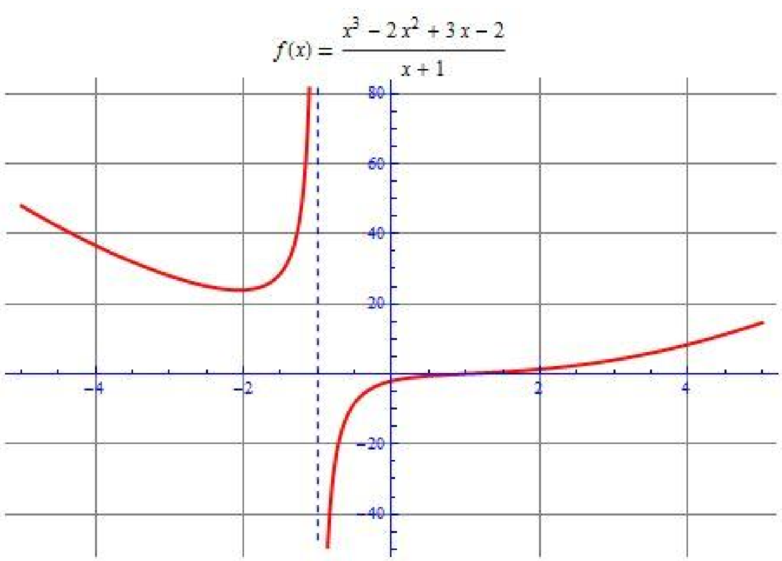
Advanced Functions

Course Notes

Unit 4 – Rational Functions, Equations and Inequalities

We are learning to

- *sketch the graphs of simple rational functions*
- *solve rational equations and inequalities with and without tech*
- *apply the techniques and concepts to solve problems involving rational models*



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Unit 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

4.1 Introduction to Rational Functions and Asymptotes

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4.2 Graphs of Rational Functions

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4.4 Solving Rational Equations

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4.5 Solving Rational Inequalities

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4.1 Rational Functions, Domain and Asymptotes

Learning Goal: We are learning to identify the asymptotes of rational functions.

Definition 4.1.1

A **Rational Function** is of the form

$f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and both $p(x)$ and $q(x)$ are polynomials.

e.g. $f(x) = \frac{3x^2 - 5x + 1}{2x - 1}$ is a rational fn

$g(x) = \frac{\sqrt{2x+5}}{3x-2}$ not a rational fn. because $\sqrt{2x+5}$ is not a polynomial

Domain

Definition 4.1.2

Given a rational function $f(x) = \frac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f = \{x \in \mathbb{R} \mid q(x) \neq 0\}$ ← the zeros of $q(x)$.

Example 4.1.1

Determine the natural domain of $f(x) = \frac{x^2 - 4}{x - 3}$.

$x \neq 3$

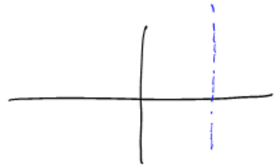
$D_f : \{x \in \mathbb{R} \mid x \neq 3\}$

$x \in (-\infty, 3) \cup (3, \infty)$

Asymptotes

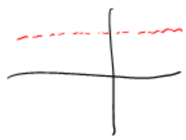
There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes



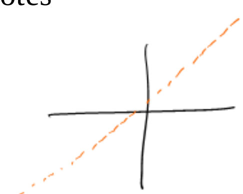
$x = \#$

2) Horizontal Asymptotes



$y = \#$

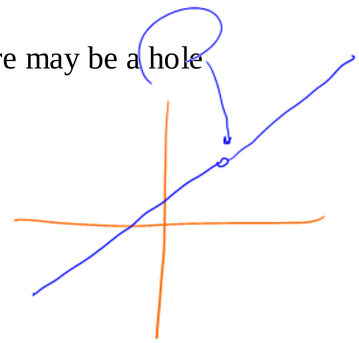
3) Oblique Asymptotes



$y = mx + b$

Vertical Asymptotes

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x) = 0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.



Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x) = \dfrac{5x}{x^2 - x - 6}$

$$f(x) = \frac{5x}{(x-3)(x+2)}$$

$x = 3$ $\quad$ $x = -2$

Since these factors stay, they are V.A.

$$x \in (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$$

b) $h(x)=\dfrac{x+3}{x^2-9}$

$$h(x)=\frac{\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$$

$x=3$ $\quad$ $\rightarrow x=-3$

$$h(x)=\frac{1}{x-3}$$

$x=3$ is a V.A.

$x=-3$ is a hole because it disappeared.

