Advanced Functions

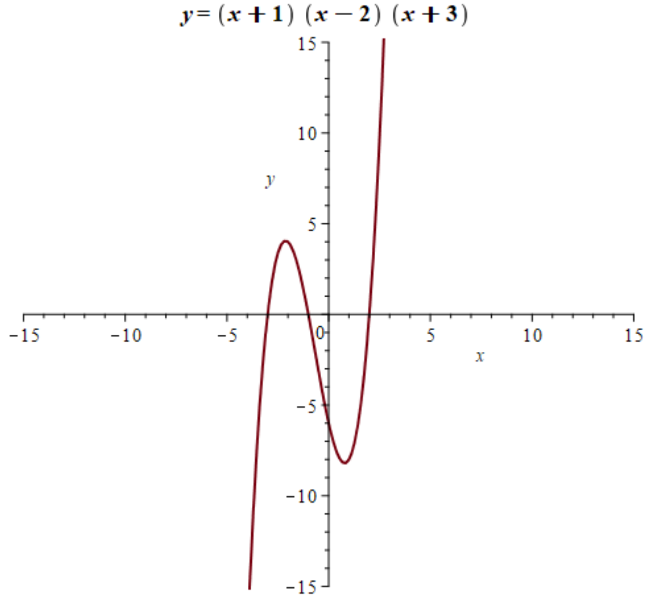
Course Notes

Chapter 2 – Polynomial Functions

***Learning Goals**: We are learning*

- *The algebraic and geometric structure of polynomial functions of degree three and higher*
- *Algebraic techniques for dividing one polynomial by another*
- *Techniques for using division to FACTOR polynomials*
- *To solve problems involving polynomial equations and inequalities*

$y = (x+1)(x-2)(x+3)$



A cubic polynomial

A∞Ω
MATH@TD

Chapter 2 – Polynomial Functions

Contents with suggested problems from the Nelson Textbook (Chapter 3)

2.1 Polynomial Functions: An Introduction

Learning Goal: We are learning to identify polynomial functions.

Definition 2.1.1

A **Polynomial Function** is of the form

constant term.

$$f(x) = \boxed{a_n x^n} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_2 x^2 + a_1 x^1 + \boxed{a_0 x^0}$$

→ n is a positive integer

→ a_n is the coefficient for the n^{th} term.

→ all exponents are positive integers

Examples of Polynomial Functions

a) $f(x) = 8x^4 - 5x^3 + 2x^2 + 3x - 5$

$a_4 = 8 \quad a_2 = 2 \quad a_0 = -5$

b) $g(x) = 7x^6 - 4x^3 + 3x^2 + 2x^1$

$a_6 = 7 \quad a_4 = 0 \quad a_1 = 2 \quad a_0 = 0$

Notes: The **TERM** $a_n x^n$ in any polynomial function (where n is the **highest power** we see) is

called the | leading term | , and then we write all the following terms

in | descending order | .

The leading term has two components:

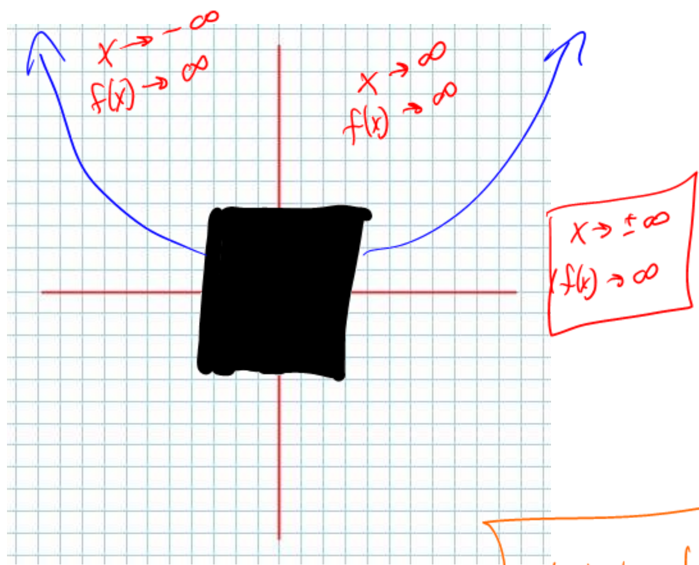
1) Leading coefficient, a_n, is either positive or negative.

2) n, the highest order/degree, it can be odd or even.

The Leading term tells us the **end behaviour** of the polynomial function.

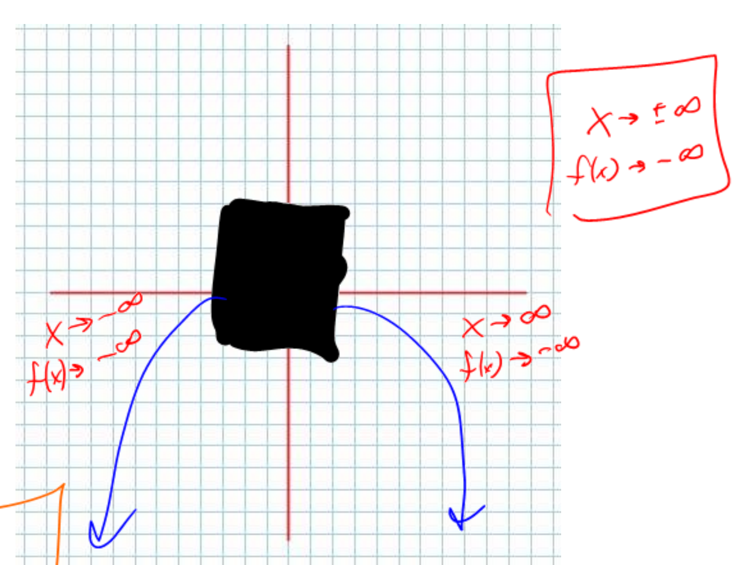
☆ all polynomial functions have 4 possible end behaviour

