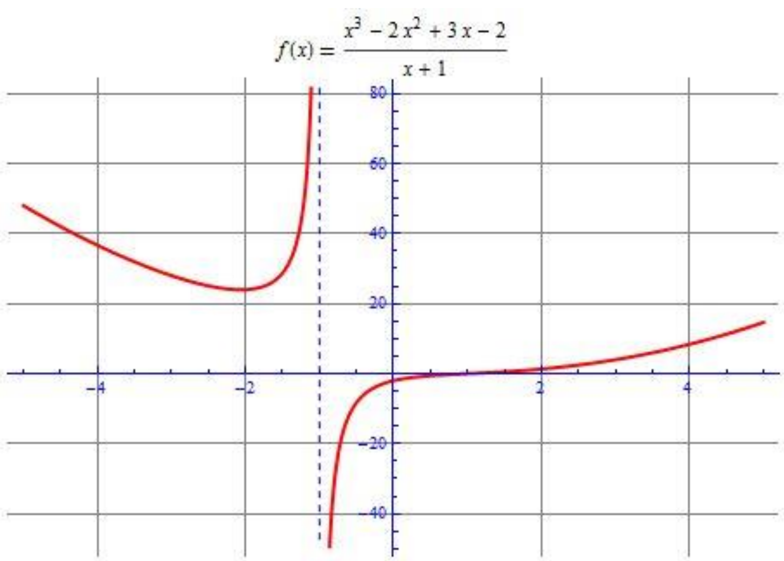
Advanced Functions

Course Notes

Unit 4 – Rational Functions, Equations and Inequalities

We are learning to

- *sketch the graphs of simple rational functions*
- *solve rational equations and inequalities with and without tech*
- *apply the techniques and concepts to solve problems involving rational models*



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Unit 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

4.1 Introduction to Rational Functions and Asymptotes
Pg. 262 #1 - 3

4.2 Graphs of Rational Functions
Pg. 272 #1, 2 (Don’t use any tables of values!), 4 – 6, 9, 10

4.4 Solving Rational Equations
Pg. 285 - 287 #2, 5 – 7def, 9, 12, 13

4.5 Solving Rational Inequalities
Pg. 295 - 297 #1, 3, 4 – 6 (def), 9, 11

4.1 Rational Functions, Domain and Asymptotes

Learning Goal: We are learning to identify the asymptotes of rational functions.

Definition 4.1.1

A **Rational Function** is of the form

$f(x) = \frac{p(x)}{q(x)}$, $\boxed{q(x) \neq 0}$ and both $p(x)$ and $q(x)$ are polynomial functions

e.g. $f(x) = \frac{3x^2 - 5x + 1}{2x - 1}$ is a rational function

$g(x) = \frac{\sqrt{2x+5}}{3x-2}$ not because $\sqrt{2x+5}$ is not a polynomial

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Domain

Definition 4.1.2

Given a rational function $f(x) = \frac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f = \{x \in \mathbb{R} \mid \underbrace{q(x) \neq 0}\}$

$\hookrightarrow$ zeros of $q(x)$.

Example 4.1.1

Determine the natural domain of $f(x) = \frac{x^2 - 4}{x - 3}$.

$x = 3$ is the zero of $q(x)$

$x \neq 3$

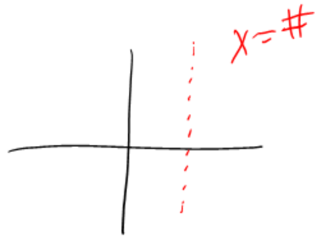
$D_f : \{x \in \mathbb{R} \mid x \neq 3\}$

$x \in (-\infty, 3) \cup (3, \infty)$

Asymptotes

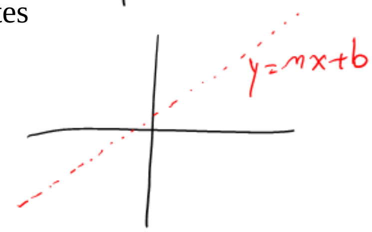
There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes



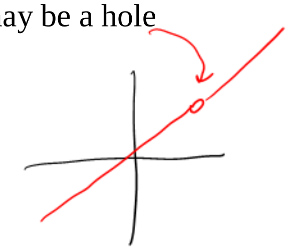
2) Horizontal Asymptotes

3) Oblique Asymptotes



Vertical Asymptotes

A rational function $f(x)=\dfrac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x)=0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.



Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x)=\dfrac{5x}{x^2-x-6}$

Factor the whole thing.

$$f(x)=\frac{5x}{(x-3)(x+2)}$$

$x=3$ V.A $\quad x=-2$ V.A.

If the factors stay, they are Vertical Asymptotes

$$x \in (-\infty,-2)\cup(-2,3)\cup(3,\infty)$$

b) $h(x)=\dfrac{x+3}{x^2-9}$

$$h(x)=\frac{\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$$

$x=3$ V.A.