$x\in(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

c) $g(x)=\dfrac{x^2-4}{x+2}$

Factor Everything

$$g(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

$\boxed{x=-2 \text{ is a hole.}}$

$$g(x)=x-2$$

$x\in(-\infty,-2)\cup(-2,\infty)$

Horizontal Asymptotes

Here we are concerned with The end behaviors of the rational fn.

i.e. We are asking, given a rational function $f(x) = \dfrac{p(x)}{q(x)}$, how is $f(x)$ behaving as $x \to \pm\infty$.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $p(x)$ > Order of $q(x)$

   e.g. $f(x) = \dfrac{x^3 - 2}{x^2 + 1}$

   When the order of p(x) is bigger than the order of q(x), there is $\boxed{\text{no}}$ horizontal asymptote

2) Order of numerator = Order of denominator

   e.g. $f(x) = \dfrac{2x^2 - 3x + 1}{3x^2 + 4x - 5}$

When the orders are the same, the H.A. is the leading coefficients.

ex: H.A. is $y = \frac{2}{3}$

e.g. Determine the horizontal asymptote of $g(x) = \dfrac{3x - 4x^5}{5x^5 + 2x - 1}$

H.A. is $y = \frac{-4}{5}$

$$f(x) = \frac{1}{x^{20}}$$

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x) = \dfrac{x^2 - 5x + 6}{x^5 + 7}$

If $p(x)$ is smaller than $q(x)$, then the H.A. is $y = 0$

Oblique Asymptotes

These occur when the order of $p(x)$ is exactly one more than the order of $q(x)$.

e.g. $f(x) = \dfrac{x^2 - 2x + 3}{x + 1}$

With Oblique Asymptotes we are still dealing with end behaviors.

O.A. have the form $y = mx + b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: By Polynomial Division.
The O.A. is the quotient.

$$f(x) = \frac{x^2 - 2x + 3}{x + 1}$$

$\hookrightarrow x = -1$

$$\begin{array}{r|rrr} -1 & 1 & -2 & 3 \\ & \downarrow & -1 & 3 \\ \hline & \underbrace{\boxed{1_x \quad -3}}_{\text{quotient}} & & \boxed{6} \text{ remainder.} \end{array}$$

$y = 1x - 3$ is the line of the O.A.

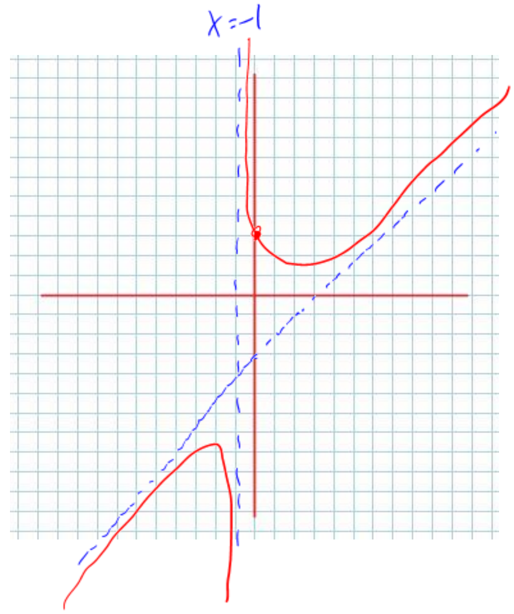
(Rough) Sketch of $f(x)=\dfrac{x^2-2x+3}{x+1}$

V.A. $x=-1$

H.A. None

O.A. $y=x-3$

y-int : $f(0)=3$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x)=\dfrac{x+2}{x^2+3x+2}$ order 1 / order 2

$$f(x)=\frac{\cancel{x+2}}{(x+1)\cancel{(x+2)}}$$

$$f(x)=\frac{1}{x+1}$$

V.A.	$x=-1$
Hole	$x=-2$
H.A.	$y=0$
O.A.	N.A.

b) $g(x)=\dfrac{\boxed{4}x^2-25}{\boxed{1}x^2-9}$ Factor All

$$g(x)=\frac{(2x+5)(2x-5)}{(x-3)(x+3)}$$

V.A.	$x=-3,\ x=3$
Hole	N.A.
H.A.	$y=\frac{4}{1}=4$
O.A.	N.A.

c) $h(x)=\dfrac{x^2+0x+0}{x+3}$

↳ $x=-3$

$$\begin{array}{r|rrr} -3 & 1 & 0 & 0 \\ & \downarrow & -3 & 9 \\ \hline & 1 & -3 & 9 \end{array}$$

∴ O.A ⇒ $y=x-3$

V.A.	$x=-3$
Hole	N.A.
H.A.	N.A.
O.A.	$y=x-3$

Example 4.1.4

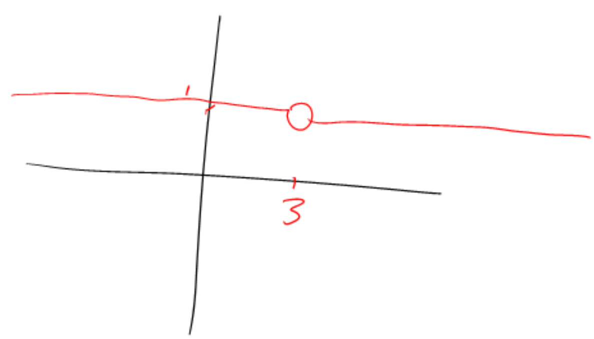
Determine an equation for a function with a vertical asymptote at $x=-3$, and a horizontal asymptote at $y=0$. → bottom bigger $\quad x+3$

$$f(x)=\frac{1}{x+3} \quad \text{or} \quad \frac{3x^2-8x+50}{(x+3)(x-5)(x-100)}$$

