Advanced Functions

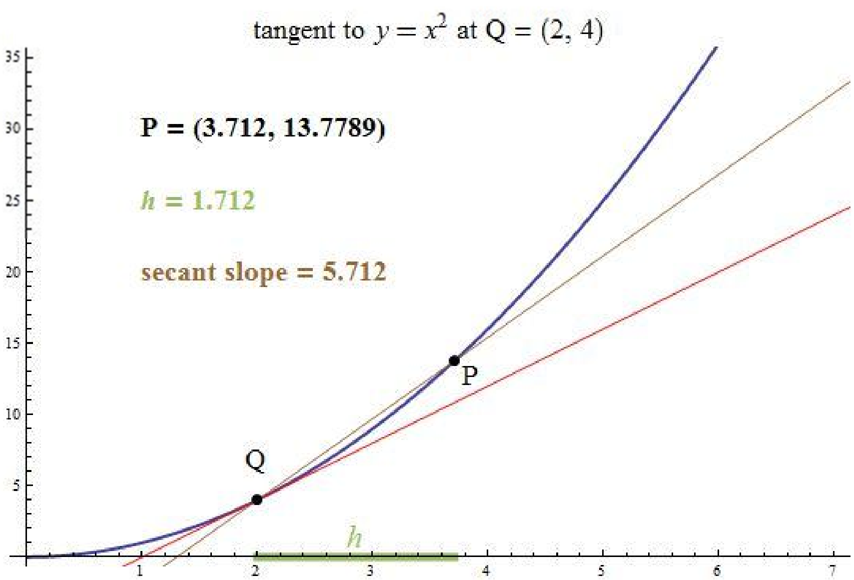
Course Notes

Unit 9 – Rates of Change

At this rate, we might as well just change to Calculus!

We will learn

- *how to calculate an average rate of change of a function, given the function as a table of values, or a sketch, or an equation*
- *how to estimate the instantaneous rate of change of a function*
- *how to interpret the meaning of the average rate of change of a function over an interval of the function's domain*
- *how to interpret the meaning of the instantaneous rate of a change of a function at a single value of the domain.*
- *how to solve problems using rates of change*



A∞Ω
MATH@TD

Chapter 9 – Rates of Change and the Tangent Problem

*Contents with suggested problems from the Nelson Textbook (**Chapter 2**)*

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

9.1 Average Rate of Change: The AROC

Pg. 76 – 77 #1 (important question), 2, 4, 9, 10

9.2 Instantaneous Rate of Change (Pt. 1)

Pg. 86 – 87 #4ac, 6, 8, 9, 10 (centered interval only)

9.3 Instantaneous Rate of Change (Pt. 2)

Various given problems

9.1 Average Rate of Change – The AROC

Learning Goal: We are learning to calculate the average rate of change over a given interval.

From Physics we learn that we can calculate the average velocity of some moving object through the formula….This is the slope of a line!!!

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} \qquad \Bigg| \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

In general we can calculate the **A**verage **R**ate **O**f **C**hange [AROC], for some given function $f(x)$, over an interval of time (the domain) $t \in [t_1, t_2]$ using the formula:

$$AROC = \frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Example 9.1.1

Consider the displacement function $s(t) = 100 - 4.9t^2$, which is being used to describe the displacement (s in m) of a falling body from the top of a 100m high cliff after t seconds.

Over the given time intervals determine the average rate of change (the AROC) of displacement for a stone dropped from the edge of the cliff:

a) $t = 0$ to $t = 1$ seconds. b) $t \in [1, 2]$ (seconds). c) $t \in [0, 3]$.

