $f(x) \to \infty$

1. $f(x)$ has order 4, therefore even
2. L.C. is positive
3. $f(x)$ has 4 zeros at $x = \frac{3}{2}$, 1, -2, and -3
4. y-int is: $f(0) = (-3)(-1)(2)(3) = 18$



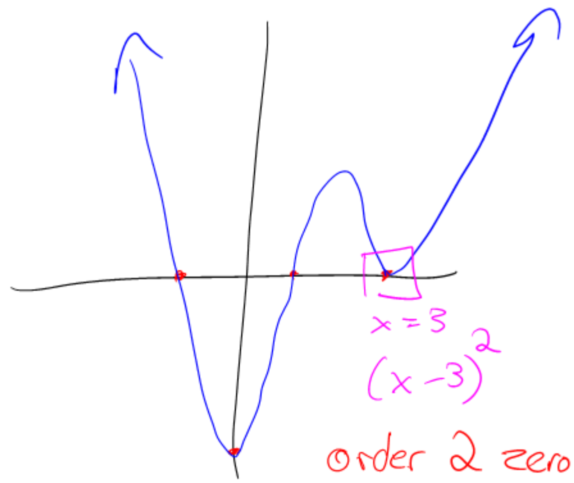
Now, consider the polynomial function $g(x)=(x-3)^2(x-1)(x+2)$

Observations: Leading term: $(x)^2(x)(x) = x^4$

1. degree is 4 ∴ even
2. L.C. is positive

$x \to -\infty,\ f(x) \to \infty$
$x \to \infty,\ f(x) \to \infty$

3. 3 zeros at $x = 3, 1,$ and -2
4. y-int: $g(0) = (-3)^2(-1)(2) = -18$



Geometric Perspective on Repeated Roots (zeros) of order **2**

Consider the quadratic in factored form: $f(x)=(x-1)^2$

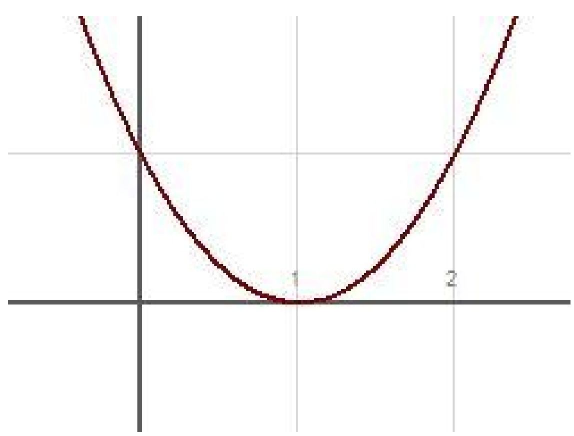
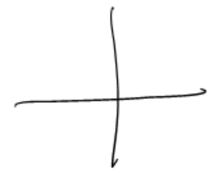


Figure 2.3.1

Consider the polynomial function in factored form: $h(t) = (t+1)^3(2t-5)$

Observations:

1. Leading Term is $(t)^3(2t) = 2t^4$



2. Degree is even and L.C. is positive

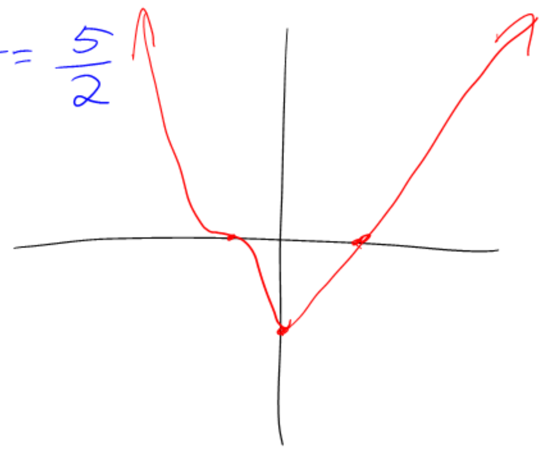
$x \to -\infty,\ f(x) \to \infty$

$x \to +\infty,\ f(x) \to \infty$

3. 2 zeros at $\boxed{t=-1}$ and $t = \frac{5}{2}$

↳ order 3 zero

4. y-int $h(0) = (1)^3(-5) = -5$



Geometric Perspective on Repeated Roots (zeros) of order **3**

Consider the function $f(x) = (x-1)^3$

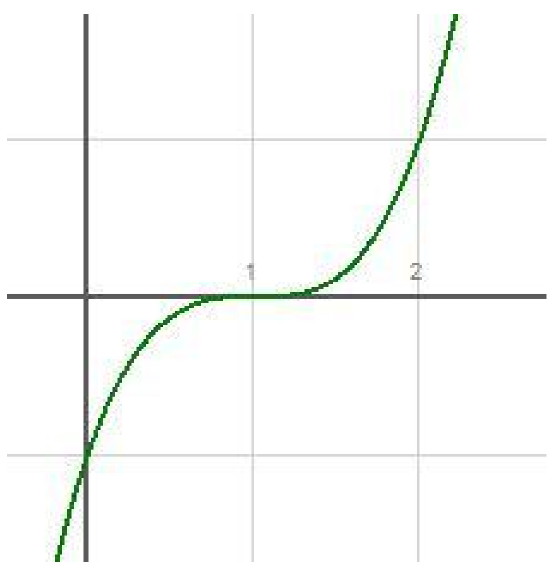
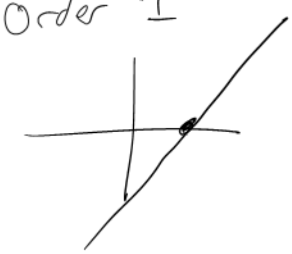
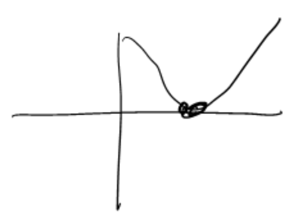


Figure 2.3.2

Order 1



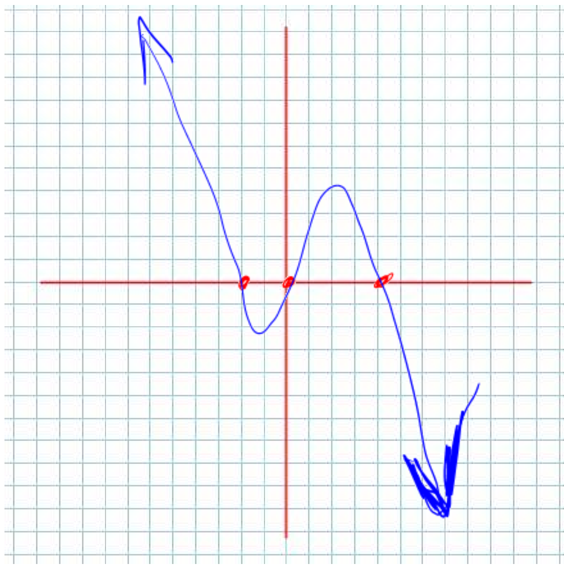
order 2



$(-2x+0)(x+1)(x-2)$

Example 2.3.1

Sketch a (**possible**) graph of $f(x) = -2x(x+1)(x-2)$



Leading Term: $-2x(x)(x) = -2x^3$

Order is 3 $\quad x \to -\infty,\ f(x) \to \infty$

L.C is negative $\quad x \to \infty,\ f(x) \to -\infty$

Zeros at $x = 0, -1, 2$

y-int: $f(0) = 0$!!

All zeros are order 1.

Families of Functions

Polynomial functions which share the same order are "broadly related" (e.g. **all** quadratics are in the "order 2 family").

Polynomial Functions which share the same order and zeros are more tightly related.

Polynomial Functions which share the same order, zeros and end behaviors are like siblings.