Example 4.1.5

Determine an equation for a function with a hole discontinuity at $x=3$.

$$f(x)=\frac{(x-3)}{(x-3)}$$



Success Criteria:

- I can identify a hole when there is a common factor between *p(x)* and *q(x)*
- I can identify a vertical asymptote as the zeros of *q(x)*
- I can identify a horizontal asymptote by studying the degrees of *p(x)* and *q(x)*
- I can identify an oblique asymptote when the degree of *p(x)* is exactly 1 greater than *q(x)*

4.2 Graphs of Rational Functions

Learning Goal: We are learning to sketch the graphs of simple rational functions.

Note: In Advanced Functions we will only consider rational functions of the form

$$f(x) = \frac{ax+b}{cx+d}$$

Rational Functions of the form $f(x) = \frac{ax+b}{cx+d}$ will have:

1) One Vertical Asymptote

$cx+d = 0$

$x = \frac{-d}{c}$, unless $c = 0$.

2) One Zero (unless $a = 0$)

x-intercept

$0 = \frac{ax+b}{\cancel{cx+d}}$ $\quad ax+b = 0 \quad x = \frac{-b}{a}$, is the x-intercept.

3) Functional Intercept $x = 0$

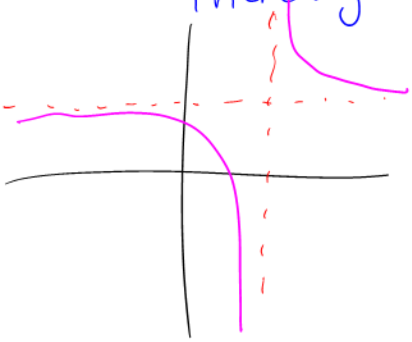
y

$f(0) = \frac{a(0)+b}{c(0)+d} = \frac{b}{d} \Rightarrow \left(0, \frac{b}{d}\right)$

4) A Horizontal Asymptote since orders are equals

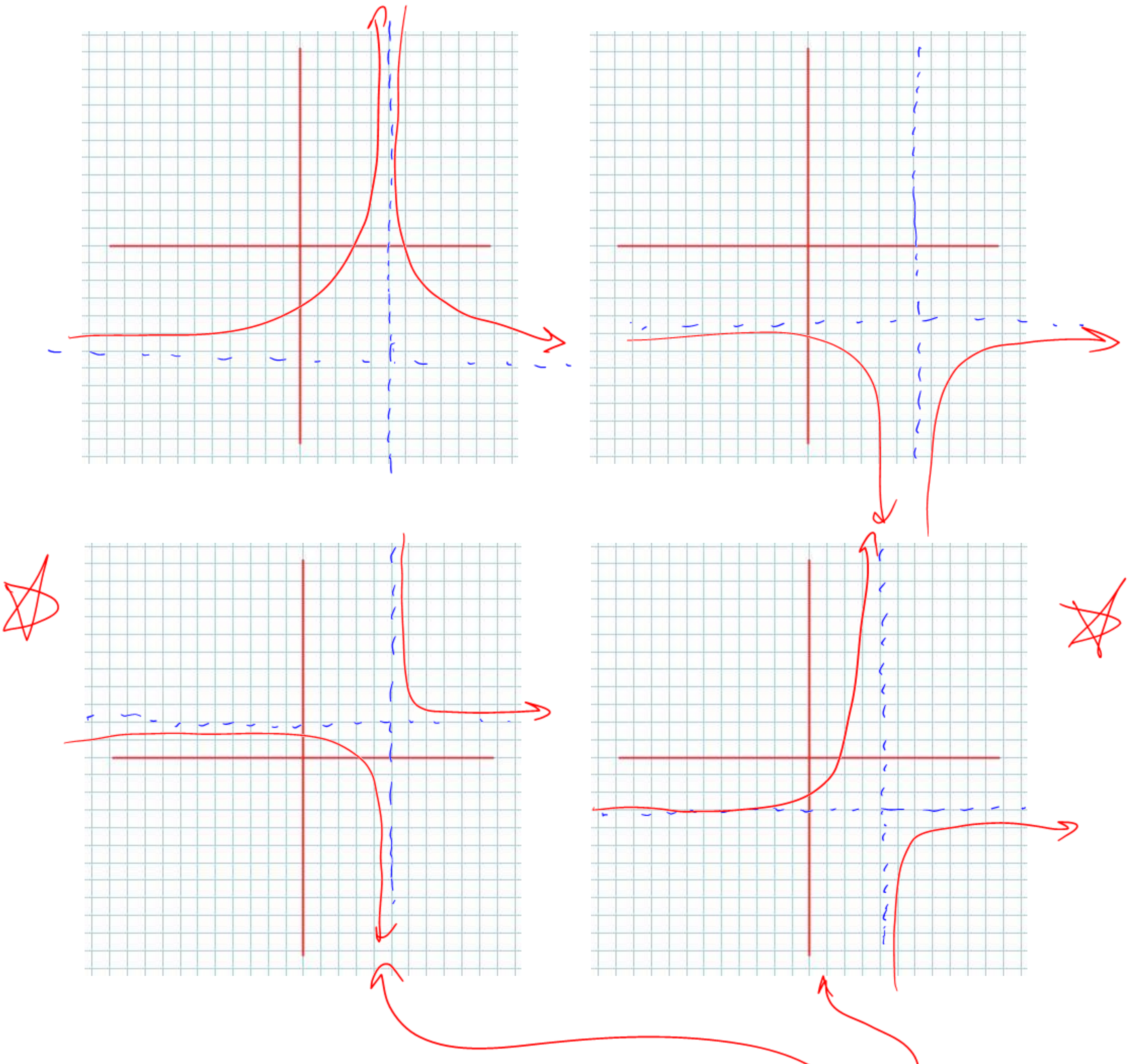
$y = \frac{a}{c}$, unless $a = 0$, then the H.A. $y = 0$