a) $AROC = \frac{s(1) - s(0)}{1 - 0}$

$= \frac{(100 - 4.9(1)^2) - (100 - 4.9(0)^2)}{1}$

$= \frac{95.1 - 100}{1}$

$= -4.9 m/s$

b) $AROC = \frac{s(2) - s(1)}{2 - 1}$

$= \frac{80.4 - 95.1}{1}$

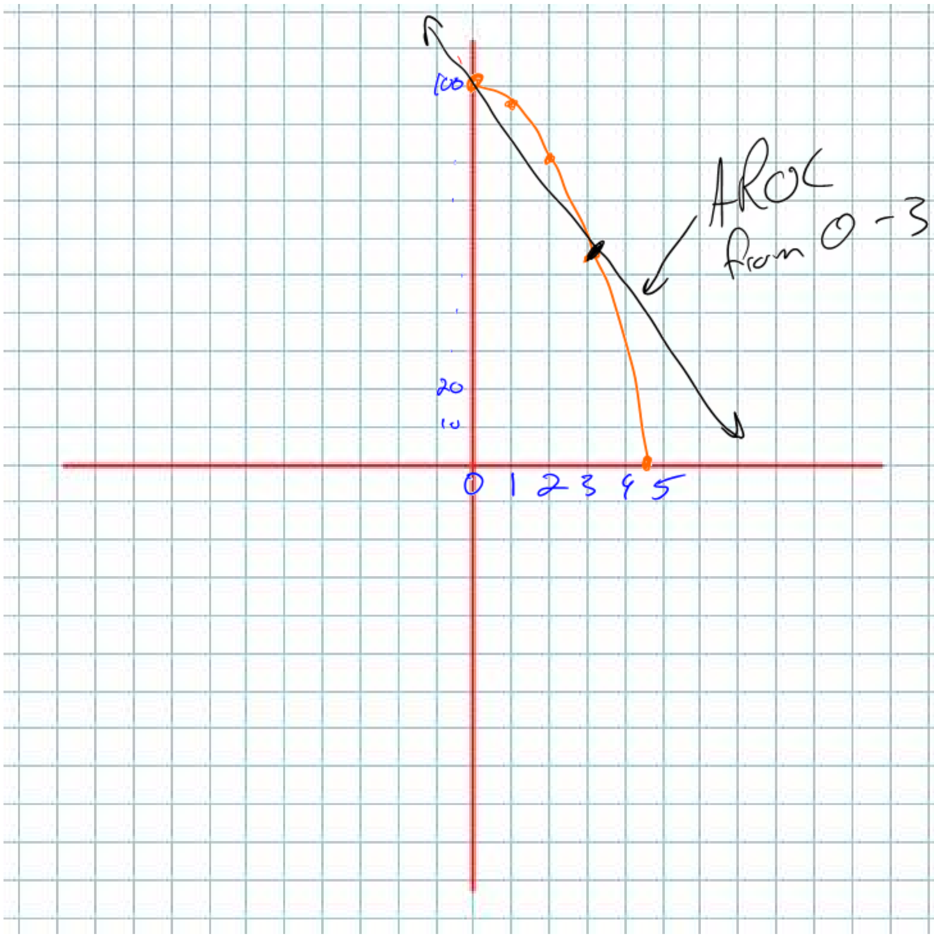
$= -14.7 m/s$

c) $AROC = \frac{s(3) - s(0)}{3 - 0}$

$= \frac{55.9 - 100}{3}$

$= -14.7 m/s$

A picture of the situation in example 9.1.1:



$0 = 100 - 4.9t^2$

$4.9t^2 = 100$

$t^2 = 20.4$

$t = \pm 4.5$

The Slope of a Secant is an AROC for some given fn over a closed interval.

Success Criteria:

- I can recognize that AROC is merely the slope of a line
- I can tell the difference between a positive and negative rate of change
- I can recognize that all linear relationships have a CONSTANT rate of change
- I can recognize that non-linear relationships do not have a constant rate of change. Average calculations over different intervals will give different answers.

9.2 Instantaneous Rate of Change – The IROC

(*part 1*)

Learning Goal: We are learning to estimate the IROC at a particular value of the independent variable.

Last day we learned that for some given function $f(x)$ we can **calculate** the AROC of that function (over some interval of the domain) as the **slope of a secant**.