$x=-3$ Hole because it disappears.

$$h(x)=\frac{1}{x-3}$$

$$x\in(-\infty,-3)\cup(-3,3)\cup(3,\infty)$$

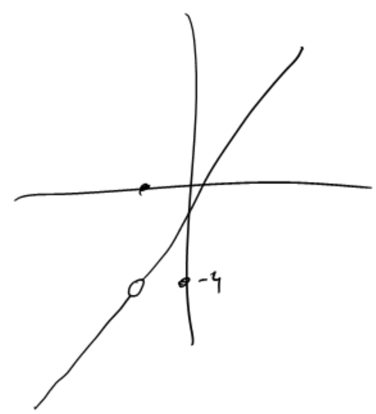
c) $g(x)=\dfrac{x^2-4}{x+2}$

$$g(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

$x=-2$ → Hole

$$g(x)=x-2$$

$$x\in(-\infty,-2)\cup(-2,\infty)$$



Horizontal Asymptotes

Here we are concerned with the end behavior of the rational function

i.e. We are asking, given a rational function $f(x) = \frac{p(x)}{q(x)}$, how is $f(x)$ behaving as $x \to \pm\infty$.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $\overset{\text{top}}{p(x)}$ $>$ Order of $\overset{\text{bottom}}{q(x)}$

   e.g. $f(x) = \frac{x^3 - 2}{x^2 + 1}$

   When the order of the top ~~&~~ is bigger than the bottom there is $\boxed{\text{no}}$ horizontal asymptote.

2) Order of numerator $=$ Order of denominator

   e.g. $f(x) = \frac{2x^2 - 3x + 1}{3x^2 + 4x - 5}$

If x is MASSIVE, the stuff behind the leading terms are inconsiquentual/irrelevant

what's left is $\frac{2\cancel{x^2}}{3\cancel{x^2}} = \boxed{\frac{2}{3} = y}$ is the horizontal asymptote.

   e.g. Determine the horizontal asymptote of $g(x) = \frac{3x - 4x^5}{5x^5 + 2x - 1}$

$$H.A = y = \frac{-4}{5}$$

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x)=\dfrac{x^2-5x+6}{x^5+7}$

$$\frac{100^2}{100^5}=\frac{10,000}{10,000,000,000}=\frac{1}{\text{Really Big}}=\text{close to zero.}$$

$f(x)=\dfrac{1}{x}$

H.A is $y=0$.

Oblique Asymptotes

These occur when the order of p(x) is exactly one bigger than p(x).

e.g. $f(x)=\dfrac{x^2-2x+3}{x+1}$ no H.A.

With Oblique Asymptotes we are still dealing with end behaviors.

O.A. have the form $y=mx+b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: By polynomial division!
Then, the O.A. is the quotient.

$f(x)=\dfrac{x^2-2x+3}{x+1}$

$x=-1$

$$\begin{array}{r|rrr} -1 & 1 & -2 & 3 \\ & & -1 & 3 \\ \hline & \underbrace{1 \quad -3}_{\text{quotient}} & & 6 \end{array}$$

$g(x)=\dfrac{x^4-3x^2+5x-2}{x^3+\ldots}$

$$x^3\ldots \overline{\left)\, x^4 \ldots \right.} \quad \overset{\text{OA}}{\boxed{x+\#}}$$

The O.A. is $y=1x-3$

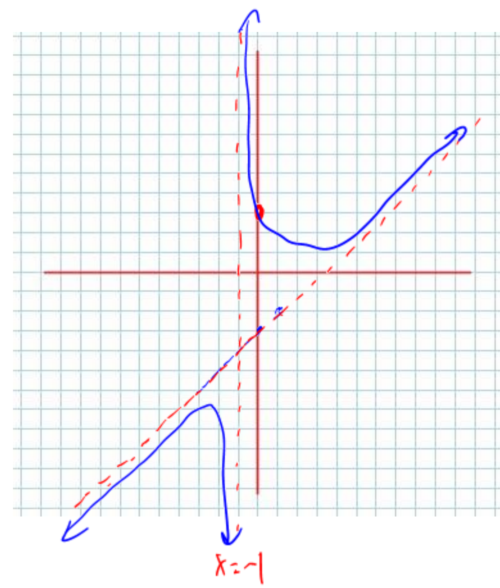
(Rough) Sketch of $f(x)=\dfrac{x^2-2x+3}{x+1}$

V.A. $x = -1$

H.A. none

O.A. $y = \overset{m}{}x \overset{b}{-3}$

$f(0) = \frac{3}{1} = 3$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x)=\dfrac{x+2}{x^2+3x+2}$ order 1 / order 2

