Advanced Functions

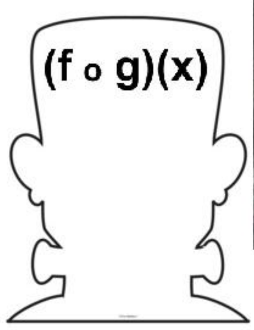
Course Notes

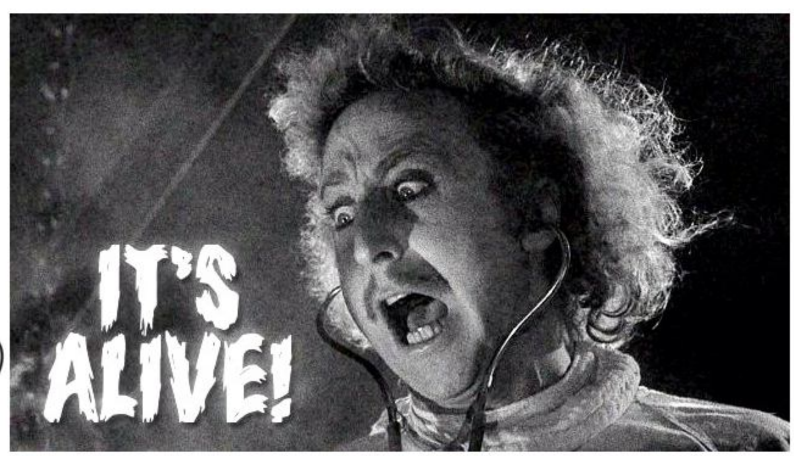
Unit 8 – Combinations of Functions

Mad Mathematicians Creating New Functions!!!

We will learn

- *how to use basic arithmetic to construct new functions from given functions*
- *how to describe the composition of two functions numerically, graphically and algebraically*
- *key characteristics of the newly created functions*





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Chapter 8 – Combinations of Functions

*Contents with suggested problems from the Nelson Textbook (**Chapter 9**)*

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

8.1 Sums and Differences of Functions

Pg. 451 #1c, 4, Finish 5 – 7, 9, 11

8.2 Product and Quotient Combinations

Pg. 537 - 539 #1bd,3,8bd,10,15

Pg. 542 # 1aef, 2 (for #1aef)

8.3 Composition of Functions

READ Ex. 4 Pg550 making sure you understand the concept of a function composed with its inverse.

Pg. 552 – 553 #1abf, 2bdf, 5bcdf (use tech to graph), 6bcd, 7bcdef, 13

8.1 Sums and Differences of Functions

Learning Goal: We are learning to combine functions through addition and subtraction.

Definition 8.1.1

Given two functions $f(x)$, and $g(x)$ with domains D_f and D_g respectively, then we can **construct** new functions:

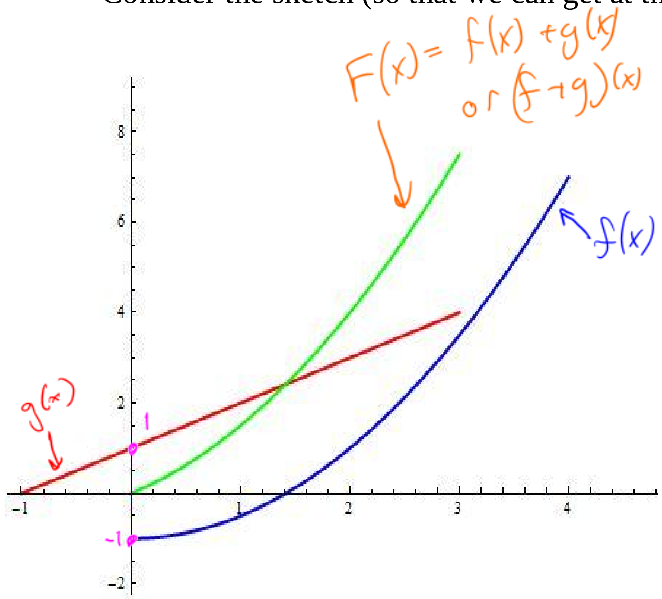
$F(x) = (f+g)(x)$ $\qquad\qquad$ $G(x) = (f-g)(x)$

where the **meaning** of the notation for the "sum" and "difference" functions is as follows:

$F(x) = f(x) + g(x)$ $\qquad\qquad$ $G(x) = f(x) - g(x)$

Example 8.1.1

Consider the sketch (so that we can get at the domain of sum and difference functions):



$D_f = [0, 4]$

$D_g = [-1, 3]$

The domain of $F(x)$ can only contain the elements common to both D_f and D_g

$D_F = [0, 3]$

Larger starting point

smaller ending point

$F(-1) = f(-1) + g(-1)$

Doesn't exist. $+ \; 0$

In general, given functions $f(x)$ and $g(x)$ with domains D_f and D_g, then the combined function

$$F(x) = (f \pm g)(x)$$

has the domain $D_F = D_f \cap D_g$ $\to$ intersection ↳ what they have in common

Example 8.1.2

Determine the domain of $F(x) = (f - g)(x)$ for $f(x) = \sqrt{x}$, and $g(x) = \log\left(\underbrace{-(x-2)}_{>0}\right)$.

$D_f = [0, \infty)$

$D_g = (-\infty, 2)$

$\therefore D_F = [0, 2)$

$-(x-2) > 0$

$x - 2 < 0$

$x < 2$

Example 8.1.3

Given $f(x) = x^3 - 4x + 1$, D_f =[-4,5] and $g(x) = 2x^2 - 1$, D_g = (0,6],

Determine: a) the order of $F(x) = (g - f)(x)$

b) D_F

c) an algebraic representation for $F(x)$

c) $F(x) = g(x) - f(x)$

$= 2x^2 - 1 - (x^3 - 4x + 1)$

$= 2x^2 - 1 - x^3 + 4x - 1$

$= -x^3 + 2x^2 + 4x - 2$

a) order of $F(x)$ is 3

b) $D_f = [-4, 5]$

$D_g = (0, 6]$

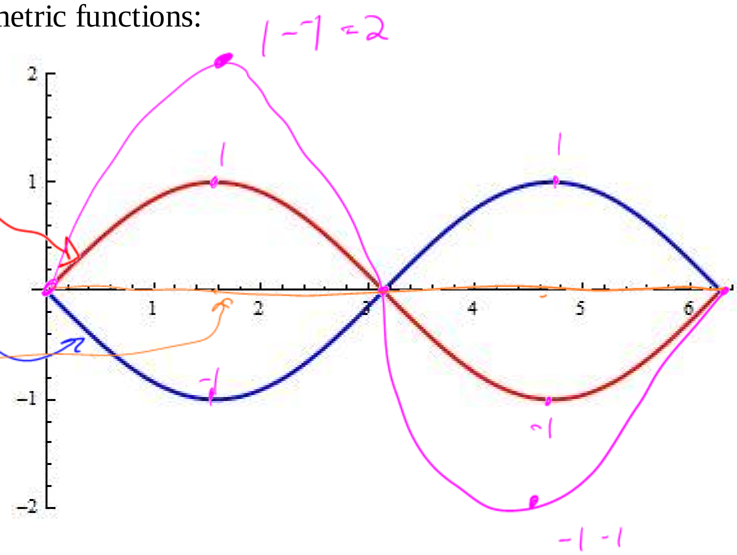
$D_F = (0, 5]$

Example 8.1.4

Consider the sketches of the trigonometric functions:

$f(x) = \sin(x),\ D_f = [0, 2\pi]$

$g(x) = \cos\left(x + \dfrac{\pi}{2}\right),\ \mathrm{D}_g = [0, 2\pi]$



Sketch a) $F(x) = (f - g)(x)$

b) $G(x) = (f + g)(x)$

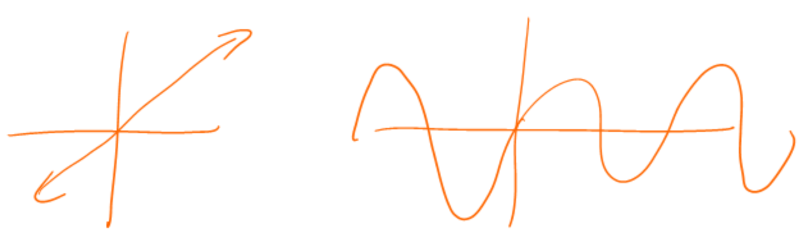
straight line at $y = 0$.

Example 8.1.5

Consider the functions $f(x) = x,\ \mathrm{g}(x) = \sin(x),\ x \in \mathbb{R}$.

What does $F(x) = (f + g)(x)$ look like? (see Ex. 3 Pg. 526)

$F(x) = x + \sin(x)$



Example 8.1.6

From your text: Pg. 528 #1ae

Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$.

Determine: a) $(f+g)(x)$ e) $(f+f)(x)$

a) $(f+g)(x) = \{(-4,6), (-2,5), (1,5), (4,10)\}$

$D_{f+g} = \{-4, -2, 1, 4\}$ Just add y's from common x-coordinates.

e) $(f+f)(x) = \{(-4,8), (-2,8), (1,6), (3,10), (4,12)\}$

Example 8.1.7

From your text: Pg. 529 #7ab

Given $f(x) = \dfrac{1}{3x-4}$ and $g(x) = \dfrac{1}{x-2}$, determine $(f+g)(x)$ and $D_{(f+g)}$

$$(f+g)(x) = \frac{1}{3x-4}(x-2) + \frac{1}{x-2}(3x-4)$$

$$= \frac{x-2+3x-4}{(3x-4)(x-2)}$$

$$= \frac{4x-6}{(3x-4)(x-2)}$$

D_f: $x \neq \frac{4}{3}$, $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$

D_g: $x \neq 2$, $(-\infty, 2) \cup (2, \infty)$

D_{f+g}: $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, 2) \cup (2, \infty)$

Success Criteria:

- I can combine functions by adding or subtracting them
- I can determine the domain and range of these combined functions

8.2 Product and Quotient Combinations

Learning Goal: We are learning to combine functions through multiplication and division.

Definition 8.2.1

Given two functions $f(x)$, and $g(x)$ with domains D_f and D_g respectively, then we can **construct** new functions:

$F(x) = (f \cdot g)(x)$

$= f(x) \cdot g(x)$

$G(x) = \left(\frac{f}{g}\right)(x)$

$= \frac{f(x)}{g(x)}$

$D_F = D_f \cap D_g$

intersection, things in common

$D_{\frac{f}{g}} = D_f \cap D_g\ ,\ g(x) \neq 0$

Example 8.2.1

Determine $(f \cdot g)(x)$ given $f(x) = \{(-2,3),(-1,5),(0,3),(1,-3),(2,-5)\}$ and $g(x) = \{(-1,4),(0,-2),(1,7),(2,-2),(3,2)\}$

$(f \cdot g)(x) = \{(-1,20),(0,-6),(1,-21),(2,10)\}$

Example 8.2.2

From your text: Pg. 537 #2 for #1e

Sketch the given pair of functions on the same set of axes. State their domains. Use your sketch to draw $(f \cdot g)(x)$. State $(f \cdot g)(x)$ and $D_{f \cdot g}$.