5) These functions will always be either increasing or decreasing, meaning no turning points



Functional Behaviour Near A Vertical Asymptote

There are **FOUR** possible functional behaviours near a V.A.:



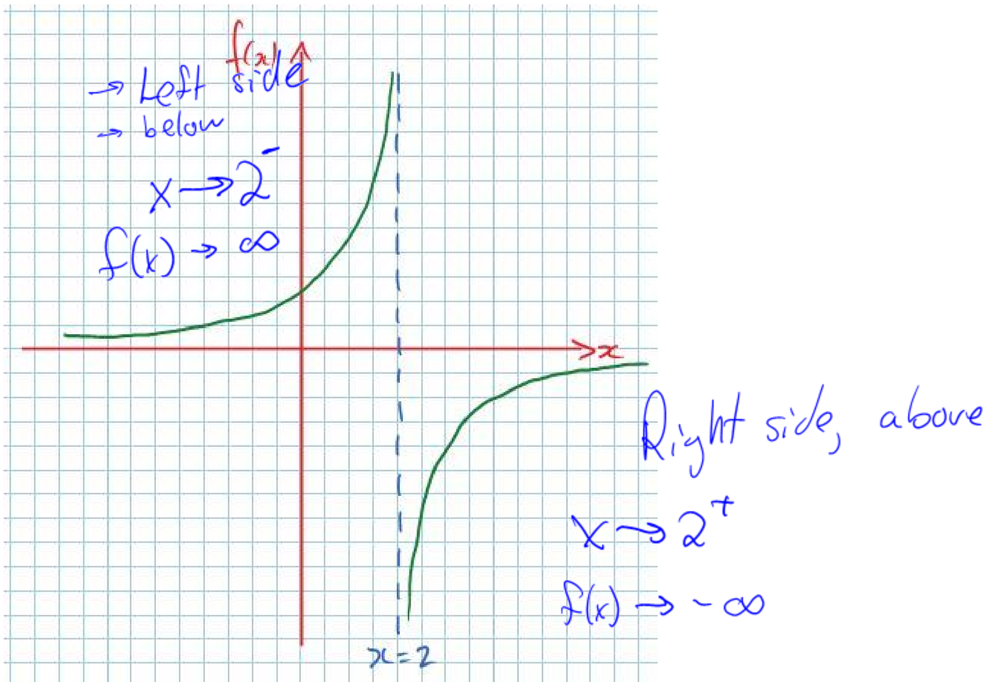
For functions of the form $f(x)=\dfrac{ax+b}{cx+d}$ we will see behaviours like

The questions is, **how do we know which**?

We need to **analyze** the function **near the V.A.**

We need to become familiar with some **Notation.**

Consider some rational function with a sketch of its graph which looks like:



Example 4.2.1

Determine the functional behaviour of $f(x)=\dfrac{2x+1}{x-3}$ near its V.A.

V.A. is $x=3$

$x \to 3^-$

$f(x) \to -\infty$

Pick an x which is less than 3 but really close to 3.

$x=2.9$, $f(2.9)=-68$

$x=2.99$, $f(2.99)=-698$

$x=2.999$, $f(2.999)=-6998$

$\therefore$ it gets smaller

$x \to 3^+$

$f(x) \to \infty$

Pick an x close to, yet bigger than 3.

$x=3.01$, $f(3.01)=702$

$x=3.001$, $f(3.001)=7002$

$\therefore$ getting bigger

We now have the tools to sketch some graphs!

