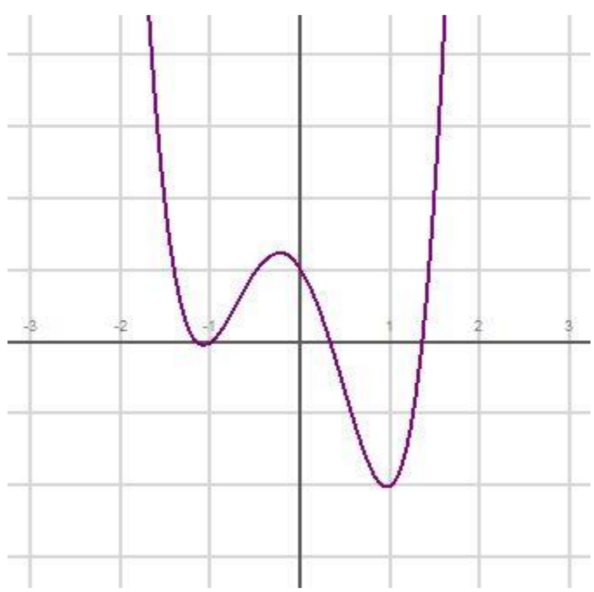
Advanced Functions

Course Notes

Chapter 3 – Polynomial Equations and Inequalities

We will learn

- *how to find solutions to polynomial equations using tech and using algebraic techniques*
- *how to solve polynomial inequalities with and without tech*
- *how to apply the techniques and concepts to solve problems involving polynomial models*



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Chapter 3 – Polynomial Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 4). You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

3.1 Solving Polynomial Equations

Learning Goal: We are learning to solve polynomial equations using a variety of strategies.

Before embarking on this wonderful journey, it seems to me that it would be prudent to make some (seemingly silly) opening statements.

Seemingly Silly Opening Statements

1) Polynomial equations ARE NOT polynomial functions!
2) Solving any equation MEANS finding a **SOLUTION** (if a solution exists)!
3) Solving a polynomial equation is ALWAYS equivalent to finding the zeros of some polynomial function!

Example 3.1.1 (***back to Grade 9***)

Solve the linear equation

$$3(x-5)+2=5x+6$$

$$3x\boxed{-15+2}=5x+6$$

$$-2x=19$$

$$x=\frac{-19}{2}$$

Example 3.1.2 (***remember grade 11?***)

Solve the quadratic equation

$$5x(x-1)+7=2x^2+9$$

Goal: "stuff" = 0

$$5x^2-5x+7=2x^2+9$$

① Factor

② Quadratic Formula.

$$3x^2-5x-2=0$$

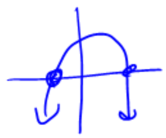
by factoring

$$(3x+1)(x-2)=0 \quad \therefore x=\frac{-1}{3} \text{ and } 2.$$

Geometrically speaking, solving a quadratic **equation** is equivalent to finding the zeros of a quadratic **function**.
Solving the **equation** in **Example 3.1.2** means the same thing as finding the zeros of the **function**

Note further that quadratic **functions** can have

2 zeros



1 zero

0 zeros

Thus quadratic **equations** can have **2 solutions**, **1 solution** or **no solutions**!

Comments about Higher Order Polynomial Equations

Consider the cubic **EQUATION** $x^3 + 2x^2 - 5 + 1 = 0$.
Q. How many zeros can this equation have?
Ans. 3, 2, 1. $\quad$ * odd must have one zero.

Consider the quartic equation $4x^4 - 3x^3 + 5x^2 - 3x + 1 = 0$.
Q. How many zeros can this equation have?
Ans. 0, 1, 2, 3, 4.

Example 3.1.3

Solve the polynomial equation by factoring:

$$4x^3 - 3x = 1$$

$$4x^3 + 0x^2 - 3x^1 - 1 = 0 \qquad \pm 1$$

$$f(1) = 4(1)^3 - 3(1) - 1$$
$$= 4 - 3 - 1$$
$$= 0 \checkmark$$

$\therefore (x-1)$ is a factor

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

$$\therefore (x-1)(4x^2 + 4x + 1) \qquad m: 4 \quad A: 4 \qquad 2, 2$$

$$\therefore (x-1)(2x+1)^2$$

> ***Note***: Solving Polynomial Equations requires writing the equation in **Standard Form**, which is: "polynomial = 0"

Shortcut.