Step one: Factor

$f(x) = \dfrac{\cancel{x+2}}{(\cancel{x+2})(x+1)}$

$f(x) = \dfrac{1}{x+1}$

V.A.	$x=-1$
Hole	$x=-2$
H.A.	$y=0$
O.A.	None

b) $g(x)=\dfrac{4x^2-25}{x^2-9}$

$g(x) = \dfrac{(2x+5)(2x-5)}{(x-3)(x+3)}$

V.A	$x=-3, 3$
Hole	none
H.A	$y=4$
O.A	None

c) $h(x)=\dfrac{1x^2+0+0}{x+3}$ order 2, order 1

$$\begin{array}{r|rrr} -3 & 1 & 0 & 0 \\ & & -3 & +9 \\ \hline & 1 & -3 & 9 \end{array}$$

V.A.	$x=-3$
Hole	none
H.A.	none
O.A.	$y=1x-3$

Example 4.1.4

→ denominator have $(x+3)$

Determine an equation for a function with a vertical asymptote at $x=-3$, and a horizontal asymptote at $y=0$.

→ order of $q(x)$ is larger.

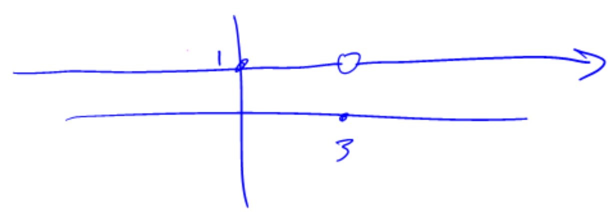
$$f(x)=\frac{5}{(x+3)}$$

Example 4.1.5

$(x-3)$

Determine an equation for a function with a hole discontinuity at $x=3$.

$$g(x)=\frac{(x-3)}{(x-3)}$$



Success Criteria:

- I can identify a hole when there is a common factor between *p(x)* and *q(x)*
- I can identify a vertical asymptote as the zeros of *q(x)*
- I can identify a horizontal asymptote by studying the degrees of *p(x)* and *q(x)*
- I can identify an oblique asymptote when the degree of *p(x)* is exactly 1 greater than *q(x)*

4.2 Graphs of Rational Functions

Learning Goal: We are learning to sketch the graphs of rational functions.

Note: In Advanced Functions we will only consider rational functions of the form

$$f(x)=\frac{ax+b}{cx+d}$$

Rational Functions of the form $f(x)=\frac{ax+b}{cx+d}$ will have:

1) One Vertical Asymptote comes from the denominator

$cx+d=0 \quad \therefore VA = \frac{-d}{c}$

2) One Zero (unless $a=0$)

comes from numerator

$ax+b=0 \quad \therefore$ zero is $x=\frac{-b}{a}$

3) Functional Intercept
y

$f(0)=\frac{a(0)+b}{c(0)+d}=\frac{b}{d} \qquad \left(0, \frac{b}{d}\right)$

4) A Horizontal Asymptote the orders are the same

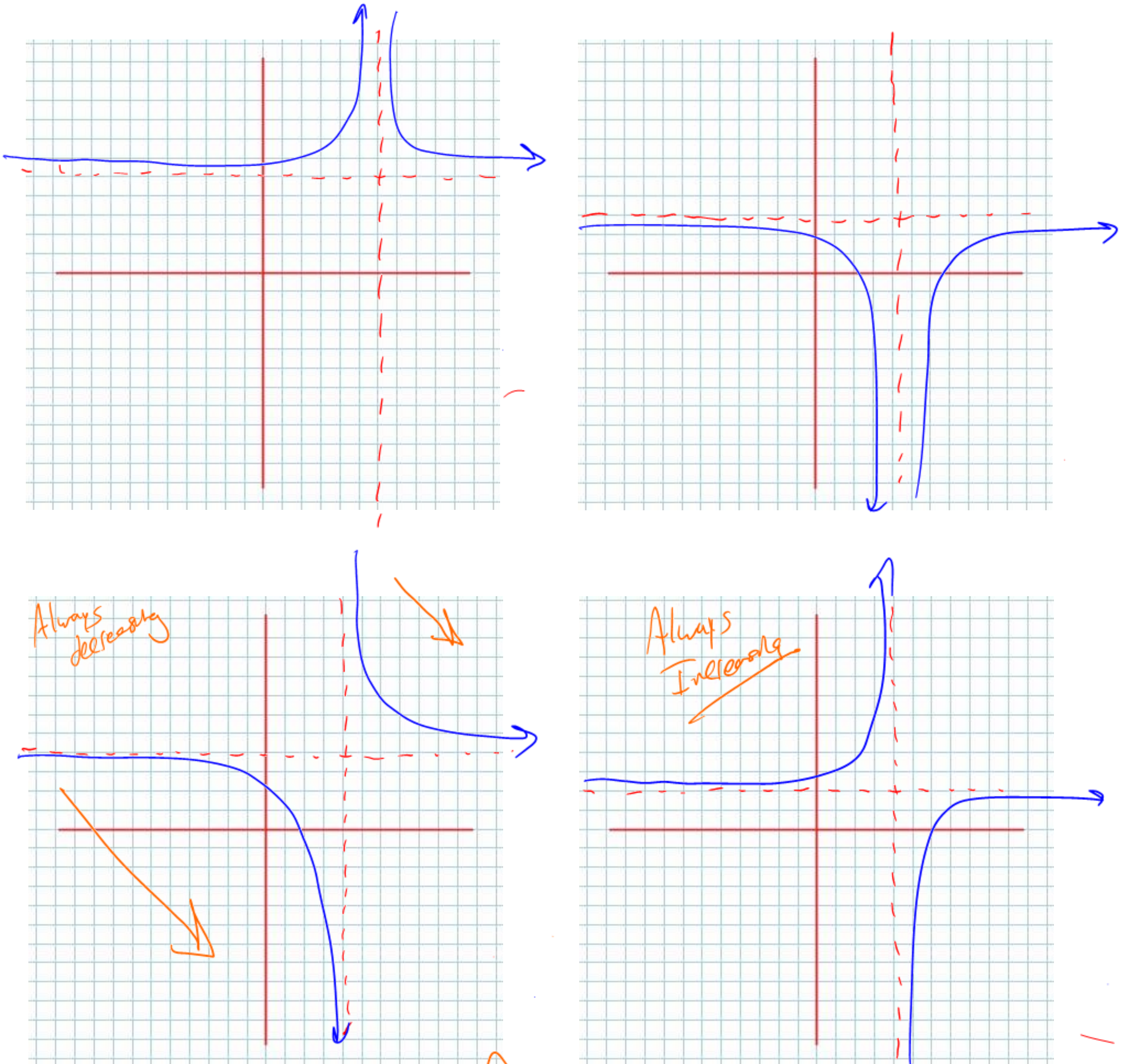
$\therefore$ H.A. is $y=\frac{a}{c}$. If $a=0$, H.A. $y=0$.