Example 4.2.2

Sketch the graph of the given function. State the domain, range, intervals of increase/decrease and where the function is positive and negative.

a) $f(x) = \dfrac{2x+1}{x-1}$ $= 0$

V.A.: $x = 1$

H.A.: $y = \frac{2}{1} = 2$

x-int: $x = \frac{-1}{2}$

y-int: $y = \frac{1}{-1} = -1$

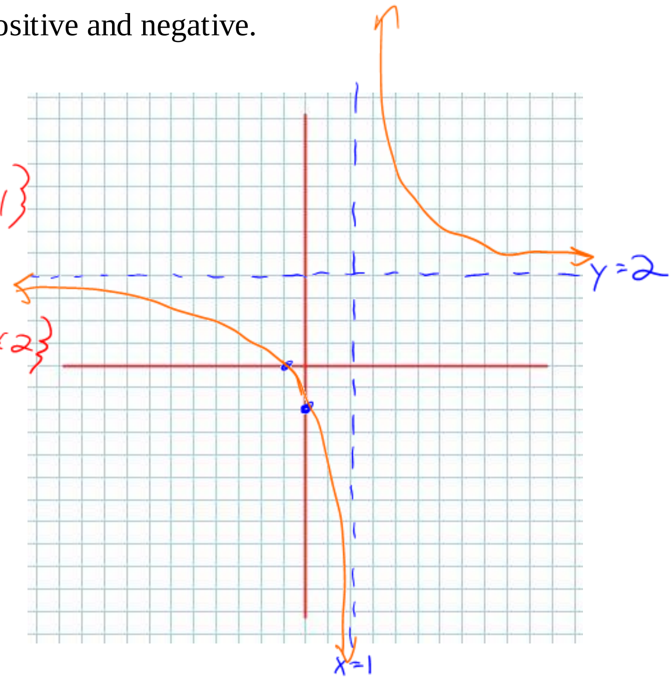
D_f: $(-\infty, 1) \cup (1, \infty)$ $\{x \in \mathbb{R} \mid x \neq 1\}$

R_f: $(-\infty, 2) \cup (2, \infty)$ $\{f(x) \in \mathbb{R} \mid f(x) \neq 2\}$

Decreasing on $(-\infty, 1) \cup (1, \infty)$

$f(x) > 0$ $(-\infty, -\frac{1}{2}) \cup (1, \infty)$

$f(x) < 0$ $(-\frac{1}{2}, 1)$



b) $g(x) = \dfrac{3x-2}{2x+5}$

V.A.: $x = \frac{-5}{2}$

H.A.: $y = \frac{3}{2}$

x-int: $x = \frac{2}{3}$

y-int: $y = \frac{-2}{5}$

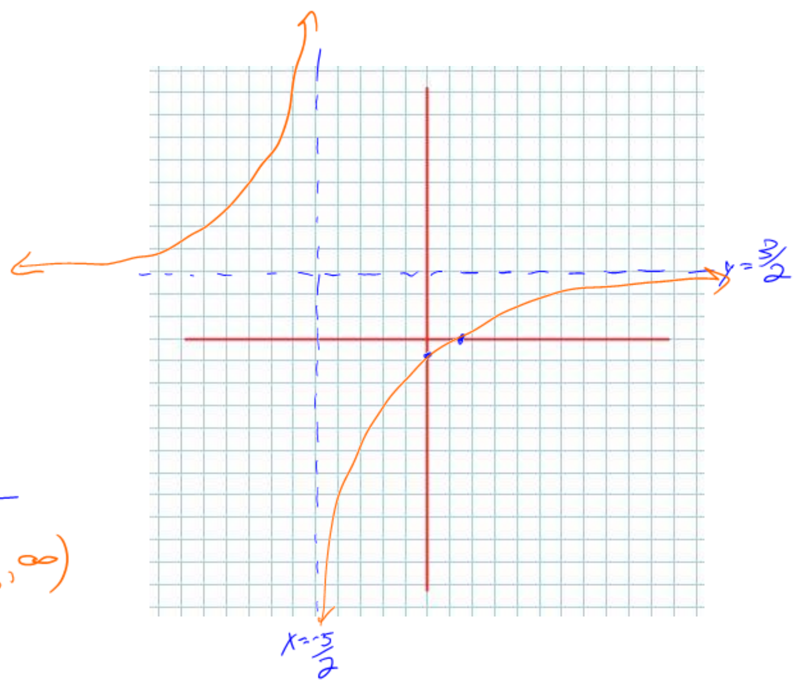
D_g: $x \in (-\infty, \frac{-5}{2}) \cup (\frac{-5}{2}, \infty)$