Example 2.3.2

The family of functions of order 4, with zeros $x = -1$, 0, 3, 5 can be expressed as:

$f(x) = a(x+1)(x+0)(x-3)(x-5)$

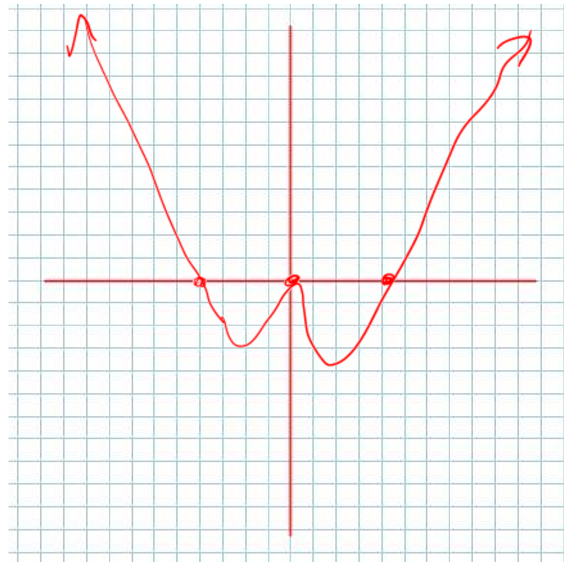
L.C. distinguishes from family members.

Example 2.3.3

Sketch a graph of $g(x) = 4x^4 - 16x^2$

$g(x) = 4x^2(x^2 - 4)$

$g(x) = 4\boxed{x^2}(x-2)(x+2)$



1. order 4, even L.C.

$\therefore$ $x \to -\infty$, $f(x) \to \infty$

$x \to \infty$, $f(x) \to \infty$

2. zeros at: $x = 0$ order 2

$x = -2$

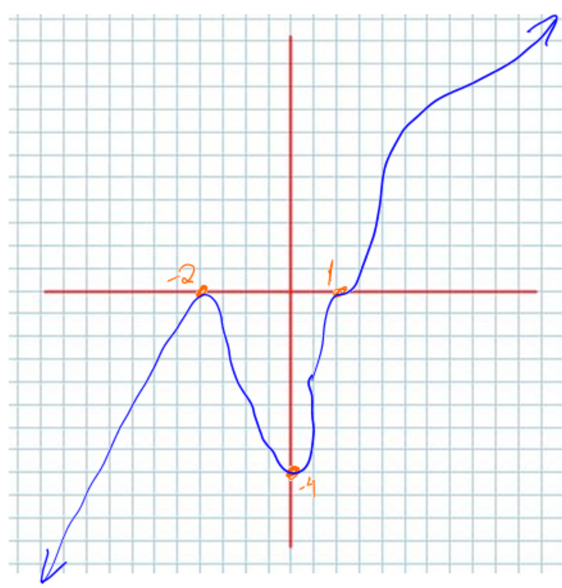
$x = 2$

3. y-int $g(0) = 0$

Example 2.3.4

Sketch a (possible) graph of $h(t) = (t-1)^3(t+2)^2$

L. Term is $(t)^3(t)^2 = t^5$



Odd and positive $\therefore$ $x \to -\infty$, $f(x) \to -\infty$

$x \to \infty$, $f(x) \to \infty$

Zeros at $x = 1$ order 3

$x = -2$ order 2

y-int $h(0) = (-1)^3(2)^2 = -4$

Example 2.3.5

Determine **the** quartic function, $f(x)$, with zeros at $x=-2, 0, 1, 3$, if $f(-1)=-2$. $= f(x)$

order 4

$$f(x) = a(x+2)(x+0)(x-1)(x-3)$$

$$-2 = a(-1+2)(-1+0)(-1-1)(-1-3)$$

$$-2 = a(1)(-1)(-2)(-4)$$

$$-2 = -8a$$

$$\frac{1}{4} = a$$

$$\therefore f(x) = \frac{1}{4}x(x+2)(x-1)(x-3)$$

Success Criteria:

- I can determine the equation of a polynomial function in factored form
- I can determine the behaviour of a zero based on the order/exponent of that factor

2.4a Dividing a Polynomial by a Polynomial

(The Hunt for Factors)

Learning Goal: We are learning to divide a polynomial by a polynomial using long division

Note: In this course we will almost always be dividing a polynomial by a ~~monomial~~ linear divisor → $x+1$ or $2x-5$

Before embarking, we should consider some "basic" terms (and notation):

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

dividend: the thing you are dividing

divisor: the thing you are dividing by

quotient: the "answer"

remainder: the stuff left over

The division Statement:

$$\boxed{\text{dividend} = (\text{quotient})(\text{divisor}) + \text{remainder}}$$

Note: The Divisor and the Quotient will both be FACTORS

IF the Remainder is zero.