5) These functions will always be either

always increasing or always decreasing

Functional Behaviour Near A Vertical Asymptote

There are **FOUR** possible functional behaviours near a V.A.:



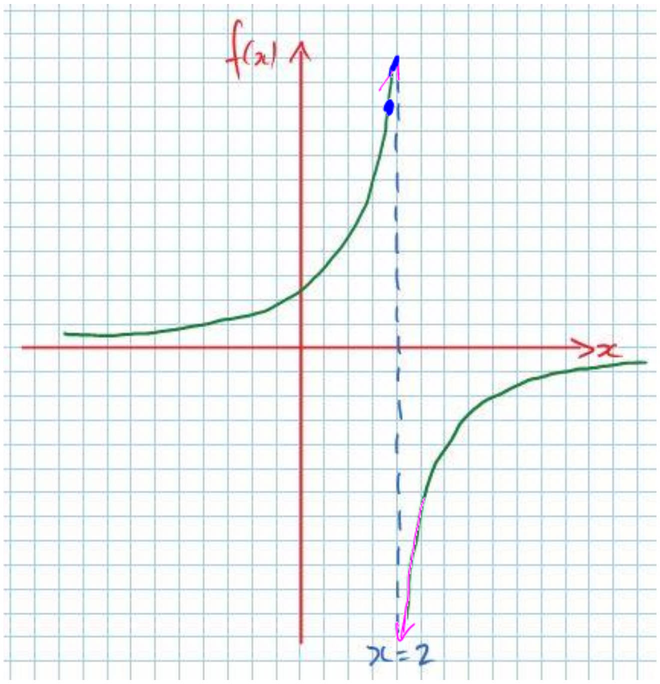
For functions of the form $f(x)=\dfrac{ax+b}{cx+d}$ we will see behaviours

The questions is, **how do we know which**?

We need to **analyze** the function **near the V.A**.

We need to become familiar with some **Notation.**

Consider some rational function with a sketch of its graph which looks like:



end behavior: $x \to -\infty$

$f(x) \to \dots$

$x \to 2^-$ Left side

$f(x) \to \infty$

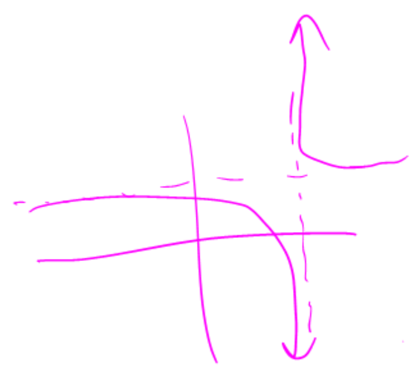
$x \to 2^+$ right side

$f(x) \to -\infty$

Example 4.2.1

Determine the functional behaviour of $f(x) = \dfrac{2x+1}{x-3}$ near its V.A.

$x = 3$



$x \to 3^-$

$f(x) \to \cancel{?}\ -\infty$

Test: $x = 2.99$ and $x = 2.999$

$f(2.99) = -698$ $\quad f(2.999) = -6998$

$x \to 3^+$

$f(x) \to ?\ \infty$

Test $x = 3.01$ $\quad x = 3.001$

$f(3.01) = 702$ $\quad f(3.001) = 7002$

We now have the tools to sketch some graphs!