Example 9.2.1

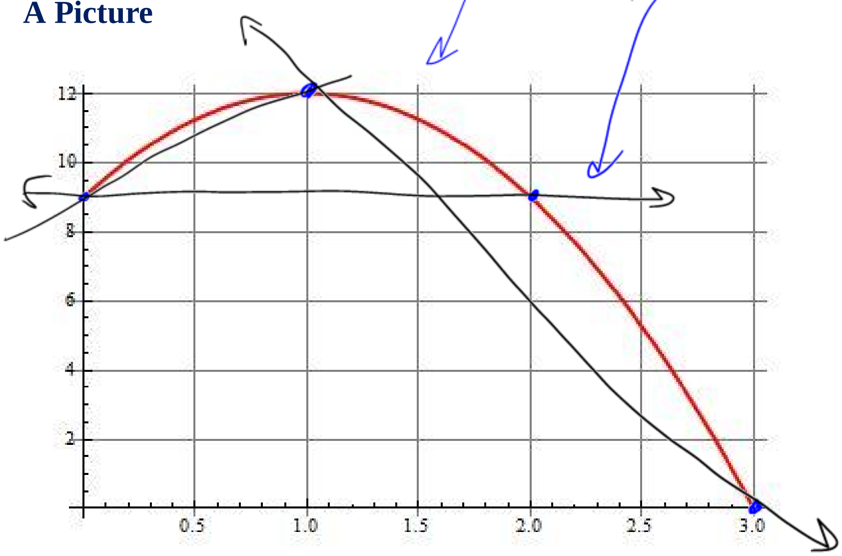
Given the displacement function $s(t) = -3(t-1)^2 + 12$, determine the AROC of a water-balloon tossed from a 3rd floor balcony, over the (time) intervals:

a) $t \in [0,1]$ $\quad$ b) $t \in [1,3]$ $\quad$ c) $t \in [0,2]$

$\frac{12-9}{1} = 3$

AROC = 0!

A Picture



b) $\frac{0-12}{3-1}$

$= \frac{-12}{2}$

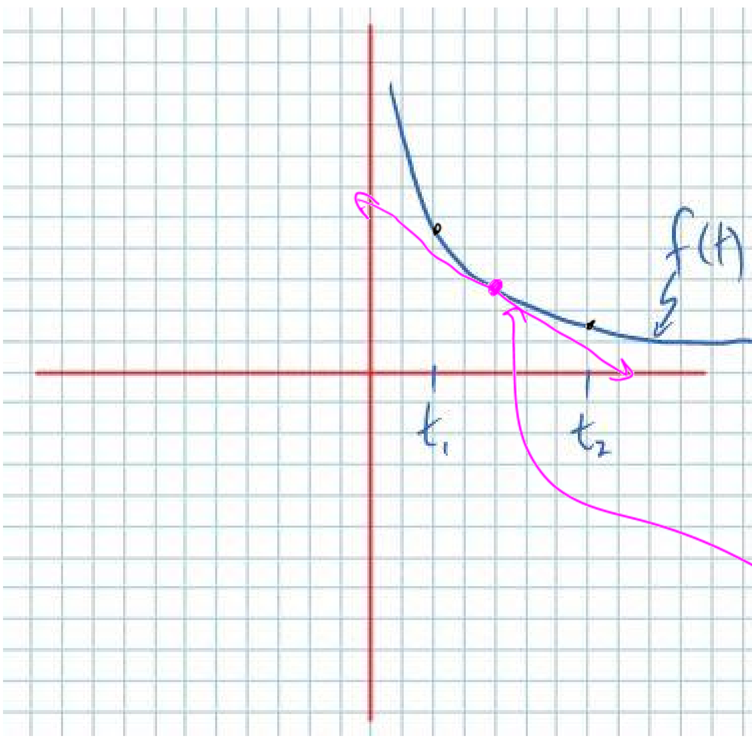
$= -6$

Note: Each AROC is a number which represents

Consider the question:

How can we calculate the velocity of the balloon at the **instant** $t = 2$ seconds?
Note: We CANNOT use the slope of a secant, since the secant requires two domain values, but an "instant" is at a single domain value $(t = 2$ in this case$)$.

Consider the following:



The **AROC** is the **slope** of a secant.

$$AROC = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Geometrically speaking, the **IROC** is also a **slope**, but it's the slope of a **TANGENT**

– a line which touches a curve at only one point.

The Problem is this:

You cannot calculate the slope using only one point!