$f(x) = x + 2, \quad g(x) = x^2 - 2x + 1$

$= (x-1)(x-1)$

$= (x-1)^2 + 0$

$D_f = (-\infty, \infty)$

$D_g = (-\infty, \infty)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$= (x+2)^1 (x-1)^2$

→ degree/order is 3, odd

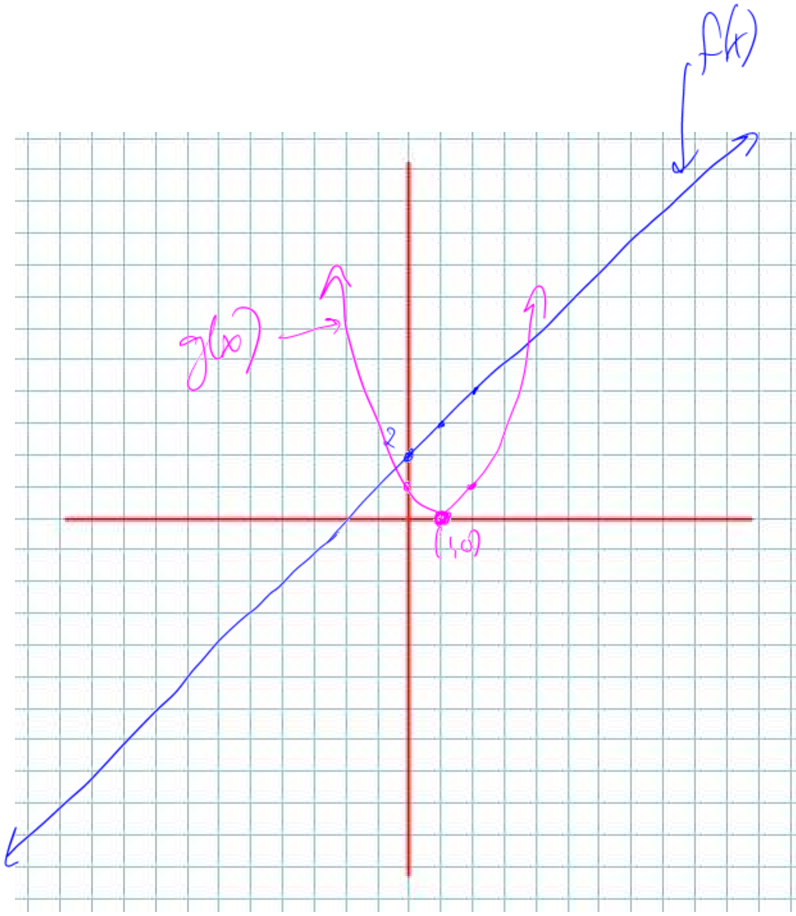
→ Leading Coefficient: 1 = positive

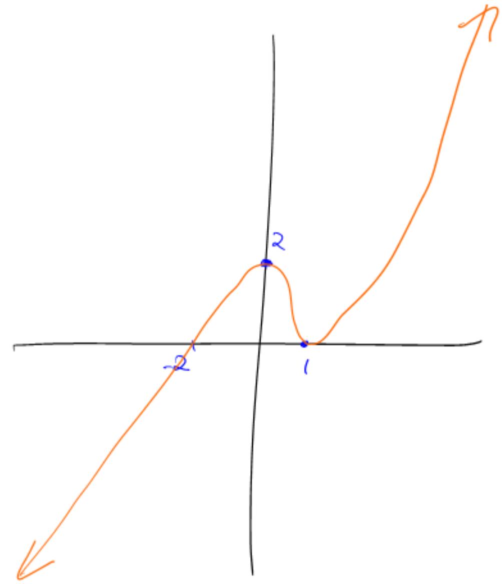
$x \to -\infty,\ f(x) \to -\infty$

$x \to \infty,\ f(x) \to \infty$

→ Zeros: $x = -2$, $x = 1$ order 2

→ y-int: $(0,2)$





Example 8.2.3

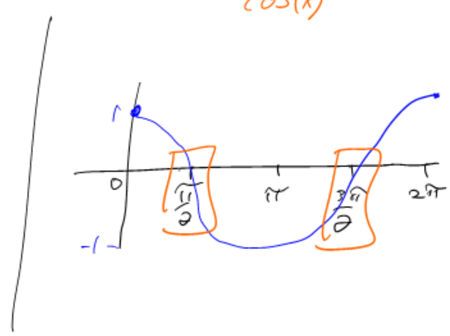
Determine the domain of $D_{f\cdot g}$ and $D_{f\div g}$ given $f(x)=\sqrt{\underbrace{2x+3}_{\geq 0}}$, and $g(x)=\sec(x) = \frac{1}{\cos(x)}$

$2x+3 \geq 0$

$x \geq -\frac{3}{2}$

$D_f = \left[-\frac{3}{2}, \infty\right)$

$D_g : \left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\right\}$



$D_{f\cdot g} \left\{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots\right\}$

$D_{\frac{f}{g}} \left\{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots\right\}$

since $g(x) = \sec(x) \neq 0$, there are no <u>new</u> restrictions

Example 8.2.4

From your text: Pg. 542 #2 for 1d.

~~Sketch the given pair of function on the same set of axes.~~ State their domains. ~~Use your sketch to draw $\left(\frac{f}{g}\right)(x)$~~. State $\left(\frac{f}{g}\right)(x)$ and $D_{\frac{f}{g}}$.

$f(x) = x + 2$ and $g(x) = \sqrt{x-2}$

$D_f = (-\infty, \infty)$

$D_g = [2, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$= \frac{x+2}{\sqrt{x-2}}$

$x = 2$ is a restriction