$(4x+2)(4x+2)$

$(2x+1)(2x+1)$

The solutions are

$x = 1$

$x = \frac{-1}{2}$

Example 3.1.4

Solve the equation by factoring:

$$12x^4 + 16x^3 - 11x = 13x^2 - 6$$

$$12x^4 + 16x^3 - 13x^2 - 11x + 6 = 0 \qquad \pm 1, \pm 2, \pm 3, \pm 6$$

$$f(-1) = 12(-1)^4 + 16(-1)^3 - 13(-1)^2 - 11(-1) + 6$$
$$= 12 - 16 - 13 + 11 + 6$$
$$= 0.$$

$\therefore (x+1)$ is a factor.

$$\begin{array}{r|rrrrr} -1 & 12 & 16 & -13 & -11 & 6 \\ & & -12 & -4 & 17 & -6 \\ \hline & 12 & 4 & -17 & 6 & 0 \end{array}$$

$$\therefore (x+1)\underbrace{(12x^3 + 4x^2 - 17x + 6)}_{g(x)}$$

Test $\pm 1, \pm 2, \pm 3, \pm 6$

$$g(\pm 1) \neq 0, \quad g(\pm 2) \neq 0, \quad g(\pm 3) \neq 0, \quad g(\pm 6) \neq 0$$

Rational Zero Test

Consider $12x^3 + 4x^2 - 17x + \overset{1}{6} = 0$.

We now, when using the factor theorem, will "test for zeros" using 2 steps:

1) Test for integer zeros using factors of the constant term.

   $\pm 1, \pm 2, \pm 3, \pm 6$ } "b" TOP

2) Test for rational zeros, where we consider $x = \frac{b}{a}$

We need the factors of 12 $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ ← "a" Bottom

The possible rational zeros are:

$\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}, \pm\frac{2}{3}, \pm\frac{3}{2}, \pm\frac{3}{4}$

24 total possible factors

$\frac{1}{3} \neq 0.3$

0.3333333

Back to **Example 3.1.4**

For $g(x) = 12x^3 + 4x^2 - 17x + 6$ the possible rational zeros are:

$$g\left(\tfrac{1}{2}\right) = 12\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{1}{2}\right)^2 - 17\left(\tfrac{1}{2}\right) + 6$$

$$= 12\left(\tfrac{1}{8}\right) + 4\left(\tfrac{1}{4}\right) - \tfrac{17}{2} + 6$$

$$= \tfrac{3}{2} + \tfrac{2}{2} - \tfrac{17}{2} + \tfrac{12}{2} = ☺$$

M: -36

A: 5

$9, -4$

$(6x+9)(6x-4)$

$(2x+3)\quad(3x-2)$

$x = \frac{1}{2}$

$$\begin{array}{c|cccc} \frac{1}{2} & 12 & 4 & -17 & +6 \\ & & 6 & 5 & -6 \\ \hline & 12 & 10 & -12 & 0 \end{array}$$

$\div 2$ $\quad 6 \quad 5 \quad -6$

$\therefore (2x-1)(6x^2+5x-6)$

$(2x-1)(2x+3)(3x-2)$

$\therefore$ The solutions are $x = -1, \frac{1}{2}, \frac{-3}{2}, \frac{2}{3}$

Example 3.1.5

Solve the equation $3x^3 - 4x + 2 = 0$.

$\pm 1, \pm 3$

$\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

Try $f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right) + 2$

$= 3\left(\frac{1}{27}\right) - \frac{4}{3} + \frac{2}{1}$

$= \frac{1}{9} - \frac{12}{9} + \frac{18}{9} = \frac{7}{9} \neq$ ☹

In fact: $f(\pm 1) \neq 0$, $f(\pm 2) \neq 0$, $f\left(\pm \frac{1}{3}\right) \neq 0$, $f\left(\pm \frac{2}{3}\right) \neq 0$

$\therefore f(x)$ does not factor.

But!! $f(x)$ is odd degree and must have one zero.

So let's graph it.

$x = -1.352$

Find all the possible factors of:

$f(x) = 8x^4 - 3x^3 + 2x^2 - 5x - 20$

1, 2, 4, 8

$\pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{4}{1}, \pm \frac{5}{1}, \pm \frac{10}{1}, \pm \frac{20}{1}$

$\pm \frac{1}{2}, \pm \frac{5}{2}$

$\pm \frac{1}{4}, \pm \frac{5}{4}$

$\pm \frac{1}{8}, \pm \frac{5}{8}$

Success Criteria:

- I can solve polynomial equations algebraically (by factoring) AND graphically
- I can recognize that only SOME polynomial equations can be solved by factoring
- I can recognize that some solutions may not make sense in the context of the question

$<, >, \leq, \geq$

3.2 Linear Inequalities

easy

Learning Goal: We are learning to solve linear inequalities.