Example 2.4.1

Use **LONG DIVISION** for the following division problem:

$$\frac{5x^4+3x^3-2x^2+6x-7}{x-2}$$

Please read Example 1 (Part A) on Pgs. 162 – 163 in your textbook.

$$\begin{array}{r} 5x^3+13x^2+24x+54 \\ x-2\,\overline{)\,5x^4+3x^3-2x^2+6x-7} \\ -(5x^4-10x^3) \\ \hline 13x^3-2x^2 \\ -(13x^3-26x^2) \\ \hline 24x^2+6x \\ -(24x^2-48x) \\ \hline 54x-7 \\ -(54x-108) \\ \hline 101 \end{array}$$

Force "x" to be equal to the first term you are work on.

$(x)(5x^3) = 5x^4$

$(x)(13x^2) = 13x^3$

$(x)(24x) = 24x^2$

$(x)(54) = 54x$

$\therefore\ 5x^4+3x^3-2x^2+6x-7 = (x-2)(5x^3+13x^2+24x+54) + 101$

KEY OBSERVATION:

$(x-2)$ is not a factor.

Example 2.4.2

$0x^4 \quad 0x^2$

Using Long Division, divide $\dfrac{2x^5+3x^3-4x-1}{x-1}$.

$$
\begin{array}{rl}
 & 2x^4+2x^3+5x^2+5x+1 \\
x-1 \,\big)\!\! & 2x^5+0x^4+3x^3+0x^2-4x-1 \\
 & -(2x^5-2x^4) \\
 & \quad 2x^4+3x^3 \\
 & \quad -(2x^4-2x^3) \\
 & \qquad 5x^3+0x^2 \\
 & \qquad -(5x^3-5x^2) \\
 & \qquad\quad 5x^2-4x \\
 & \qquad\quad -(5x^2-5x) \\
 & \qquad\qquad 1x-1 \\
 & \qquad\qquad -(x-1) \\
 & \qquad\qquad\quad 0
\end{array}
$$

$(x)(2x^4) = 2x^5$

$\therefore 2x^5+3x^3-4x-1 = (x-1)(2x^4+2x^3+5x^2+5x+1)$

KEY OBSERVATION: $(x-1)$ is a factor :)

Classwork: Pg. 169 #5 **(Yep, that's it for today)**

Success Criteria:

- I can use long division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after long division, there is no remainder

One more example:

$$2x^3 + 4x^2 - 3x + 24$$

$$x^2 + 2x - 4 \,\big)\, 2x^5 + 8x^4 - 3x^3 + 2x^2 - 10x + 8$$

$$-(2x^5 + 4x^4 - 8x^3)$$

$$4x^4 + 5x^3 + 2x^2$$

$$-(4x^4 + 8x^3 - 16x^2)$$

$$-3x^3 + 18x^2 - 10x$$

$$-(-3x^3 - 6x^2 + 12x)$$

$$24x^2 - 22x + 8$$

$$-(24x^2 + 48x - 96)$$

$$-70x + 104$$

$$\therefore \boxed{} = (x^2 + 2x - 4)(2x^3 + 4x^2 - 3x + 24) + (-70x + 104)$$

2.4b Dividing a Polynomial by a Polynomial

(The Hunt for Factors – Part 2)

Learning Goal: We are learning to divide a polynomial by a polynomial using synthetic division

Here we will examine an alternative form of polynomial division called **Synthetic Division**. Don't be fooled! This is not "fake division". You're thinking with the wrong meaning for "synthetic". (Do a search online and see if you can come up with the meaning I am taking!)

In Synthetic Division we concern ourselves with the coefficients of the ~~divisor~~ dividend and the zero of the ~~dividend~~ divisor.