Pictures

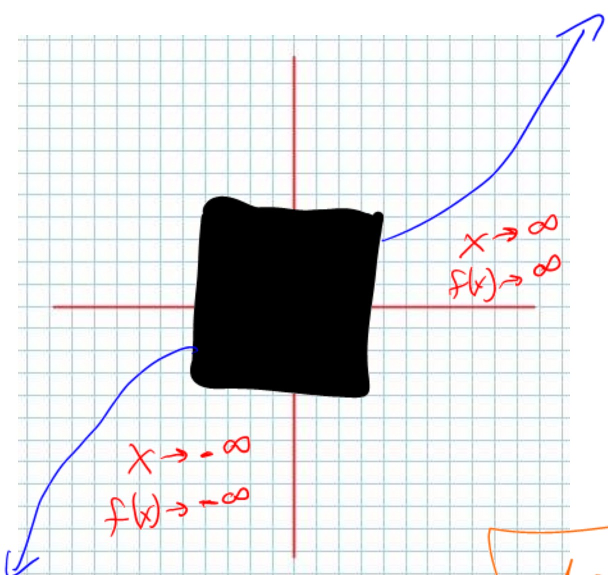


n is even
a_n is positive

☆ Think of a parabola

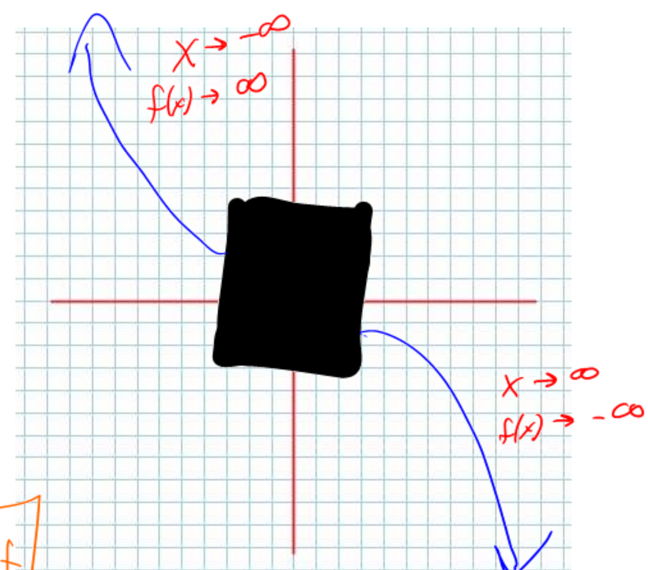


n is even
a_n is negative



n is odd
a_n is positive

☆ Think of a line



n is odd
a_n is negative

Definition 2.1.2

The **order** of a polynomial is the value of the highest power, or just the degree of the leading term.

ex: $g(x) = 2x^3 + 3x^2 \boxed{- 8x^5} + 1$

The order of $g(x)$ is 5

Determine the end behavior of:

$h(x) = 2(x-3)^2(2x+8)^3(4x+5)$

aside: $(x^2 + x - 1)^5$

All we need is the Leading Term.

$2(x)^2(2x)^3(4x)$

$= 2(x^2)(8x^3)(4x)$

$= 64x^6$ → even

↳ positive

$\therefore$ $x \to -\infty$, $f(x) \to \infty$; $x \to \infty$, $f(x) \to \infty$

Success Criteria:

- I can justify whether a function is polynomial or not
- I can identify the degree of a polynomial function
- I can recognize that the domain of a polynomial is the set of all real numbers
- I can recognize that the range of a polynomial function may be the set of all real numbers, or it may have an upper/lower bound
- I can identify the shape of a polynomial function given its degree

2.2 Characteristics (Behaviours) of Polynomial Functions

Today we open, and look inside the black box of mystery

Learning Goal: We are learning to determine the turning points and end behaviours of polynomial functions.

Consider the sketch of the graph of some function, $f(x)$:

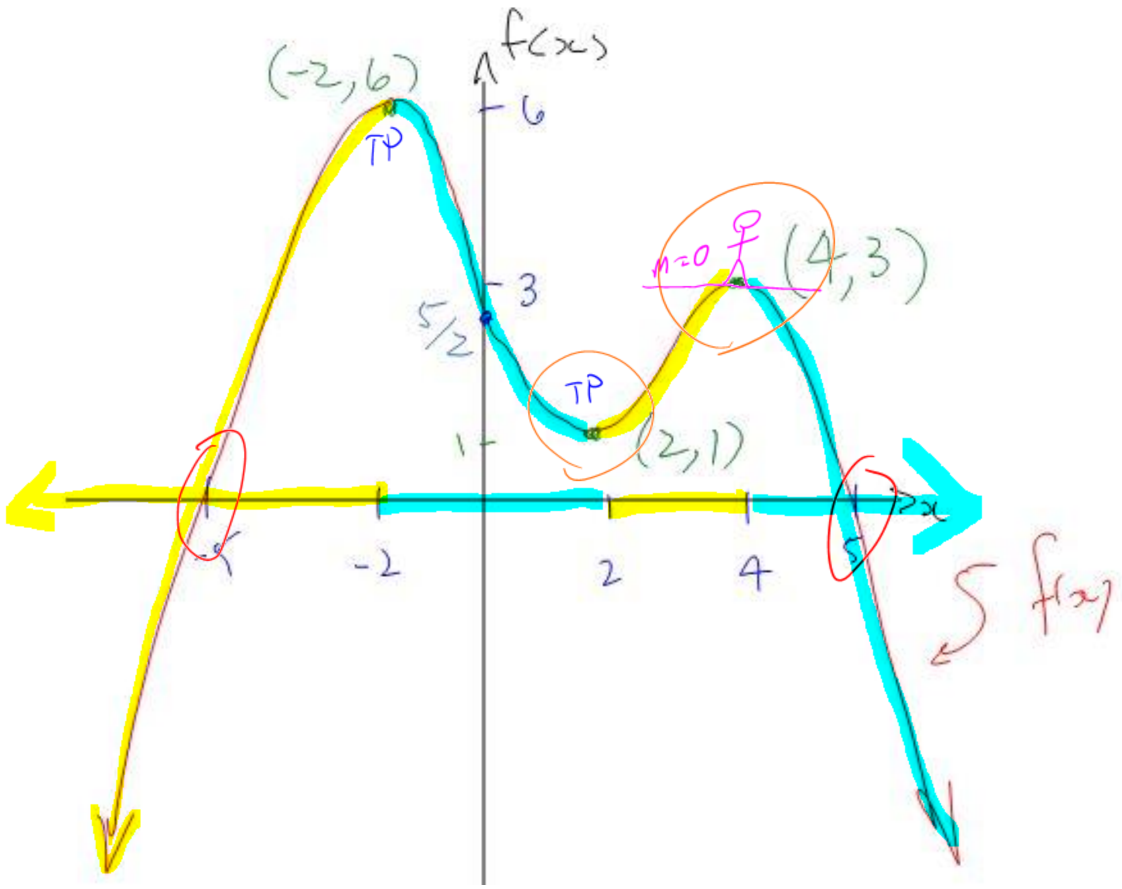


Figure 2.2.1

Observations about $f(x)$:

1) $f(x)$ is a polynomial of even order (degree). The end behaviors are the same.

2) The leading coefficient is negative (going down)

3) $f(x)$ has 3 turning points (where the functional behaviour of INCREASING/DECREASING switches from one to the other.)

4) $f(x)$ has 2 zeros (x-intercepts)

$f(-5)=0$ and $f(5)=0$

Zeros at $x=-5$ and $x=5$

5) $f(x)$ is increasing on $(-\infty,-2)\cup(2,4)$

only refers to the x-values/domain

$f(x)$ is decreasing on $(-2,2)\cup(4,\infty)$

6) $f(x)$ has a global max ~~functional~~ y value of 6.

This max is called global because it is the absolute highest value.

☆ only even polynomial functions have a global max/min.

7) $f(x)$ has a local minimum at $(2,1)$

and a local maximum at $(4,3)$

Consider the sketch of the graph of some function $g(x)$:

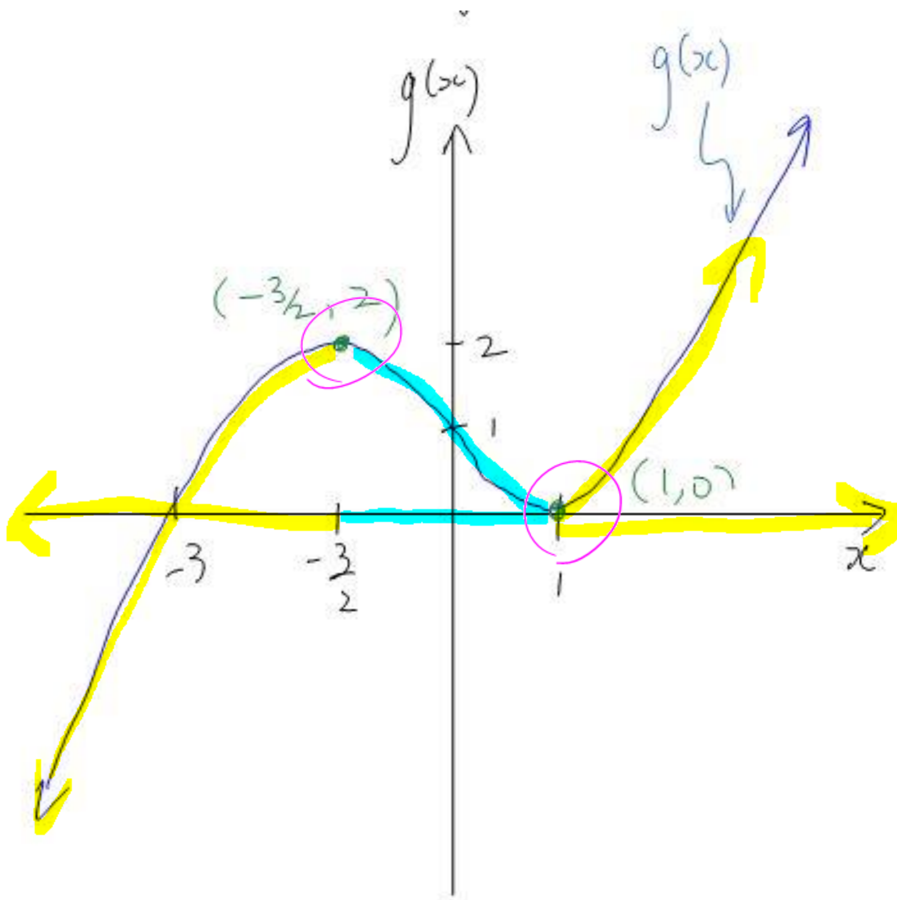


Figure 2.2.2

Observations about $g(x)$:

(1) Local max at $\left(\frac{-3}{2}, 2\right)$ and a local min at $(1, 0)$ → 2 turning points

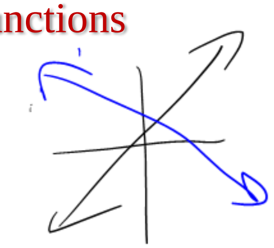
(2) Increasing from $\left(-\infty, \frac{-3}{2}\right) \cup (1, \infty)$
Decreasing from $\left(-\frac{3}{2}, 1\right)$