Example 4.2.2

Sketch the graph of the given function. State the domain, range, intervals of increase/decrease and where the function is positive and negative.

a) $f(x) = \dfrac{2x+1}{x-1}$

H.A.: $y = \frac{2}{1} = 2$

V.A: $x = 1$

x-int: $x = \frac{-1}{2}$

y-int: $y = \frac{1}{-1} = -1$

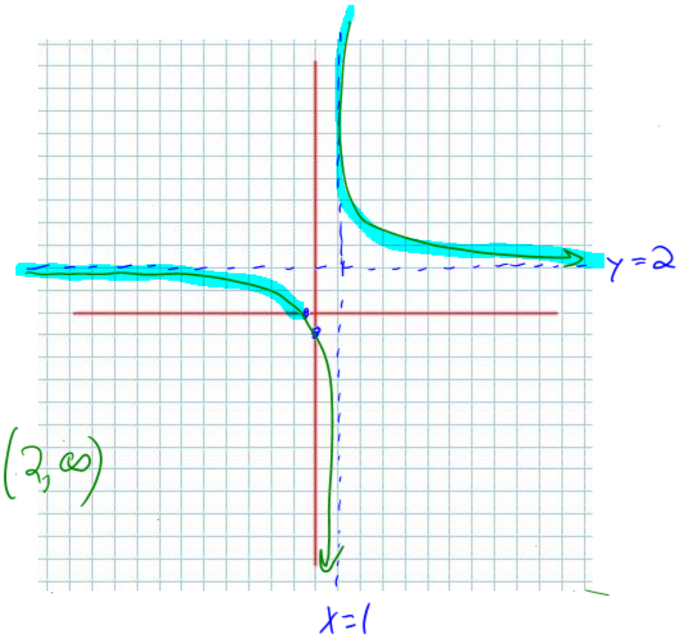
$D_f: x \in (-\infty, 1) \cup (1, \infty)$

$R_f: f(x) \in (-\infty, 2) \cup (2, \infty)$

f(x) is always decreasing.

$f(x) > 0$ when $x \in (-\infty, -\frac{1}{2}) \cup (2, \infty)$

$f(x) < 0$ when $x \in (-\frac{1}{2}, 1)$



b) $g(x) = \dfrac{3x-2}{2x+5}$

H.A: $y = \frac{3}{2}$

V.A.: $x = \frac{-5}{2}$

x-int: $x = \frac{2}{3}$

y-int: $y = \frac{-2}{5}$

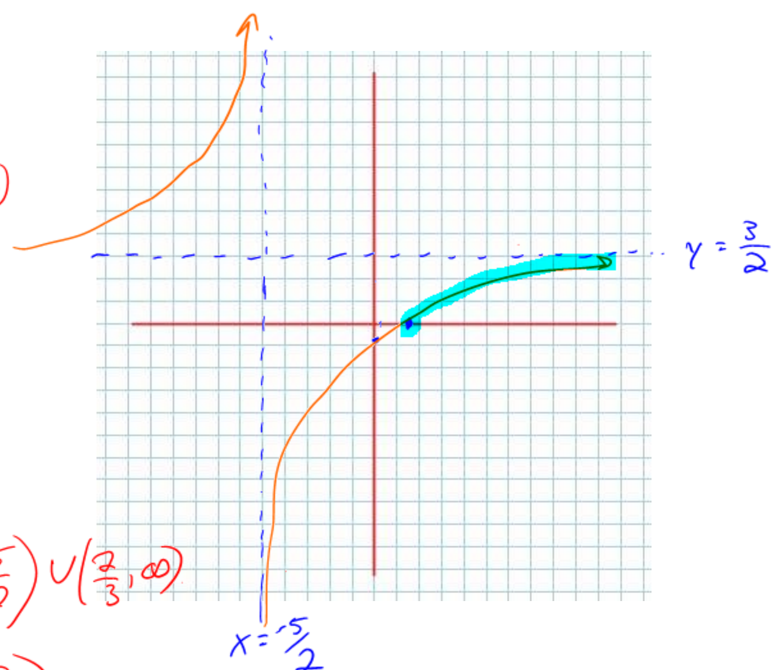
$D_g: x \in (-\infty, \frac{-5}{2}) \cup (\frac{-5}{2}, \infty)$

$R_g: g(x) \in (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

g(x) is always increasing.