$D_{\frac{f}{g}} : (2, \infty)$

cannot ~~accept~~ use 2!

Success Criteria:

- I can combine functions by multiplying or dividing them
- I can determine the domain and range of these combined functions
  - The domain is the intersection of the domains of f and g

8.3 Composition of Functions

Learning Goal: We are learning to combine functions by inserting one into the other.

In sections 8.1 and 8.2 we examined how to combine functions (constructing new functions) through the standard algebraic operations of addition, subtraction, multiplication and division. Here we will learn another method for combining functions, but **we won't be using a standard algebraic operation.**

The concept we define as **Composition of Functions** is **very useful for Calculus** (among other things) as some of you will see next semester.

The basic idea is that given two functions $f(x)$ and $g(x)$, we can define the composition of the two by

inserting one function into the other.

The "algebraic" notation may seem a little weird, but don't make fun. Math has feelings too.

Definition 8.3.1

Given two functions $f(x)$ and $g(x)$ we write the composition of $f(x)$ and $g(x)$ as

$$(f \circ g)(x)$$

We can also write the composition of $g(x)$ and $f(x)$ as

$$(g \circ f)(x)$$

The "Algebraic Meaning" of Composition

$$(f \circ g)(x) = f(g(x))$$

inner

$$(g \circ f)(x) = g(f(x))$$

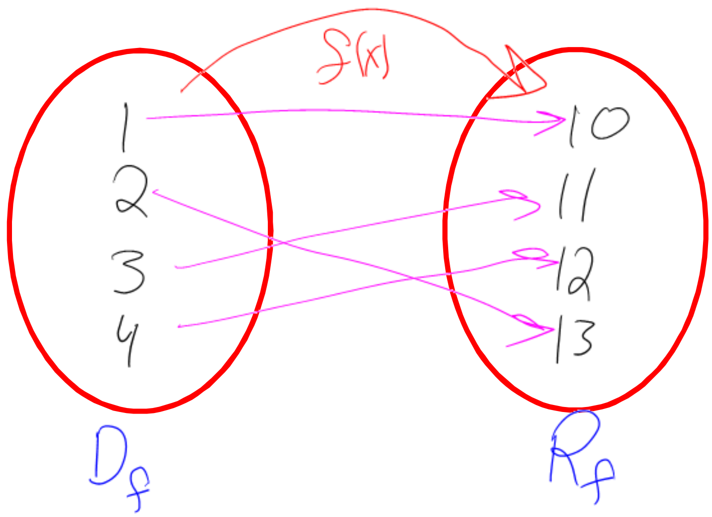
outer

Note: It is **very helpful** to keep in mind the distinction between the **inner** and **outer** functions