We will consider two techniques for **ESTIMATING** the so-called

INSTANTANEOUS RATE OF CHANGE (IROC)

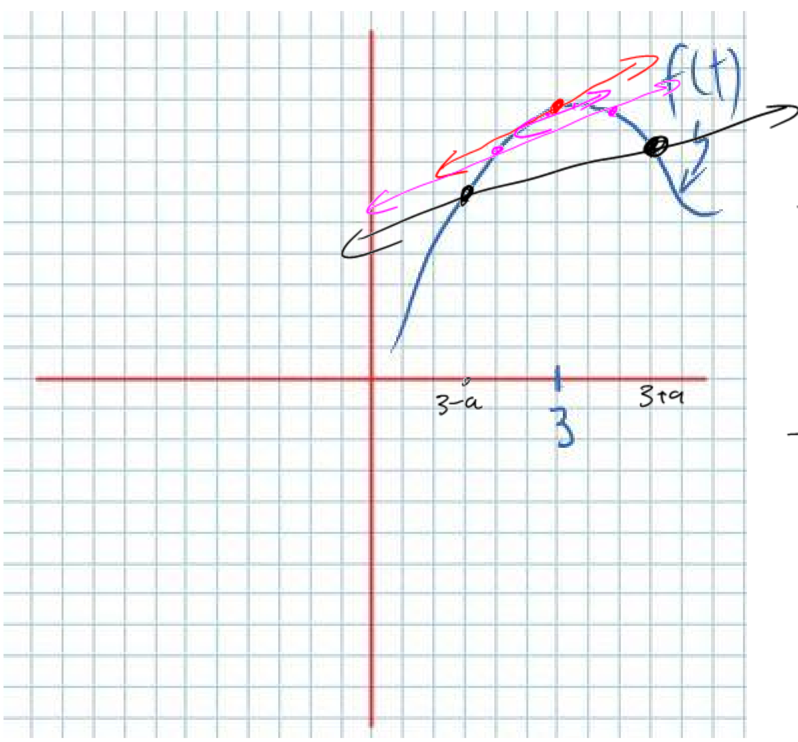
Note: To estimate the IROC (a number we CAN'T calculate) we will be calculating the slopes of secants because we can do that calculation!

1) Using a "**centered interval**" and **squeezing the interval** to get better and better **estimates**

A Geometric View

Consider the picture:

IROC at $x = 3$.



- Pick a number "a".

Go down "a" from 3

Go up "a" from 3

- calculate that AROC,

the squeeze the interval,

making it smaller and smaller.

An Algebraic View

Example 9.2.2

Estimate the IROC for $s(t) = -2(t-1)^3 + 3$ at $t = 2$.

$[2-a,\ 2+a]$

Consider $a = 1$ $\therefore$ AROC of $[1, 3]$

$$= \frac{s(3) - s(1)}{3 - 1} = \frac{-13 - (3)}{2} = \frac{-16}{2} \cancel{-8}$$

$= -8$ units

This "a" is much too big

Consider $a = 0.1$ $\therefore$ AROC of $[1.9, 2.1]$

$$= \frac{s(2.1) - s(1.9)}{2.1 - 1.9} = \frac{0.338 - 1.542}{0.2}$$

$= -6.02$ units

Consider $a = 0.001$ $\therefore$ AROC of $[1.99, 2.01]$

$$= \frac{s(2.01) - s(1.99)}{2.01 - 1.99} = -6.002$$

$\therefore$ The IROC ~ -6 units at $x = 2$

approximately

Example 9.2.3

From your text: Pg. 87 #5

Using a centered interval approach, determine an estimate for the IROC at $x = 3$sec of the height (in *m*) of an object, which is moving according to $h(x) = -5x^2 + 3x + 65$.

use $a = 0.01$, $[2.99, 3.01]$

$$\text{IROC} \sim \text{AROC} = \frac{h(3.01) - h(2.99)}{3.01 - 2.99}$$

$$= \frac{28.7295 - 29.2695}{0.02}$$

$$= -27 \text{ m/s}$$

Success Criteria:

- I can estimate the IROC of the dependent variable using pictures, or the centered-interval approach by using smaller and smaller values of *h*
- I can recognize that the best estimate for IROC occurs when the interval used (*h*) is as small as possible

9.3 Instantaneous Rate of Change – The IROC (*part 2*)

The Difference Quotient

Learning Goal: We are learning to estimate IROC by using the difference quotient method.