Once again, it seems a good idea to begin with a couple of opening statements.

Absolutely Non-Silly Opening Statements

1) The **algebra** of inequalities is the **SAME** as the algebra on equality (i.e. solving equations), with two exceptions:

   a) If you multiply or divide **by a negative**, then you MUST FLIP THE SIGN.

   $$\frac{-2x}{-2} > \frac{6}{-2}$$
   $$x < -3$$

   $$-2x + 2x > 6 + 2x$$
   $$-6 + 6 > 2x - 6$$
   $$\frac{-6}{2} > \frac{2x}{2}$$
   $$-3 > x$$

   b) We can have 2 sided inequalities – e.g.

   $$3 \leq x \leq 5 \quad \text{or} \quad 2 > x > 10$$

2) The **Solution Set** of inequalities is infinite.

Example 3.2.1

Solve the (linear) inequality $3x - 2 > 4$.

$$3x - 2 + 2 > 4 + 2$$
$$\frac{3x}{3} > \frac{6}{3}$$
$$x > 2$$

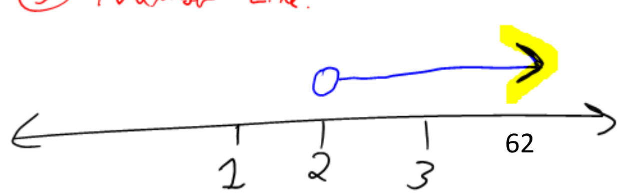
① Set Notation:

$$\{x \in \mathbb{R} \mid x > 2\}$$

② Interval notation:

$$x \in (2, \infty)$$

③ Number Line.



Focus on the middle, do the "math" every where.

Example 3.2.2

Solve the two sided inequality $-2 > -4x + 5 \geq -3$.

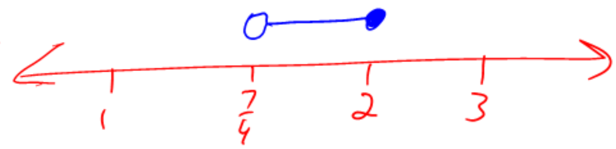
$-5 \quad -5 \quad -5$

$$\frac{-7}{-4} > \frac{-4x}{-4} \geq \frac{-8}{-4}$$

$$\frac{7}{4} < x \leq 2$$

Interval: $x \in \left(\frac{7}{4}, 2\right]$

Graph:



Example 3.2.3

Solve $5 \leq \boxed{3(x-2) - 4(x+3)} \leq 12$

simplify

$$5 \leq 3x - 6 - 4x - 12 \leq 12$$

$$5 \leq -x - 18 \leq 12$$

$+18 \quad +18 \quad +18$

$$\frac{23}{-1} \leq \frac{-x}{-1} \leq \frac{30}{-1}$$

$$-23 \geq x \geq -30$$

arrange from small to big

$$-30 \leq x \leq -23 \qquad x \in [-30, -23]$$

Example 3.2.4

Write the following sketch of a solution set in interval and set notation:

$x < -4$ or $\boxed{-4 > x \quad x \geq 3}$

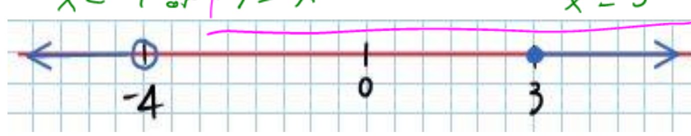


Figure 3.2.4

Interval: $x \in (-\infty, -4) \cup [3, \infty)$

Set: $\{x \in \mathbb{R} \mid -4 > x \geq 3\}$

Graphical Views of (non-linear) Polynomial Inequalities

(the Algebra is tough…)

Example 3.2.5
Consider the sketch of the graph of some mystery cubic function.

Q. $\underbrace{\text{When (or better WHERE)}}_{\text{domain or } x\text{'s}}$ are *the* $\underbrace{\text{functional values}}_{y\text{'s}}$ positive?