The Domain of a Composition of Functions

Recall the basic “**machinery**” of any function: **“Plug a (domain) number into the function, and get a (range) number out.”**

Picture

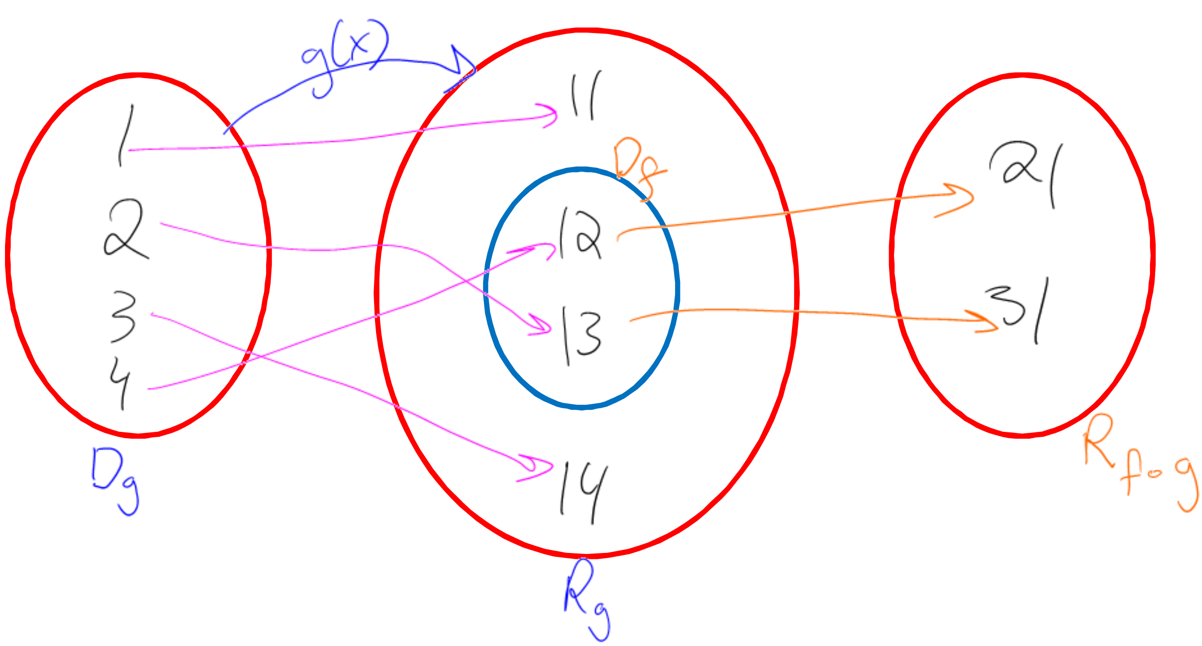


Recall further that many functions cannot claim “all real numbers” as their (natural) domain.

e.g. Determine the domain of $f(x)=\sqrt{x+1}$ $\quad D_f: [-1, \infty)$, $x \geq -1$

Now we consider the “machinery” for the composition function $(f \circ g)(x) = f(\underbrace{g(x)}_{\text{first}})$

Picture:



Algebraic Definition of the Domain of a Composition of Two Functions

Given two functions $f(x)$ and $g(x)$ with domains D_f and D_g respectively, then the domain of the composition of $f(x)$ and $g(x)$ is given by:

range of g(x)

$$D_{(f \circ g)} = \{x \in \mathbb{R} \mid x \in D_g \quad \text{such that } g(x) \in D_f \}$$

outer ← inner

Using words we might write that the **domain of a composition of functions** $(f \circ g)(x) = f(g(x))$ is **the set of all *x* values** which belong to the domain of the