Suppose we wish to calculate the Instantaneous Rate of Change of some function, $f(x)$, at $x = 2$. Last day we saw three things:

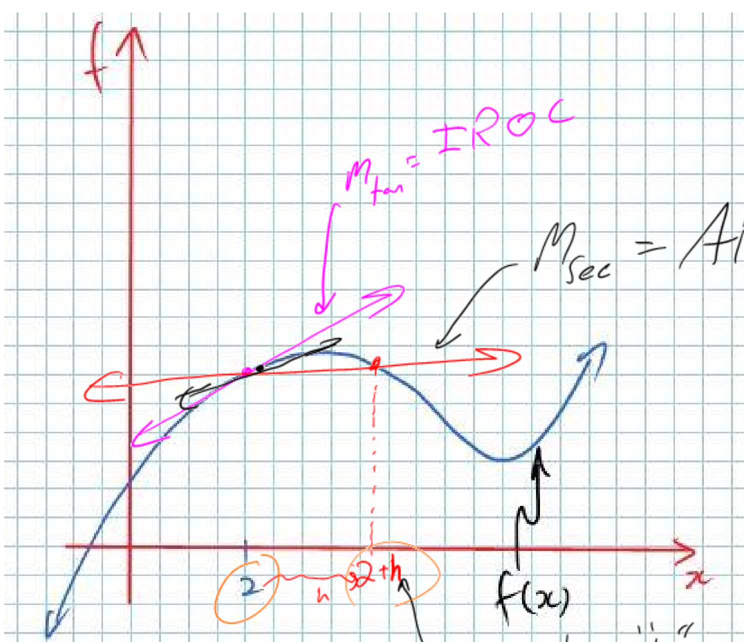
1) IROC = the slope of the tangent at $x = 2$

2) We cannot calculate the slope of a tangent using standard techiques.

3) We can estimate the IROC using the AROC

Rather than using a "centered interval" approach, we now consider the so-called **Difference Quotient** (which can be much more useful than the centered interval approach).

Consider the sketch:



= AROC, by making "h" really small, we will approximate the IROC

make this "h" really small

Now, we understand that

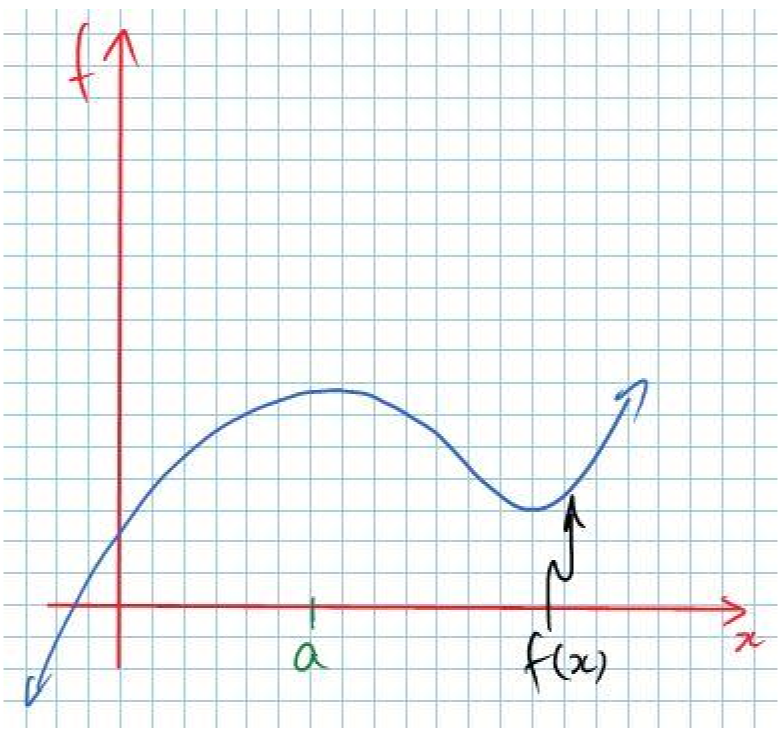
$$IROC \sim AROC = m_{\sec} = \frac{f(2+h) - f(2)}{\cancel{2}+h \cancel{-2}}$$