↳ > 0

> 0

$x \in (-1, 2) \cup (4, \infty)$

$= 0$

< 0

Figure 3.2.5

Example 3.2.6

Consider the sketch of the quartic $g(x)$, and determine where

a) $g(x) \leq 0$

b) $g(x) > 2$

c) $-1 \leq g(x) \leq 2$

a) $x \in [-4, 1] \cup [3]$

b) $x \in (-\infty, -5) \cup (5, \infty)$

c) $x \in [-5, -3] \cup [0, 5]$

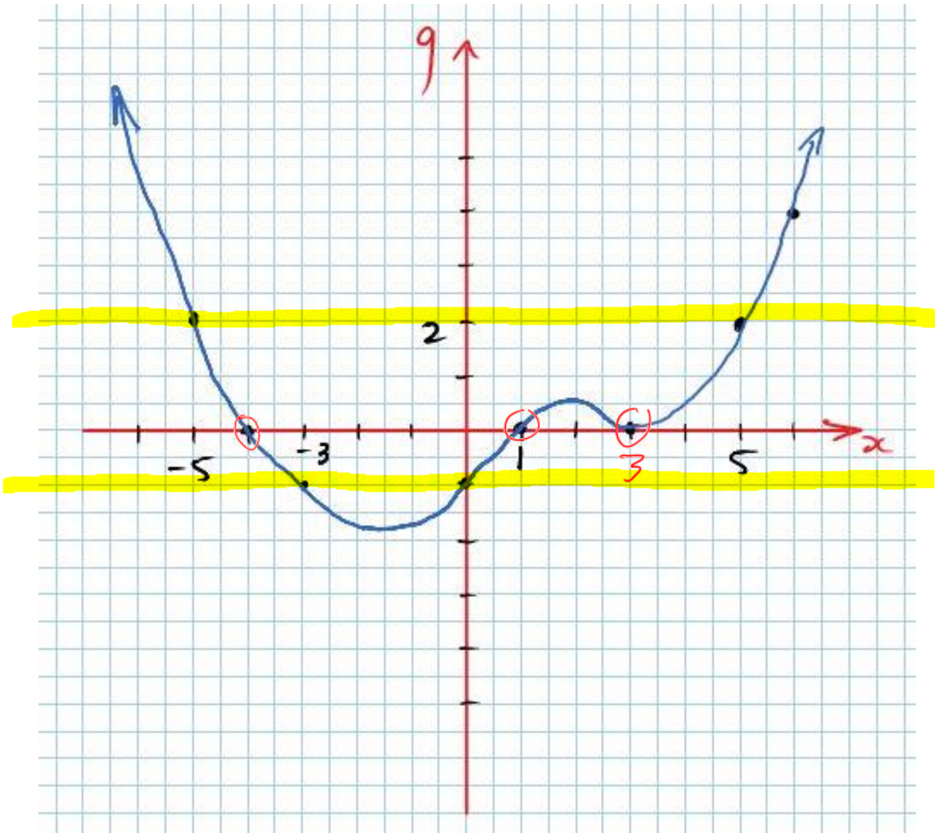


Figure 3.2.6

Success Criteria:

- I can solve a linear inequality by using inverse operations
- I can recognize that when you multiply/divide by a negative number, you MUST reverse the inequality sign
- I can recognize that linear inequalities have many solutions
- I can express the solution to a linear inequality on a number line

3.3 Solving Polynomial Inequalities

Learning Goal: We are learning to solve polynomial inequalities.

For this section, no opening statements are required….

Non-Required Opening Statement

Solving non-linear polynomial inequalities can be accomplished in two ways:

1) Graphically (sometimes called Geometrically)
2) Algebraically (which tends to be more useful)

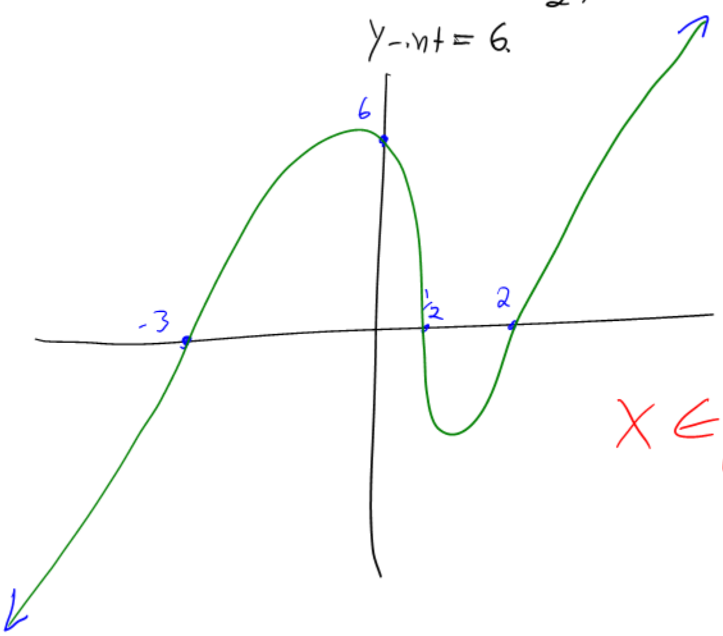
Example 3.3.1

Solve $(2x-1)(x-2)(x+3) \geq 0$ **REMEMBER**: ***F**ACTORED **F**ORM **I**S **Y**OUR **F**RIEND*