(3) 2 zeros at $x=-3$ and $x=1$

(4) $g(x)$ is odd. The end behaviors are different

(5) The L.C. is positive

General Observations about the Behaviour of Polynomial Functions



1) The Domain of all Polynomial Functions is $x \in (-\infty, \infty)$

2) The Range of ODD ORDERED Polynomial Functions is
$f(x) \in (-\infty, \infty)$

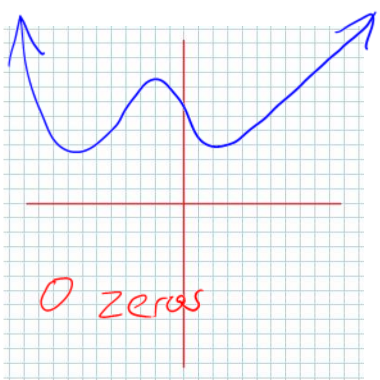
3) The Range of EVEN ORDERED Polynomial Functions
1. The sign of the L.C.
2. The y-value of the global max/min

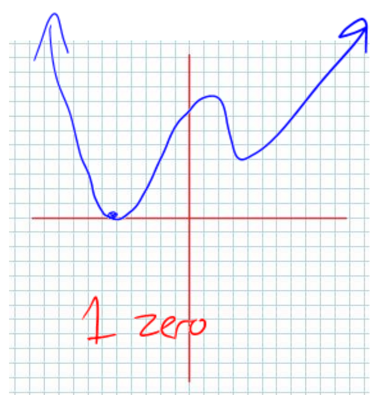
If L.C. > 0, $[\#, \infty)$ $[\text{min}, \infty)$
L.C. < 0, $(-\infty, \#]$ $(-\infty, \text{max}]$

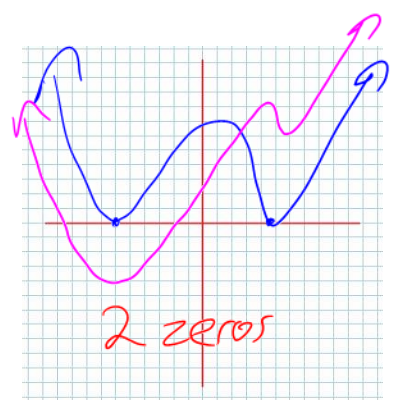
Even Ordered Polynomials

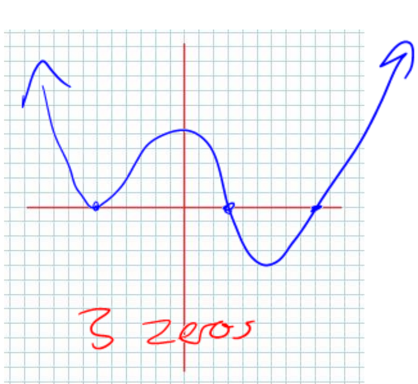
Zeros: A Polynomial Function, $f(x)$, with an even degree of "n" (i.e. $n = 2, 4, 6...$) can have 0 zeros, 1, 2, 3, 4, ... n zeros

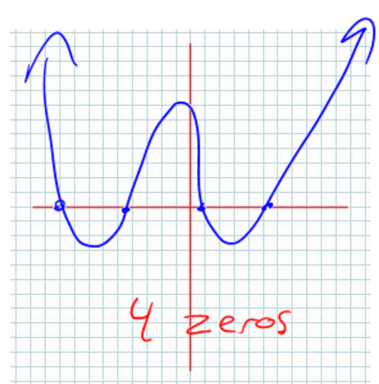
e.g. A degree 4 Polynomial Function (with a positive leading coefficient) can look like:

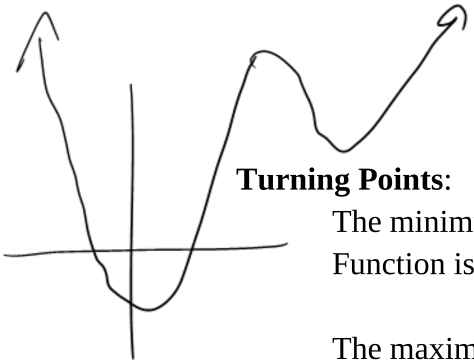












Turning Points:

The minimum number of turning points for an Even Ordered Polynomial Function is one, because you must turn.

The maximum number of turning points for a Polynomial Function of (even) order n is

$n-1$

Odd Ordered Polynomials

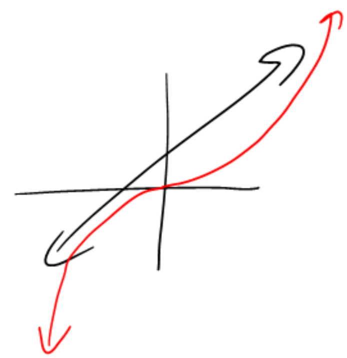
Zeros: min is one because the range is $(-\infty, \infty)$

max is n

Turning Points:

min # of T.P. is zero.

max # is $n-1$



Example 2.2.1 (*#2, for #1b, from Pg. 136*)

Determine the minimum and maximum number of zeros and turning points the given function may have: $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

$g(x)$ is odd and positive.