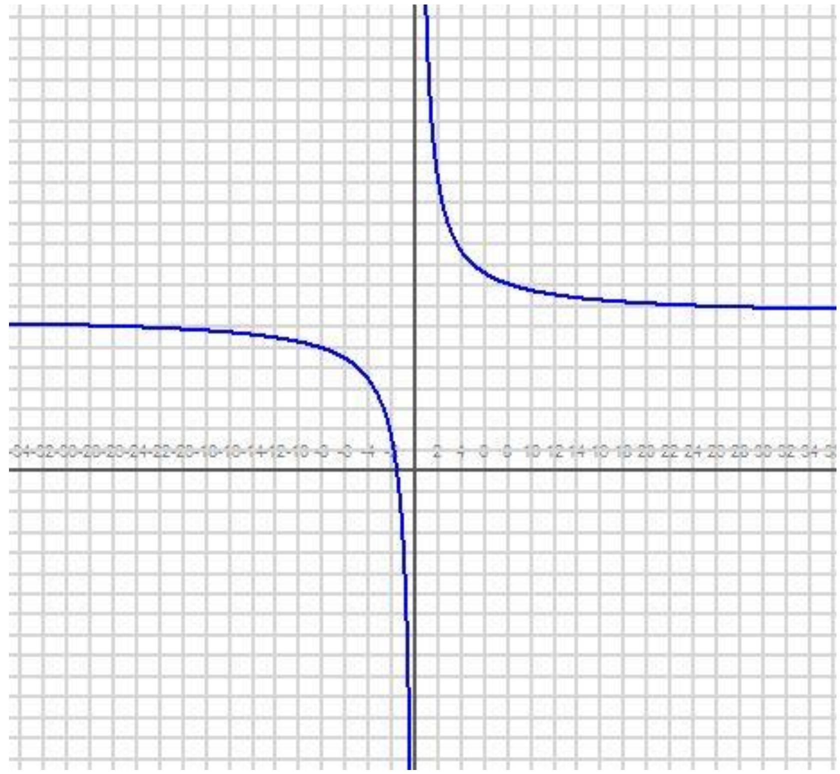
$g(x) > 0$ when $x \in (-\infty, \frac{-5}{2}) \cup (\frac{2}{3}, \infty)$

$g(x) < 0$ when $x \in (\frac{-5}{2}, \frac{2}{3})$



Example 4.2.3

Consider question #9 on page 274: $I(t) = \dfrac{15t + 25}{t}$



Success Criteria:

- I can identify the horizontal asymptote as $\frac{a}{c}$
- I can identify the vertical asymptote as $-\frac{d}{c}$
- I can identify the y-intercept as $\frac{b}{d}$
- I can identify the x-intercept as $-\frac{b}{a}$

4.4 Solving Rational Equations

Learning Goal: We are learning to solve rational equations. Think rationally!

Solving a Rational Equation is **VERY MUCH** like solving a Polynomial Equation. Thus, this stuff is so much fun it should be illegal. But it isn't illegal unless you break a rule of algebra. Math Safe!

KEY (*this is a major key for you music buffs*)

Multiplying by the Multiplicative Inverse of the Common Denominator is wonderful to use **WHEN YOU HAVE** something like:

$$\text{RATIONAL}_1 + \text{RATIONAL}_2 = \text{RATIONAL}_3$$

$$\boxed{(x-2)(2)(x)}$$

e.g. $$\frac{3}{x-2} + \frac{3}{2} = \frac{4(x+5)}{x}$$

$$\frac{(x-2)(2)(x)}{1}\left(\frac{3}{x-2}\right) + \frac{(x-2)(2)(x)}{1}\left(\frac{3}{2}\right) = \frac{(x-2)(2)(x)}{1}\left(\frac{4(x+5)}{x}\right)$$

$$6x + 3x(x-2) = 8(x-2)(x+5)$$

Expand, polynomial = 0, then solve.

Make Sure To Keep **RESTRICTIONS ON X** In Mind

like the V.A.

This means that restrictions cannot be solutions

Example 4.4.1

a) Solve $\frac{x}{5} = \frac{9}{18}$ $(5)(18)$

No restrictions

$18x = 45$

$\frac{18x}{18} = \frac{45}{18}$

$x = \frac{5}{2}$

b) Solve $\frac{1}{x} - \frac{5x}{3} = \frac{2}{5}$ $(3)(5)(x)$

RESTRICTIONS

$x \neq 0$ | $C(1)$. $(3)(5)(x)$

$15 - 25x^2 = 6x$

$0 = 25x^2 + 6x - 15$

by technology, or Quad Formula

$x = -0.9$ and $x = 0.66$

c) Solve $\left(\frac{3}{x}+\frac{4}{x+1}=\frac{2}{1}\right)(x)(x+1)$

RESTRICTIONS

$x \neq 0$
$x \neq -1$

CD: $(x)(x+1)$

$3(x+1)+4(x)=2x(x+1)$

$\overset{-3x}{3x}+3+\overset{-4x}{4x}=2x^2+\overset{-7x}{2x}$

$0=2x^2-5x-3$

$0=(x-3)(2x+1)$

M: -6 A: -5

$-6, +1$

$(2x-6)(2x+1)$

$(x-3)(2x+1)$

$x=3$ ✓, $x=\frac{-1}{2}$ ✓

d) Solve $\left(\frac{10}{\underbrace{x^2-2x}_{x(x-2)}}+\frac{4}{x}=\frac{5}{x-2}\right)(x)(x-2)$