Graphically: LT: $2x^3$ 

Zeros: $x = \frac{1}{2}, 2, -3$

y-int = 6



Note: Solving an inequality graphically is rather easy, **BUT** solving algebraically requires more work

$x \in \left[-3, \frac{1}{2}\right] \cup [2, \infty)$

Example 3.3.1 (Continued)

Solve $(2x-1)(x-2)(x+3) \ge 0$

Algebraically

For this technique we will construct an "**Interval Chart**", which can also be thought of as a "**table of signs**" (and wonders?)

Note: It is often helpful to remember that in mathematics we are dealing with **NUMBERS**.

Numbers have signs: Positive or Negative

e.g. $(x-2)$ is a **NUMBER** whose sign switches from +'ve to –'ve at $x=2$ (*i.e. the sign switches at the zero of the factor*)

The Interval Chart looks like:

Intervals	Split the Domain $(-\infty,\infty)$ at all **ZEROS** of the Factors		
Test Values	Choose a Domain	value inside each	Interval
Sign on 1st Factor			
Sign on 2nd Factor			
Sign on 3rd Factor			
Sign on the Product of Factors	Find the Intervals	with the **sign** we	want to answer the question

zeros, $x = -3, \frac{1}{2}, 2$

For our problem above, our chart will look like:

Intervals	$(-\infty,-3)$	$(-3,\frac{1}{2})$	$(\frac{1}{2},2)$	$(2,\infty)$
Test Values	-5	0	1	10
$(x+3)$	$-$	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
$(2x-1)$	$-$	$-$	$+$	$+$
Product	$-$	$+$	$-$	$+$

≥ 0

$x \in [-3, \frac{1}{2}] \cup [2, \infty)$

$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Example 3.3.2

Solve algebraically $4x^4 + 16x^3 + x^2 - 39x - 18 < 0$.

Wait a second….where is your friend and mine…

factored form.

Try f(-2) says Jesus.

$= 4(-2)^4 + 16(-2)^3 + (-2)^2 - 39(-2) - 18$

$= 4(16) + 16(-8) + 4 + 78 - 18$

$= 64 - 128 + 4 + 78 - 18$

$= 0 \quad \therefore (x+2)$ is a factor.

-2	4	16	1	-39	-18
		-8	-16	30	18
	4	8	-15	-9	0

$\pm 1, \pm 3, \pm 9$

$\therefore (x+2)\underbrace{(4x^3 + 8x^2 - 15x - 9)}_{g(x)}$

$g(\pm 1) \neq 0 \quad g(-3) = 4(-3)^3 + 8(-3)^2 - 15(-3) - 9$

$= 4(-27) + 8(9) + 45 - 9$

$= -108 + 72 + 45 - 9 = 0!!!$ woot.

$\therefore (x+3)$

-3	4	8	-15	-9
		-12	12	9
	4	-4	-3	0

m: -12 A: -4

$(4x-6)(4x+2)$

$(2x-3)(2x+1)$

$\therefore (x+2)(x+3)(4x^2 - 4x - 3)$

$(x+2)(x+3)(2x-3)(2x+1)$

The zeros are $x = -3, -2, -\frac{1}{2}$ and $\frac{3}{2}$

Intervals	$(-\infty, -3)$	$(-3, -2)$	$(-2, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{3}{2})$	$(\frac{3}{2}, \infty)$
Test Values	-4	-2.5	-1	0	2
$x+3$	$-$	$+$	$+$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$2x+1$	$-$	$-$	$-$	$+$	$+$
$2x-3$	$-$	$-$	$-$	$-$	$+$
Product	$+$	$-$	$+$	$-$	$+$

$$\therefore\ 4x^4+16x^3+x^2-39x-18<0$$

$$\text{when } x \in (-3,-2) \cup \left(-\tfrac{1}{2}, \tfrac{3}{2}\right)$$

Success Criteria:

- I can solve polynomial inequalities algebraically by
  1. Moving all terms to one side of the inequality
  2. Factoring to find the zeros of the corresponding polynomial
  3. Creating a number line, graph, or an interval chart
  4. Determining the intervals on which the polynomial is positive or negative
- I can solve polynomial inequalities graphically