Zeros: min $= 1$, max $= 5$ (n)

Turning Points: min $= 0$, max $= 4$ $(n-1)$

Example 2.2.2 *(#4d from Pg. 136)*

Describe the end behaviour of the polynomial function using the order and the sign on the leading coefficient for the given function: $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$

↳ negative and even

$x \to -\infty$

$f(x) \to -\infty$

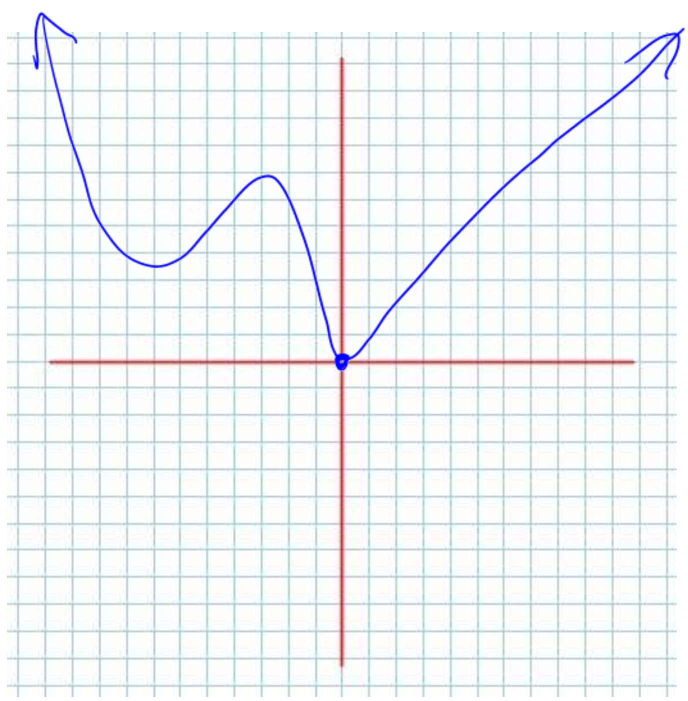
$x \to \infty$

$f(x) \to -\infty$

Example 2.2.3 (*#7c from Pg. 137*)

Sketch a graph of a polynomial function that satisfies the given set of conditions:

Degree 4 - positive leading coefficient - 1 zero - 3 turning points.



Success Criteria:

- I can differentiate between an even and odd degree polynomial
- I can identify the number of turning points given the degree of a polynomial function
- I can identify the number of zeros given the degree of a polynomial function
- I can determine the symmetry (if present) in polynomial functions

2.3 Zeros of Polynomial Functions

(*Polynomial Functions in Factored Form*)

Today we take a deeper look inside the Box of Mystery, ***carefully examining Zeros of Polynomial Functions***

Learning Goal: We are learning to determine the equation of a polynomial function that describes a particular situation or graph and vice-versa.

We'll begin with an **Algebraic Perspective**:

Consider the polynomial function in factored form:

$$f(x) = (2x-3)^1(x-1)^1(x+2)^1(x+3)^1$$

Observations: Leading Term: $2x^4$

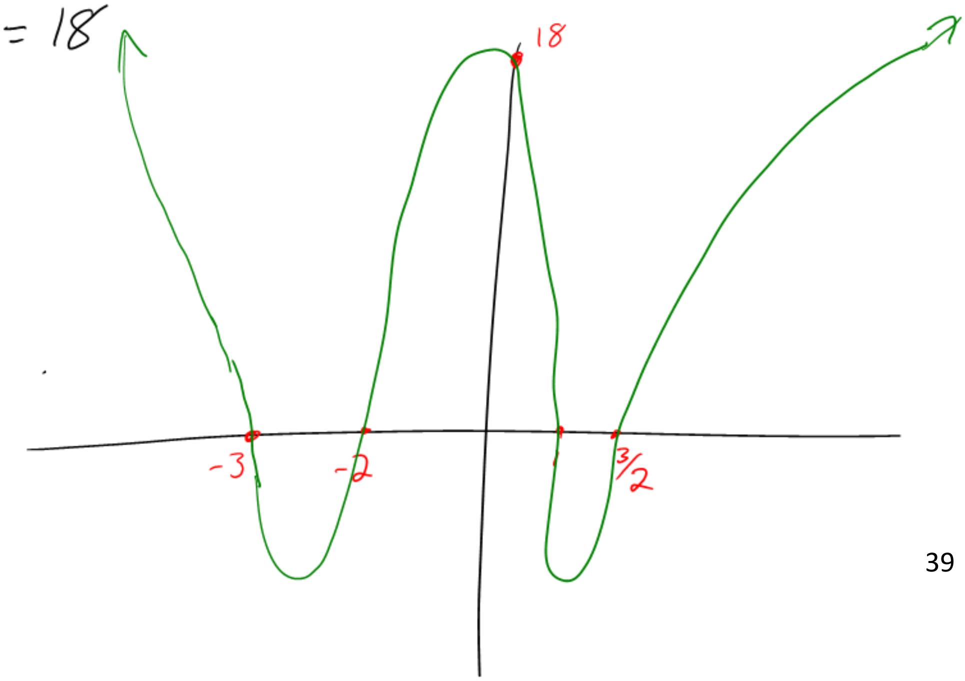
1. $f(x)$ is even and positive $\therefore$ $x \to \pm\infty$, $f(x) \to \infty$

2. Order/degree is 4

3. 4 zeros at $x = \frac{3}{2}, 1, -2, -3$

4. y-int: $f(0) = (2(\cancel{0})-3)(\cancel{0}-1)(\cancel{0}+2)(\cancel{0}+3)$

$= 18$



$(x-3)^2 = 0$

$x - 3 = 0$

$x = 3$

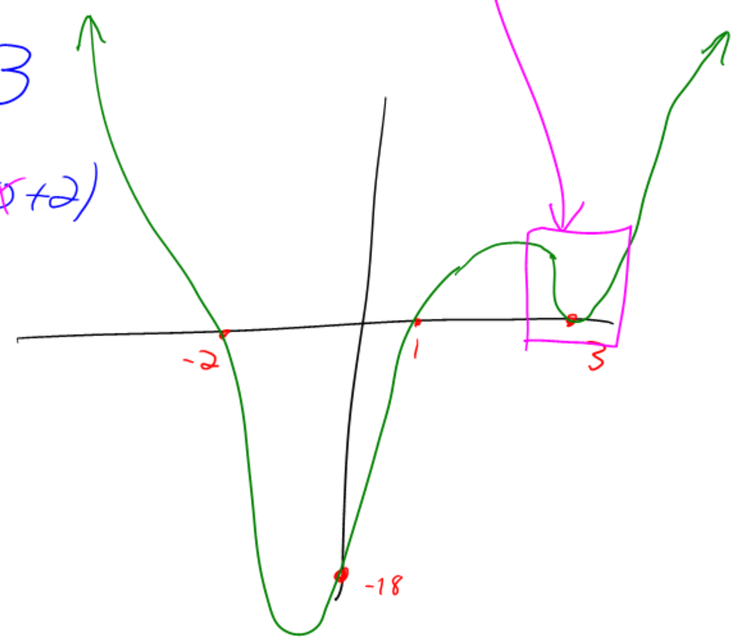
Now, consider the polynomial function $g(x) = \boxed{(x-3)^2}(x-1)^1(x+2)^1$