R_g: $g(x) \in (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

Increasing on D_g

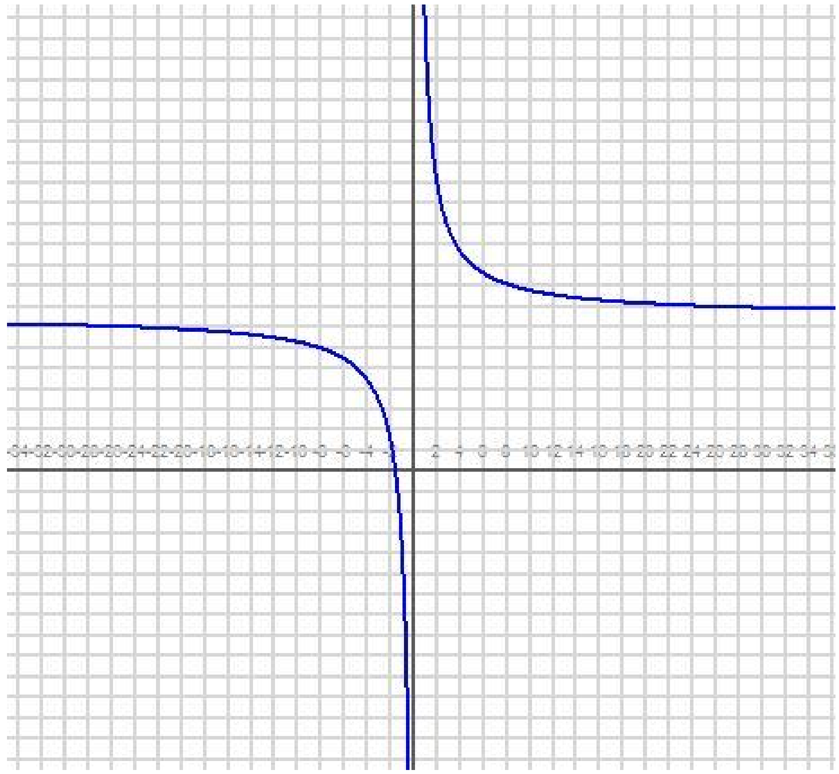
$f(x) > 0$ on $(-\infty, \frac{-5}{2}) \cup (\frac{2}{3}, \infty)$

$f(x) < 0$ on $(\frac{-5}{2}, \frac{2}{3})$



Example 4.2.3

Consider question #9 on page 274: $I(t) = \dfrac{15t + 25}{t}$



Success Criteria:

- I can identify the horizontal asymptote as $\frac{a}{c}$
- I can identify the vertical asymptote as $-\frac{d}{c}$
- I can identify the y-intercept as $\frac{b}{d}$
- I can identify the x-intercept as $-\frac{b}{a}$

4.4 Solving Rational Equations

Learning Goal: We are learning to solve rational equations. Think rationally!

Solving a Rational Equation is **VERY MUCH** like solving a Polynomial Equation. Thus, this stuff is so much fun it should be illegal. But it isn't illegal unless you break a rule of algebra. Math Safe!

KEY (*this is a major key for you music buffs*)

Multiplying by the Multiplicative Inverse of the Common Denominator is wonderful to use **WHEN YOU HAVE** something like:

$$\text{RATIONAL}_1 + \text{RATIONAL}_2 = \text{RATIONAL}_3$$

e.g.

$$\boxed{(2)(x)(x-2)}$$

$$\frac{3}{x-2} = \frac{4(x+5)}{x} + \frac{3}{2}$$

$$(2)(x)\cancel{(x-2)}\left(\frac{3}{\cancel{x-2}}\right) = (2)\cancel{(x)}(x-2)\left(\frac{4(x+5)}{\cancel{x}}\right) + \cancel{(2)}(x)(x-2)\left(\frac{3}{\cancel{2}}\right)$$

$$6x = 8(x-2)(x+5) + 3x(x-2)$$

Expand, "stuff" = 0, solve.

Make Sure To Keep **RESTRICTIONS ON X** In Mind

this means that restrictions cannot be solutions.

Example 4.4.1

cross multiply (if you remember...)

No restrictions

a) Solve $\frac{x}{5}=\frac{9}{18}$

$18x=45$

$x=\frac{45}{18}$

$x=\frac{5}{2}$

b) Solve $\left(\frac{1}{x}-\frac{5x}{3}=\frac{2}{5}\right)(5)(3)(x)$

RESTRICTIONS

$x \neq 0$

C.D.
$15x$ or $(5)(3)(x)$

$15-25x^2=6x$

$0=25x^2+6x-15$ D.N.F

use Quadratic Formula

$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x=-0.9$ and $x=0.66$

c) Solve $\left(\frac{3}{x}+\frac{4}{x+1}=\frac{2}{1}\right)(x)(x+1)$

RESTRICTIONS

$x \neq 0$ | C.D.