Synthetic Division uses only numbers, no variables.

There are three operations:
1. bring down
2. times
3. add

Note: Synthetic division uses only linear divisors.

ex: $2x+3$, $x-8$, $\cancel{x^2-4}$

The Set-up

all powers must be present.

Zero of divisor	Coefficients of the dividend	
	blank	numbers arising from "times/add"
	coefficients of the quotient	remainder

Example 2.4.3

Divide using synthetic division:

$$(4x^3-5x^2+2x-1)\div(x-\boxed{2})$$

① Bring Down

② Times

③ Add

$$\begin{array}{r|rrrr}
2 & 4 & -5 & 2 & -1 \\
+ & & 8 & 6 & 16 \\
\hline
 & 4 & 3 & 8 & 15 \\
 & x^2 & x^1 & x^0 & \text{remainder}
\end{array}$$

$$\therefore\ 4x^3-5x^2+2x-1 = (x-2)(4x^2+3x+8)+15$$

Example 2.4.4

Divide using synthetic division:

$$\frac{4x^4+3x^2-2x+1}{x+1} \quad x=-1$$

$$\begin{array}{r|rrrrr}
-1 & 4 & 0 & 3 & -2 & 1 \\
+ & \downarrow & -4 & 4 & -7 & 9 \\
\hline
 & 4 & -4 & 7 & -9 & \boxed{10}\ \text{remainder}
\end{array}$$

$$\therefore\ 4x^4+3x^2-2x+1 = (x+1)(4x^3-4x^2+7x-9)+10$$

Example 2.4.5

Divide using your choice of method *(and you choose synthetic division…amen)*

$$(2x^3-9x^2+x+12)\div(2x-3)$$

$x=\frac{3}{2}$

$$\begin{array}{c|cccc} \frac{3}{2} & 2 & -9 & 1 & 12 \\ & \downarrow & 3 & -9 & -12 \\ \hline & 2 & -6 & -8 & 0 \\ & \downarrow & \downarrow & \downarrow & \\ & 1 & -3 & -4 & \end{array}$$

when you have a fraction, divide by the denominator

no remainder!!

$$2x^3-9x^2+x+12=\underbrace{(2x-3)}_{\text{Factor!!}}(x^2-3x-4)$$

$$=(2x-3)(x-4)(x+1)$$

Example 2.4.6

Is $3x-1$ a factor of the function $f(x)=6x-x^3+2+3x^4$? $\Rightarrow 3x^4-x^3+0x^2+6x+2$

$x=\frac{1}{3}$

$$\begin{array}{c|ccccc} \frac{1}{3} & 3 & -1 & 0 & 6 & 2 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & 3 & 0 & 0 & 6 & \boxed{4} \end{array}$$

$\therefore$ 3x-1 is not a factor.

Example 2.4.7 (OK…this is a lot of examples!)

Consider again (from **Example 2.4.6**) $f(x) = 3x^4 - x^3 + 6x + 2$, and calculate $f\left(\frac{1}{3}\right)$.

$$f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^4 - \left(\frac{1}{3}\right)^3 + 6\left(\frac{1}{3}\right) + 2$$

$$= \frac{\cancel{3}^{1}}{1}\left(\frac{1}{\cancel{81}_{27}}\right) - \frac{1}{27} + 2 + 2$$

$$= 4$$

WAIT!!! This is the same remainder when dividing by $3x-1$!

Example 2.4.8

Consider **Example 2.4.5**. Let $g(x) = 2x^3 - 9x^2 + x + 12$, and calculate $g\left(\frac{3}{2}\right)$.

$$g\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$$

$$= \frac{\cancel{2}^{1}}{1}\left(\frac{27}{\cancel{8}_{4}}\right) - 9\left(\frac{9}{4}\right) + \frac{3}{2} + \frac{12}{1}$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{6}{4} + \frac{48}{4} = \frac{0}{4} = 0!$$

The Remainder Theorem

Given a polynomial function, $f(x)$, divided by a linear binomial, $x-k$, then the remainder of the division is the value $f(k)$.