Observations: Leading Term is x^4

1. $g(x)$ is positive and even $\therefore$ $x \to \pm\infty$, $g(x) \to \infty$

2. zeros at $x = 1, -2, 3$

3. y-int: $g(0) = (\cancel{0}-3)^2(\cancel{0}-1)(\cancel{0}+2)$
$= (9)(-1)(2)$
$= -18$



Geometric Perspective on Repeated Roots (zeros) of order **2**

Consider the quadratic in factored form: $f(x) = \underbrace{(x-1)^2}(\quad)(\quad)$

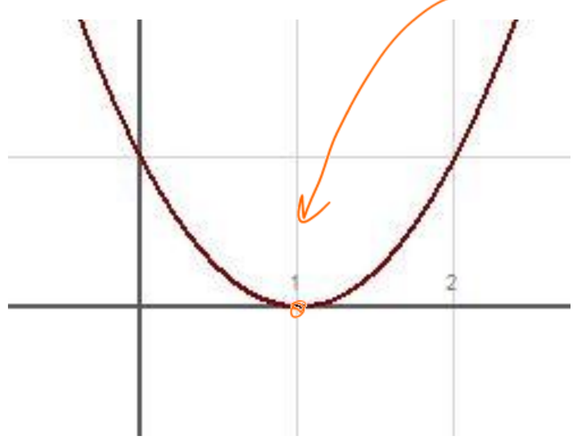


Figure 2.3.1

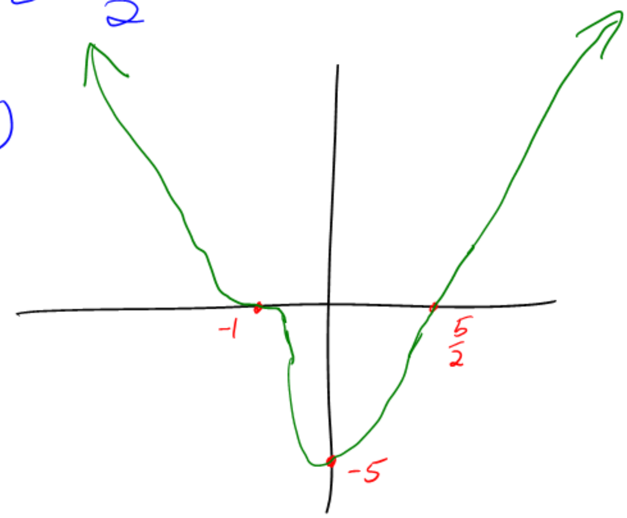
Consider the polynomial function in factored form: $h(t) = (t+1)^3(2t-5)$

Observations: Leading Term $2t^4$

1. Positive and even, $\overset{t}{x} \to \pm\infty$, $\overset{h(t)}{f(x)} \to \infty$

2. Zeros at $\boxed{t=-1}$ order 3 and $t = \frac{5}{2}$

3. y-int: $h(0) = (\cancel{0}+1)^3(2\cancel{(0)}-5)$
$= -5$



Geometric Perspective on Repeated Roots (zeros) of order **3**

Consider the function $f(x) = (x-1)^3$

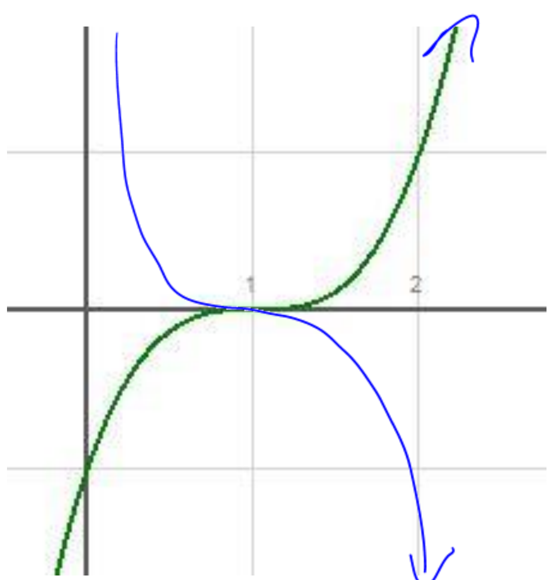
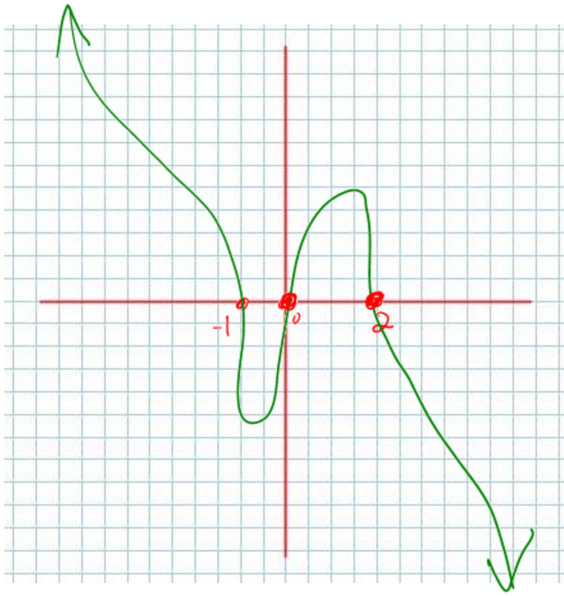


Figure 2.3.2

$$-2x = 0$$
$$x = 0$$

Example 2.3.1

Sketch a (**possible**) graph of $f(x) = -2x(x+1)(x-2)$



L.T. is $-2x^3$

↳ odd and negative

$\therefore x \to -\infty, f(x) \to \infty$

$x \to \infty, f(x) \to -\infty$

Zeros: $x = -1, 2, 0$

order 1

y-int is 0.