$x \neq -1$ | $(x)(x+1)$

$3(x+1)+4x=2x(x+1)$

$\boxed{3x+3+4x}=2x^2+2x \quad -7x \quad -3$

$0=2x^2-5x-3$

M: -6 A: -5 $\boxed{-6, +1}$

$0=\underbrace{2x^2-6x}_{2x}\underbrace{+1x-3}_{+1}$

$0=(2x+1)(x-3)$

$\therefore x=\frac{-1}{2}$

$x=3$

d) Solve $\left(\frac{10}{\cancel{x^2-2x}}+\frac{4}{x}=\frac{5}{x-2}\right)(x)(x-2)$

$x(x-2)$

RESTRICTIONS

$x \neq 0$ | C.D.

$x \neq 2$ | $x(x-2)$

$10+4(x-2)=5x$

$10+4x-8=5x \quad -4x$

$2=x$

However, $x \neq 2$, $\therefore$ there are no solutions to this equation.

e) Solve $\left(\frac{16x}{1}-\frac{5}{x+2}=\frac{15}{x-2}-\frac{60}{(x-2)(x+2)}\right)(x-2)(x+2)$

Restrictions
$x \neq -2$
$x \neq 2$

C.D.
$(x-2)(x+2)$

$$16x\underbrace{\boxed{(x-2)(x+2)}}_{(x^2-4)}-5(x-2)=15(x+2)-60$$

$$16x^3-64x-5x+10=\boxed{15x+30-60}$$

$$\frac{16x^3-84x+40=0}{4}$$

$$4x^3-21x+10=0 \longrightarrow \pm 1, \pm 2, \pm 5, \pm 10$$

Let $f(x)=4x^3-21x+10$ Try $f(2)$

$f(2)=4(2)^3-21(2)+10$

$=32-42+10=0!$

$\therefore (x-2)$ is a factor

$$\begin{array}{r|rrrr} 2 & 4 & 0 & -21 & 10 \\ & & 8 & 16 & -10 \\ \hline & 4 & 8 & -5 & 0 \end{array}$$

$=4x^2+8x-5$

M: -20
A: 8 $\quad 10, -2$

$$(x-2)(4x^2+8x-5)=0$$

$$(x-2)(\underbrace{4x^2+10x}_{2x}\underbrace{-2x-5}_{-1})=0$$

$$(x-2)(2x-1)(2x+5)=0$$

$\therefore$ ~~$x=2$~~ inadmissable

$x=\frac{1}{2}$

$x=\frac{-5}{2}$

Example 4.4.2

From your Text: Pg. 285 #10

The Turtledove Chocolate factory has two chocolate machines. Machine A takes *m* minutes to fill a case with chocolates, and machine B takes *m + 10* minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

Work or a rate problem → $\frac{\text{a job}}{\text{time}} = \frac{\text{filling a case}}{\text{minutes}}$

A: $\frac{1}{m}$ B: $\frac{1}{m+10}$ Together: $\frac{1}{15}$

$$\left(\frac{1}{m} + \frac{1}{m+10} = \frac{1}{15}\right) 15(m)(m+10)$$

Restrictions: $m \neq 0$, $m \neq -10$

C.D. $(15)(m)(m+10)$

$$15(m+10) + 15m = m(m+10)$$

$$15m + 150 + 15m = m^2 + 10m \quad -30m \quad -150$$

$$0 = m^2 - 20m - 150 \quad \text{D.N.F.}$$

$\therefore$ by the Q.F.

$$m = 25.8$$

Machine A takes 25.8 min

Machine B takes 35.8 min

Success Criteria:

- I can recognize that the zeros of a rational function are the zeros of the numerator
- I can solve rational equations by multiplying each term by the lowest common denominator, then solving the resulting polynomial equation
- I can identify inadmissible solutions based on the context of the problem

4.5 Solving Rational Inequalities

Learning Goal: We are learning to solve rational inequalities using algebraic and graphical approaches.

The joy, wonder and peace these bring is really quite amazing

e.g. Solve $\left(\frac{x-2}{7} \geq 0\right)7$

$x-2 \geq 0$

$x \geq 2$

Note: For Rational Inequalities, with a variable in the denominator, you **CANNOT** multiply by the multiplicative inverse of the common denominator!!!!

Why? If the factor, $x+3$, is negative, the inequality would need to flip. Since we do not know the sign of "$x+3$", we cannot multiply by it.

Example 4.5.1

Solve $\frac{x-2}{x+3} \geq 0$

Zero: $x=2$, V.A. at $x=-3$

We solve by using an Interval Chart

For the intervals, we split $(-\infty, \infty)$ at all zeros (**where the numerator is zero**), and all restrictions (**where the denominator is zero**) of the (SINGLE) rational expression. Keep in mind that it may take a good deal of algebraic manipulation to get a SINGLE rational expression…

Intervals	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
T.V.	-4	0	3
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
Ratio	$+$	$-$	$+$

You can never equal a restriction

$\therefore \frac{x-2}{x+3} \geq 0$ when $x \in (-\infty, -3) \cup [2, \infty)$

Example 4.5.2

Solve $\frac{1}{x+5} < 5$ $\quad -5 \quad -5$

DO NOT CROSS MULTIPLY (or else)