RESTRICTIONS

$x \neq 0$
$x \neq 2$

CD: $(x)(x-2)$

$10+4(x-2)=5x$

$10+4x-8=5x$

$2=x$

However, $x \neq 2$ $\therefore$ there are no solutions to this equation.

e) Solve $\left(\frac{16x}{1}-\frac{5}{x+2}=\frac{15}{x-2}-\frac{60}{(x-2)(x+2)}\right)(x-2)(x+2)$

Restrictions: $x \neq -2$, $x \neq 2$

CD: $(x-2)(x+2)$

$16x(x-2)(x+2)-5(x-2)=15(x+2)-60$

$16x(x^2-4)-5x+10=15x+30-60$

$16x^3-64x-5x+10=15x-30$

$16x^3-84x+40=0$

$\div 4$

$\underbrace{4x^3-21x+10}_{f(x)}=0$ $\quad 10 \to \pm1, \pm2, \pm5, \pm10$

Test $x=-2$, $f(-2)=4(-2)^3-21(-2)+10$

$=-32+42+10$

$\neq 0$

$f(2)=4(2)^3-21(2)+10$

$=32-42+10$

$=0 \therefore (x-2)$ is a factor.

$$\begin{array}{r|rrrr} 2 & 4 & 0 & -21 & 10 \\ & & 8 & 16 & -10 \\ \hline & 4 & 8 & -5 & 0 \end{array}$$

$\therefore (x-2)(4x^2+8x-5)=0$

$(x-2)(2x+5)(2x-1)=0$

M: -20 A: 8

$10, -2$

$(4x+10)(4x-2)$

$(2x+5)(2x-1)$

$\therefore$ ~~$x=2$~~ inadmissable

$x=\frac{-5}{2}$

$x=\frac{1}{2}$

Example 4.4.2

From your Text: Pg. 285 #10

The Turtledove Chocolate factory has two chocolate machines. Machine A takes *m* minutes to fill a case with chocolates, and machine B takes $m + 10$ minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

Work or rate of work problem. $\rightarrow$ $\frac{\text{a job}}{\text{time}} = \frac{\text{filling } \boxed{a} = 1 \text{ case}}{\text{minutes}}$

Machine A: $\frac{1}{m}$ Machine B: $\frac{1}{m+10}$ Together: $\frac{1}{15}$

$$\left(\frac{1}{m} + \frac{1}{m+10} = \frac{1}{15}\right)(m)(m+10)(15)$$

Restrictions: $m \neq 0$, $m \neq -10$

CD: $(m)(m+10)(15)$

$$15(m+10) + 15m = m(m+10)$$

$$15m + 150 + 15m = m^2 + 10m$$

$$0 = m^2 - 20m - 150 \quad \text{D.N.F.}$$

by QF or ~~tech~~ tech

$m = 25.8$

∴ Machine A takes 25.8 to fill a case and Machine B takes 35.8 minutes to fill a case.

Success Criteria:

- I can recognize that the zeros of a rational function are the zeros of the numerator
- I can solve rational equations by multiplying each term by the lowest common denominator, then solving the resulting polynomial equation
- I can identify inadmissible solutions based on the context of the problem

4.5 Solving Rational Inequalities

Learning Goal: We are learning to solve rational inequalities using algebraic and graphical approaches.

The joy, wonder and peace these bring is really quite amazing

e.g. Solve $7\left(\frac{x-2}{7} \geq 0\right)$

$x-2 \geq 0$

$x \geq 2$

Note: For Rational Inequalities, with a variable in the denominator, you **CANNOT** multiply by the multiplicative inverse of the common denominator!!!!

Why? If the factor, $x+3$, is negative, the inequality would need to flip.

Example 4.5.1

Solve $\frac{x-2}{x+3} \geq 0$

We solve by using an Interval Chart

For the intervals, we split $(-\infty,\infty)$ at all zeros (**where the numerator is zero**), and all restrictions (**where the denominator is zero**) of the (SINGLE) rational expression. Keep in mind that it may take a good deal of algebraic manipulation to get a SINGLE rational expression…

zero at $x=2$ and a restriction at $x=-3$

Interval	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
Test Value	-4	0	3
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
Ratio	$+$	$-$	$+$

$\therefore \frac{x-2}{x+3} \geq 0$ when $x \in (-\infty, -3) \cup [2, \infty)$

Example 4.5.2

Solve $\frac{1}{x+5} < 5$

DO NOT CROSS MULTIPLY (or else)

- Get everything on one side
- Simplify into a single Rational Expression using a common denominator
- Interval Chart it up

$$\frac{1}{x+5} - \frac{5(x+5)}{1(x+5)} < 0$$

$$\frac{1-5(x+5)}{x+5} < 0$$

$$\frac{1-5x-25}{x+5} < 0$$

$$\frac{-5x-24}{x+5} < 0$$

zero at $x = \frac{-24}{5} = -4.8$

restriction at $x = -5$

Intervals	$(-\infty, -5)$	$(-5, -4.8)$	$(-4.8, \infty)$
T.V.	-6	-4.9	0
$-5x-24$	$+$	$+$	$-$
$x+5$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$

The chart solves the single R.E.