Proof of the Remainder Theorem

Consider $f(x) \div (x-k)$

$$\therefore f(x) = \underbrace{(x-k)}_{\text{divisor}}\underbrace{(q(x))}_{\text{quotient}} + \underbrace{r}_{\text{remainder}}$$

$$f(k) = \cancel{\underbrace{(k-k)}_{0}(q(k))} + r$$

$$f(k) = r \quad \square$$

Example 2.4.9

Determine the remainder of $\dfrac{5x^4 - 3x^3 - 50}{x-2}$. $= f(x)$

WAIT!!!! We MUST have a FUNCTION

$$f(2) = 5(2)^4 - 3(2)^3 - 50$$

$$= 80 - 24 - 50$$

$$= 6$$

$\therefore$ the remainder is 6.

Success Criteria:

- I can appreciate that synthetic division is "da bomb"
- I can use synthetic division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after synthetic division, there is no remainder (The Remainder Theorem)

2.5 The Factor Theorem

(Factors have been FOUND)

Learning Goal: We are learning the connections between a polynomial function and its remainder when divided by a binomial

The Factor Theorem

Given a polynomial function, $f(x)$, then $x-\boxed{a}$ is a factor of $f(x)$ if and only if $f(a)=0$.

Example 2.5.1

Use the Theorem to factor $x^3+2x^2-5x\boxed{-6}$.

WAIT!!!! We need a FUNCTION

$f(x) = x^3+2x^2-5x-6$

Test the possible factors of -6:

$\pm 1, \pm 2, \pm 3, \pm 6$

Test $x=1$ or $(x-1)$

$f(1) = 1^3+2(1)^2-5(1)-6 = -8$

Test $x=-1$ or $(x+1)$

$f(-1) = (-1)^3+2(-1)^2-5(-1)-6$

$= -1+2+5-6=0$

$\therefore (x+1)$ is a factor!

Now divide

$f(x) = (x\ a)(x\ b)(x\ c)$

$(a)(b)(c) = -6$

$\therefore$ the factors must divide -6.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \\ & x^2 & x & x^0 & \end{array}$$

$\therefore x^3+2x^2-5x-6 = (x+1)(x^2+x-6)$

$= (x+1)(x-2)(x+3)$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Example 2.5.2

Factor **fully** $x^4 - x^3 - 16x^2 + 4x + 48$

Raven says try $x = 2$, $(x-2)$

$f(2) = (2)^4 - (2)^3 - 16(2)^2 + 4(2) + 48$

$= 16 - \cancel{8} - 64 + \cancel{8} + 48$

$= 0!!$

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -16 & 4 & 48 \\ & & 2 & 2 & -28 & -48 \\ \hline & 1 & 1 & -14 & -24 & 0 \end{array}$$

$g(x) = x^3 + x^2 - 14x - 24$

Jessica says try $x = 3$ $(x-3)$

$g(3) = (3)^3 + (3)^2 - 14(3) - 24$

$= 27 + 9 - 42 - 24 \neq 0$:(

Katie says try $x = 4$ $(x-4)$

$g(4) = 4^3 + 4^2 - 14(4) - 24$

$= 64 + 16 - 56 - 24$

$= 80 - 80 = 0!!$:)

$$\begin{array}{r|rrrr} 4 & 1 & 1 & -14 & -24 \\ & & 4 & 20 & 24 \\ \hline & 1 & 5 & +6 & 0 \end{array}$$

$\therefore x^4 - x^3 - 16x^2 + 4x + 48 = (x-2)(x-4)(x^2 + 5x + 6)$

$= (x-2)(x-4)(x+3)(x+2)$

Example 2.5.3 (*Pg 177 #6c in your text***)**

Factor fully $x^4+8x^3+4x^2-48x$

test factors of 48.

$$f(x)=x(\underbrace{x^3+8x^2+4x\boxed{-48}}_{g(x)})$$

Try $x=2$ $(x-2)$

$$g(2)=2^3+8(2)^2+4(2)-48$$

$$=8+32+8-48=0$$

$$\begin{array}{r|rrrr} 2 & 1 & 8 & 4 & -48 \\ & & 2 & 20 & 48 \\ \hline & 1 & 10 & 24 & 0 \end{array}$$

$$\therefore x^4+8x^3+4x^2-48x=x(x-2)(x^2+10x+24)$$

$$=x(x-2)(x+6)(x+4)$$

Example 2.5.4 (*Pg 177 #10*)

When ax^3-x^2+2x+b is divided by $\boxed{x-1}$ the remainder is 10. When it is divided by $x-2$ the remainder is 51. Find *a* and *b*.