- Get everything on one side
- Simplify into a single Rational Expression using a common denominator
- Interval Chart it up

hard part

$$\frac{1}{x+5} - \frac{5}{1}\frac{(x+5)}{(x+5)} < 0$$

$$\frac{1-5(x+5)}{x+5} < 0$$

$$\frac{1-5x-25}{x+5} < 0$$

$$\frac{-5x-24}{x+5} < 0$$

zero: $x = \frac{-24}{5} = -4.8$

restrictions: $x \neq -5$

all round brackets

Interval	$(-\infty, -5)$	$(-5, -4.8)$	$(-4.8, \infty)$
TV.	-6	-4.9	0
$-5x-24$	$+$	$+$	$-$
$(x+5)$	$-$	$+$	$+$
ratio	$-$	$+$	$-$

$\therefore \frac{1}{x+5} < 5$ for $x \in (-\infty, -5) \cup (-4.8, \infty)$

Example 4.5.3

Solve $\dfrac{x^2+3x+2}{x^2-16} \geq 0$

FACTORED FORM IS YOUR FRIEND

$$\frac{(x+1)(x+2)}{(x-4)(x+4)} \geq 0$$

zeros: $x = -2, -1$

restrictions: $x \neq -4, 4$

Intervals	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
T.V.	-5	-3	-1.5	0	5
$x+1$	$-$	$-$	$-$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$	$+$	$+$
Ratio	$+$	$-$	$+$	$-$	$+$

$$\therefore \frac{x^2+3x+2}{x^2-16} \geq 0 \text{ for } x \in (-\infty, -4) \cup [-2, -1] \cup (4, \infty)$$

This is bigger, so move things over to that side.

$$3x+5 \geq 4x-8$$

Example 4.5.4

Solve $\boxed{\frac{3}{x+2}} \leq x$

$$0 \leq \frac{x(x+2)}{1(x+2)} - \frac{3}{x+2}$$

$$0 \leq \frac{x^2+2x-3}{x+2}$$

$$0 \leq \frac{(x+3)(x-1)}{x+2}$$

Zeros: $x=-3, 1$

restriction: $x \neq -2$

Interval	$(-\infty,-3)$	$(-3,-2)$ $[-3,-2)$	$(-2,1)$ $[-2,1)$	$(1,\infty)$
T.V.	-4	-2.5	0	2
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$	$+$

$\therefore \frac{3}{x+2} \leq x$ for $x \in [-3,-2) \cup [1,\infty)$

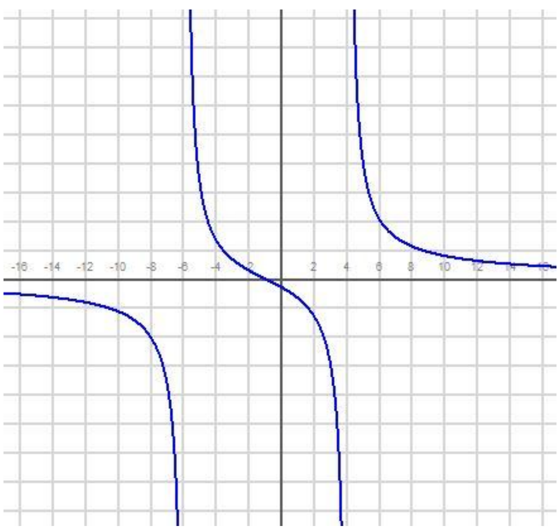
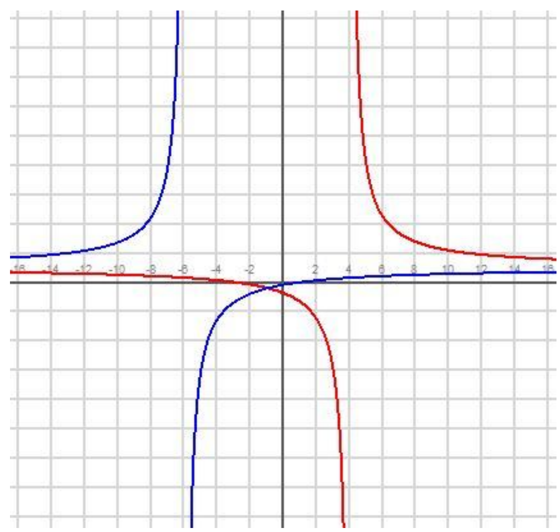
Example 4.5.5

From your Text: Pg. 296 #6a

Using **Graphing Tech**

Solve $\dfrac{x+3}{x-4} \geq \dfrac{x-1}{x+6}$

Note: There are **TWO** methods, both of which require a **FUNCTION** (let $f(x) = \ldots$ returns)

1) Get a Single Function (on one side of the inequality)	2) Use Two Functions (one for each side)
	

Success Criteria:

- I can recognize that an inequality has many possible intervals of solutions
- I can solve an inequality algebraically, using an interval chart
- I can solve an inequality graphically