Definition of the Difference Quotient

In general, if we wish to approximate the IROC of $f(x)$ at some (general) domain value $x=a$, then

$$IROC \sim AROC = \frac{f(a+h) - f(a)}{h}, \text{ for small } h.$$

Note: "h" can be either positive or negative. Consider the sketch:



Example 9.3.1

Given $s(t) = 2t^2 - 3t - 5,$ determine a difference quotient which will estimate the IROC of $s(t)$ at $t = a$. Use that difference quotient to estimate the IROC at $t = 3$ using $h = 0.0001$.

$$\text{IROC} \approx \text{AROC} = \frac{s(a+h) - s(a)}{h}$$

For $a = 3$, $h = 0.0001$

$$\text{IROC} \approx \frac{s(3.0001) - s(3)}{0.0001}$$

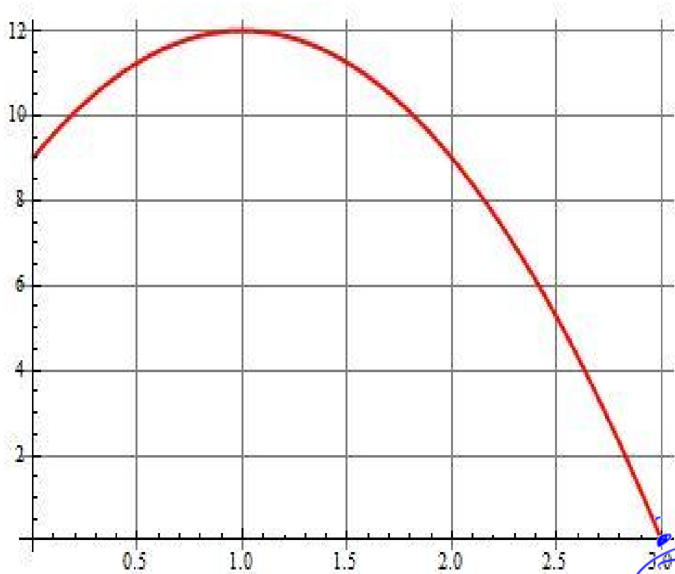
$$= \frac{4.0009002 - 4}{0.0001}$$

$$= 9.0002$$

$\therefore$ The IROC at $t = 3$ is 9 units.

Example 9.3.2

Consider the water-balloon problem from Example 9.2.1. The water-balloon "flies" according to the function $s(t) = -3(t-1)^2 + 12$. Estimate the instantaneous velocity (the IROC) of the balloon when it hits the ground (at $t = 3$ sec).



Let's use $h = -0.001$

$$IROC \approx \frac{s(2.999) - \cancel{s(3)}^{\to 0}}{-0.001}$$

$$= \frac{0.011997}{-0.001}$$

$$= -11.997$$

$\therefore$ The IROC at $t=3$ is -12 m/s

Class/Homework

Determine an estimate for the IROC of the given function at the indicated domain value using a difference quotient. Use $h = 0.001$ for your estimation.

a) $f(x) = x^2 - 3x + 1$ at $x = 2$ *IROC* ~ 1

b) $h(t) = 2^t - 3$ at $t = 0$ *IROC* ~ 0.693

c) $g(x) = \sin(x)$ at $x = \pi$ *IROC* ~ –1

d) $s(t) = \dfrac{t+1}{t-2}$ at $t = 3$ *IROC* ~ –3

e) $g(x) = x^3 + 2$ at $x = 3$ *IROC* ~ 27

Success Criteria:

- I can estimate the IROC of the dependent variable using the difference quotient method using smaller and smaller values of *h*

$$IROC \sim AROC = \frac{f(a+h)-f(a)}{h},$$ for a *very very very* small *h* ☺

- I can recognize that the best estimate for IROC occurs when the interval used (*h*) is as small as possible