$\therefore \frac{1}{x+5} < 0$ when $x \in (-\infty, -5) \cup (-4.8, \infty)$

Example 4.5.3

Solve $\dfrac{x^2+3x+2}{x^2-16} \geq 0$

FACTORED FORM IS YOUR FRIEND

$$\frac{(x+1)(x+2)}{(x+4)(x-4)} \geq 0$$

Zeros at $x=-1,-2$

Restrictions at $x=-4,4$

Intervals	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
T.V.	-5	-3	-1.5	0	5
$(x+1)$	$-$	$-$	$-$	$+$	$+$
$(x+2)$	$-$	$-$	$+$	$+$	$+$
$(x-4)$	$-$	$-$	$-$	$-$	$+$
$(x+4)$	$-$	$+$	$+$	$+$	$+$
Ratio	$+$	$-$	$+$	$-$	$+$

$\therefore \dfrac{x^2+3x+2}{x^2-16} \geq 0$ when $x \in (-\infty, -4) \cup [-2, -1] \cup (4, \infty)$

$$3x+5 \geq 4x-8$$
$$13 \geq x$$

Larger side/higher order.

Example 4.5.4

Solve $\dfrac{3}{x+2} \leq x$

$$0 \leq \frac{x(x+2)}{1(x+2)} - \frac{3}{x+2}$$

$$\frac{-x^2-2x+3}{x+2} \leq 0$$

$$\frac{-1(x^2+2x-3)}{x+2} \geq 0 \quad \div -1$$

$$0 \leq \frac{x^2+2x-3}{x+2}$$

$$0 \leq \frac{(x+3)(x-1)}{x+2}$$

Zeros at $x = 1, -3$

Restriction at $x = -2$

Intervals	$(-\infty, -3)$	$[-3, -2)$	$(-2, 1]$	$[1, \infty)$
T.V.	-4	-2.5	0	2
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$	$+$

$\therefore \dfrac{3}{x+2} \leq x$ when $x \in [-3, -2) \cup [1, \infty)$

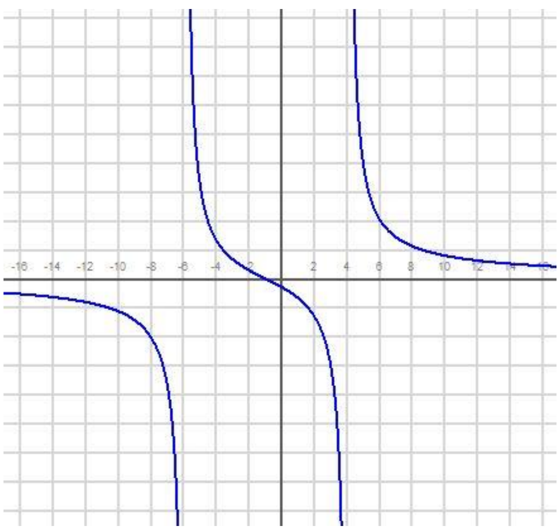
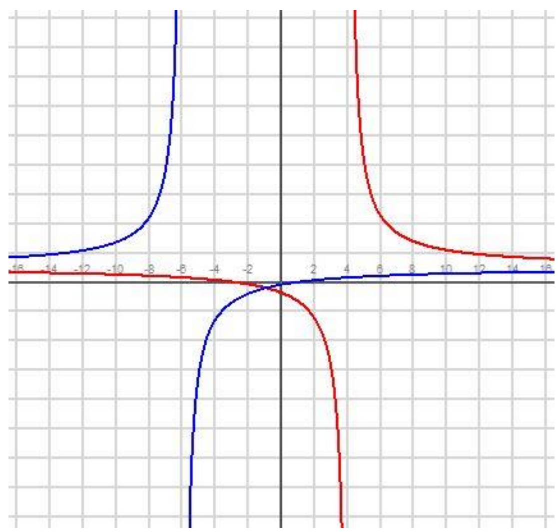
Example 4.5.5

From your Text: Pg. 296 #6a

Using **Graphing Tech**

Solve $\frac{x+3}{x-4} \geq \frac{x-1}{x+6}$

Note: There are **TWO** methods, both of which require a **FUNCTION** (let $f(x) = \ldots$ returns)

1) Get a Single Function (on one side of the inequality)	2) Use Two Functions (one for each side)
	

Success Criteria:

- I can recognize that an inequality has many possible intervals of solutions
- I can solve an inequality algebraically, using an interval chart
- I can solve an inequality graphically