Families of Functions

Polynomial functions which share the same order are "broadly related" (e.g. **all** quadratics are in the "order 2 family").

Polynomial Functions which share the same order and zeros are more tightly related.

Polynomial Functions which share the same order, zeros and end behaviors. are like siblings.

$$f(x) = -2(x-3)^2(x+1)$$
$$g(x) = -5(x-3)^2(x+1)$$

Example 2.3.2

The family of functions of order 4, with zeros $x = -1, 0, 3, 5$ can be expressed as:

$$f(x) = \boxed{a}(x+1)(x+0)(x-3)(x-5)$$

↳ this is what distinguishes family members

Example 2.3.3

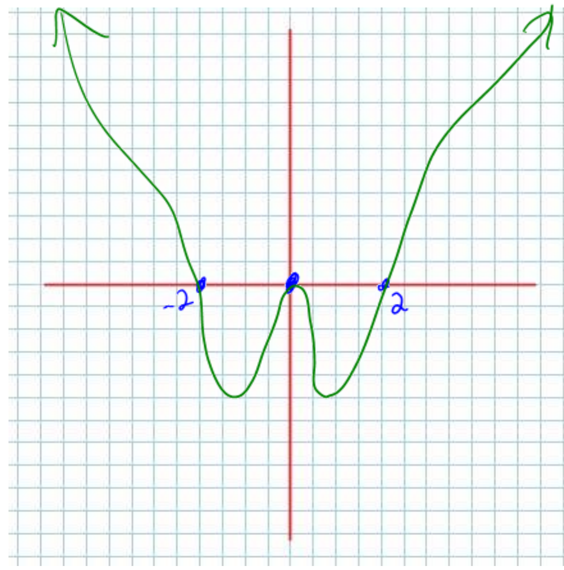
Sketch a graph of $g(x) = 4x^4 - 16x^2$

Leading Term: $4x^4$

↳ positive, even,

$x \to \pm\infty$

$g(x) \to \infty$



Factor to get zeros.

$g(x) = 4x^2(x^2 - 4)$

$g(x) = 4x^2(x-2)(x+2)$

Zeros: $x = 0, -2, 2$

order 2

Example 2.3.4

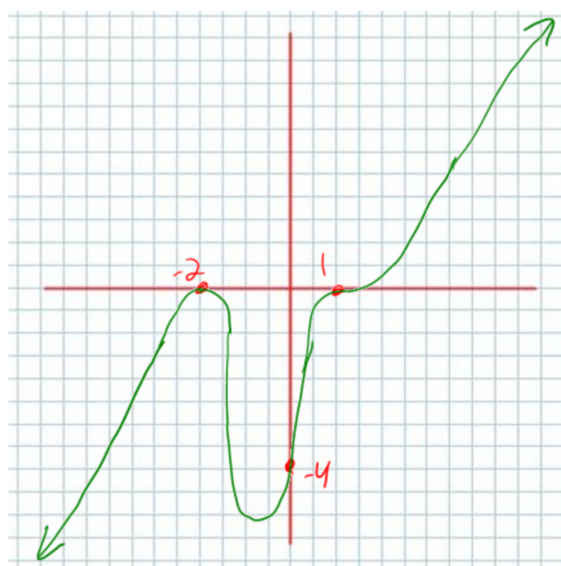
Sketch a (possible) graph of $h(t) = (t-1)^3(t+2)^2$

Leading Term: t^5

↳ positive, odd

$x \to -\infty,\ f(x) \to -\infty$

$x \to \infty,\ f(x) \to \infty$



Zeros $t = 1$ order 3

$t = -2$ order 2

y-int: $h(0) = (0-1)^3(0+2)^2$

$= (-1)(4)$

$= -4$

Example 2.3.5

order 4

Determine **the** quartic function, $f(x)$, with zeros at $x=-2, 0, 1, 3$, if $\boxed{f(-1)=-2}$.

x

$$f(x) = a(x+2)(x+0)(x-1)(x-3)$$

$$-2 = a(-1+2)(-1+0)(-1-1)(-1-3)$$

$$-2 = a(1)(-1)(-2)(-4)$$

$$-2 = a(-8)$$

$$\frac{-2}{-8} = a$$

$$\frac{1}{4} = a$$

$$\therefore f(x) = \frac{1}{4}(x+2)\boxed{(x+0)}(x-1)(x-3)$$

$$f(x) = \frac{1}{4}x(x+2)(x-1)(x-3)$$

Success Criteria:

- I can determine the equation of a polynomial function in factored form
- I can determine the behaviour of a zero based on the order/exponent of that factor

2.4a Dividing a Polynomial by a Polynomial

(The Hunt for Factors)

Learning Goal: We are learning to divide a polynomial by a polynomial using long division

> *Note: In this course we will almost always be dividing a polynomial by a ~~monomial~~* linear divisor

$$\begin{array}{r} 49 \\ 5\overline{)248} \\ -20\downarrow \\ \hline 48 \\ -45 \\ \hline 3 \end{array}$$

Before embarking, we should consider some "basic" terms (and notation):

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$\therefore 248 = 5(49) + 3$

the thing you are dividing

the thing you are dividing by

the answer

the leftovers

$$\boxed{\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}}$$

The division statement.