$x=1$ $f(1)=10$

This problem is very instructive.

$$f(x)=ax^3-x^2+2x+b$$

$$f(1)=a(1)^3-(1)^2+2(1)+b=10$$

$$a\boxed{-1+2}+b=10$$

$$\boxed{a+b=9}$$

$$f(2)=a(2)^3-(2)^2+2(2)+b=51$$

$$8a-4+4+b=51$$

$$8a+b=51$$

$$-(a+b=9)$$

$$7a=42$$

$$a=6$$

use elimination and subtract the equations

$$\boxed{\therefore a=6,\ b=3}$$

Success Criteria:

- I can use test values to find the factors of a polynomial function
- I can factor a polynomial of degree three or greater by using the factor theorem
- I can recognize when a polynomial function is not factorable

2.6 Factoring Sums and Differences of Cubes

patternspatternspatternspatternspatternspatternspatternspatternspatternspatterns
atternspatternspatternspatternspatternspatternspatternspatternspatternspatternsp
tternspatternspatternspatternspatternspatternspatternspatternspatternspatternspa
ternspatternspatternspatternspatternspatternspatternspatternspatternspatternspat
ernspatternspatternspatternspatternspatternspatternspatternspatternspatternspatt
rnspatternspatternspatternspatternspatternspatternspatternspatternspatternspatte

Knowing how to factor a sum or difference of cubes is a simple matter of remembering patterns.

Learning Goal: We are learning to factor a sum or difference of cubes.

Example 2.6.1 (*Recalling the pattern for factoring a Difference of Squares*)

Factor $\sqrt{4x^2} - \sqrt{25}$

$= (2x-5)(2x+5)$

Note: Sums of Squares DO NOT factor!!

e.g. Simplify $x^2 + 4$

$8x^3 \pm 27$

Differences of Cubes

SOAP

Pattern

$(cube_1 - cube_2) = (cuberoot_1 \overset{\text{same}}{-} cuberoot_2)(cuberoot_1^2 \overset{\text{opposite}}{+} cuberoot_1 \times cuberoot_2 \overset{\text{Always Positive}}{+} cuberoot_2^2)$

$8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)$

TWO POSITIVES and ONE NEGATIVE

Sums of Cubes (These DO factor!!)

Pattern

$(cube_1 + cube_2) = (cuberoot_1 + cuberoot_2)(cuberoot_1^2 - cuberoot_1 \times cuberoot_2 + cuberoot_2^2)$

$8x^3 + 27 \quad (2x + 3)(4x^2 - 6x + 9)$

Example 2.6.2

Factor $x^3 - 8$ $= (x - 2)(x^2 + 2x + 4)$

$\sqrt[3]{x^3}$ $\sqrt[3]{8}$ $(x)^2$ $(x)(2)$ $(2)^2$

Example 2.6.3

Factor $27x^3 + 125y^3$

$= (3x + 5y)(9x^2 - 15xy + 25y^2)$

Example 2.6.4

Factor $1 - 64z^3$ $= (1 - 4z)(1 + 4z + 16z^2)$

Example 2.6.5

Factor $1000x^3 + 27$ $= (10x + 3)(100x^2 - 30x + 9)$

Example 2.6.6

Factor $x^6 - 729$ $= (x^2 - 9)(x^4 + 9x^2 + 81)$

$= (x - 3)(x + 3)(x^4 + 9x^2 + 81)$

Success Criteria:

- I can use patterns to factor a sum of cubes
- I can use patterns to factor a difference of cubes