__inner__ function which have range values which are in the domain of the

__outer__ function.

Example 8.3.1

Given $f(x) = 3x + 1$ and $g(x) = x^3 - 1$ determine:

a) $(f \circ g)(0)$

$= f(g(0))$

$= f(0^3 - 1)$

$= f(-1)$

$= 3(-1) + 1 = -2$

$\therefore (0, -2)$

b) $g(f(0))$

$= g(3(0) + 1)$

$= g(1)$

$= (1)^3 - 1$

$= 0$

$\therefore (0, 0)$

c) $(f \circ f)(1)$

$= f(f(1))$

$= f(3(1) + 1)$

$= f(4)$

$= 3(4) + 1 = 13.$

$\therefore (1, 13)$

Example 8.3.2

From your text: Pg. 552 #2 ac"g"

Given $f(x) = \{(0,1),(1,2),(2,5),(3,10)\}$ and $g(x) = \{(2,0),(3,1),(4,2),(5,3),(6,4)\}$

determine:

$D_{g \circ f} = \{1, 2\}$

$D_{f \circ g} = \{2, 3, 4, 5\}$

a) $(g \circ f)(2)$

$= g(f(2))$

$= g(5)$

$= 3$

c) $f(g(5))$

$= f(3)$

$= 10$

g) $g(f(3))$

$= g(10)$

$=$ Doesn't exist!

Something Silly but Entirely Serious

Given $f(x) = 2x^2 - 1$ determine:

a) $f(2)$

$= 2(2)^2 - 1$

$= 7$

b) $f(A)$

$2A^2 - 1$

c) $f(\square)$

$2(\square)^2 - 1$

d) $f(\square + \triangle)$

$2(\square + \triangle)^2 - 1$

Example 8.3.3

From your text: Pg 552 #6ae

Given the functions $f(x)$ and $g(x)$ determine functional equations for $f(g(x))$ and $g(f(x))$ and determine their domains.

a) $f(x) = 3x$ and $g(x) = \sqrt{x-4}$

$f(g(x)) = f(\sqrt{x-4}) = 3\sqrt{x-4}$

$D_{f \circ g}: [4, \infty)$

$g(f(x)) = g(\boxed{3x}) = \sqrt{\boxed{3x} - 4}$

$D_{g \circ f}: [\frac{4}{3}, \infty)$

e) $f(x) = 10^x$ and $g(x) = \log(x)$

$(-\infty, \infty)$

$f(g(x)) = f(\underbrace{\log(x)}_{x>0}) = 10^{\log(x)} = \boxed{x}$

$D_{f \circ g} = (0, \infty)$

$g(f(x)) = g(10^x) = \log(10^x) = x$

$D_f = (-\infty, \infty)$

$R_f = (0, \infty)$

all of this is valid in a log.

$\therefore D_{g \circ f}: (-\infty, \infty)$

Example 8.3.4

From your text: Pg. 553 #7a

Given $h(x)$ find two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$.

a) $h(x) = \sqrt{x^2+6}$

$g(x) = x^2$

$f(x) = \sqrt{\boxed{x}+6}$

$f(g(x)) = \sqrt{x^2+6}$

$g(x) = x^2+6$

$f(x) = \sqrt{x}$

$g(x) = \boxed{x^2+3}$

$f(x) = \sqrt{x+3}$

$g(x) = \sqrt{x^2+6}$

$f(x) = x$

Success Criteria:

- I can combine functions by inserting one into the other
    - $(f \circ g)(x) = f(g(x))$
- I can determine the domain and range of these combined functions
    - For domain, it is the set of values, *x*, in the domain of *g* for which *g(x)* is in the domain of *f*.