Note: The Divisor and the Quotient will both be Factors/Zeros

IF

The remainder is zero.

Example 2.4.1

Use **LONG DIVISION** for the following division problem:

$$\frac{5x^4+3x^3-2x^2+6x-7}{x-2}$$

Please read Example 1 (Part A) on Pgs. 162 – 163 in your textbook.

$$\begin{array}{r l}
 & 5x^3+13x^2+24x+54 \\
x-2 \,\big) & 5x^4+3x^3-2x^2+6x-7 \\
 & -(\cancel{5x^4}-10x^3) \\
 & \quad \cancel{13x^3}-2x^2 \\
 & \quad -(\cancel{13x^3}-26x^2) \\
 & \qquad \cancel{24x^2}+6x \\
 & \qquad -(\cancel{24x^2}-48x) \\
 & \qquad\quad \cancel{54x}-7 \\
 & \qquad\quad -(\cancel{54x}-108) \\
 & \qquad\qquad 101
\end{array}$$

$$(x)(5x^3) = 5x^4$$
$$(x)(13x^2) = 13x^3$$
$$(x)(24x) = 24x^2$$
$$(x)(54) = 54x$$

$$\therefore\ 5x^4+3x^3-2x^2+6x-7 = (x-2)(5x^3+13x^2+24x+54)+101$$

KEY OBSERVATION: $(x-2)$ is not a factor

Example 2.4.2

Using Long Division, divide $\dfrac{2x^5 + 3x^3 - 4x - 1}{x - 1}$.

$0x^4$ $0x^2$

All terms must be present.

$$
\begin{array}{r}
2x^4 + 2x^3 + 5x^2 + 5x + 1 \\
(x-1)\overline{)\,2x^5 + 0x^4 + 3x^3 + 0x^2 - 4x - 1} \\
-(\cancel{2x^5} - 2x^4) \\
\hline
\cancel{2x^4} + 3x^3 \\
-(\cancel{2x^4} - 2x^3) \\
\hline
\cancel{5x^3} + 0x^2 \\
-(\cancel{5x^3} - 5x^2) \\
\hline
\cancel{5x^2} - 4x \\
-(\cancel{5x^2} - 5x) \\
\hline
\cancel{x} - 1 \\
-(\cancel{x} - 1) \\
\hline
☺
\end{array}
$$

$\therefore 2x^5 + 3x^3 - 4x - 1 = (x-1)(2x^4 + 2x^3 + 5x^2 + 5x + 1)$

KEY OBSERVATION: $(x-1)$ is a factor

Classwork: Pg. 169 #5 (Yep, that's it for today)

Success Criteria:

- I can use long division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after long division, there is no remainder

$$2x^3$$

$$\boxed{x^2}+2x-4 \overline{\big)\, \cancel{2x^5}+8x^4-3x^3+2x^2-10x+8}$$

$$-(\cancel{2x^5}+4x^4-8x^3)$$

$$4x^4+5x^3+2x^2$$

$$(x^2)(2x^3)=2x^5$$

$$x^2+0x+5 \overline{\big)}$$

$$-70x+104$$

2.4b Dividing a Polynomial by a ~~Polynomial~~ linear function

(*The Hunt for Factors – Part 2*)

Learning Goal: We are learning to divide a polynomial by a polynomial using synthetic division

Here we will examine an alternative form of polynomial division called **Synthetic Division**. Don't be fooled! This is not "fake division". You're thinking with the wrong meaning for "synthetic". (Do a search online and see if you can come up with the meaning I am taking!)

In Synthetic Division we concern ourselves with the coefficients of the dividend and the zero of the divisor.

Synthetic Division uses

- only uses numbers
- three steps ① Bring down ② times ③ Add

> **Note:** Can only use linear divisors
>
> $x+5$ zero $= -5$
>
> $3x-8$ zero $= \frac{8}{3}$

The Set-up

all powers/terms must be present.

zero of divisor	coefficients from the dividend	
	blank	numbers arising from "times/add"
	coefficients of the quotient	remainder

$x^5 \div x = x^4$

$x^4 \div x = x^3$

$x^{99} \div x = x^{98}$

Example 2.4.3

Divide using synthetic division:

zero

$$(4x^3-5x^2+2x-1)\div(x-2)$$

$$\begin{array}{c|cccc}
2 & 4 & -5 & 2 & -1 \\
+ & \downarrow & 8 & 6 & 16 \\
\hline
 & 4 & 3 & 8 & 15 \\
 & x^2 & x^1 & x^0 & \text{remainder}
\end{array}$$

① Bring Down

② Times

③ Add

$$\therefore\ 4x^3-5x^2+2x-1=(x-2)(4x^2+3x+8)+15$$

$$3x^4-11x^3-22x^2+15x-25\div x-5$$

$$\begin{array}{c|ccccc}
5 & 3 & -11 & -22 & 15 & -25 \\
 & & 15 & 20 & -10 & 25 \\
\hline
 & 3 & 4 & -2 & 5 & 0
\end{array}$$

$$\therefore\ \sim\sim\sim = (x-5)(3x^3+4x^2-2x+5)$$

Example 2.4.4

Divide using synthetic division:

$0x^3$

$$\frac{4x^4+3x^2-2x+1}{x+1}$$

$x=-1$

$$\begin{array}{c|ccccc}
-1 & 4 & 0 & 3 & -2 & 1 \\
 & \downarrow & -4 & 4 & -7 & 9 \\
\hline
 & 4 & -4 & 7 & -9 & 10 \\
 & & & & & \text{remainder}
\end{array}$$

$$\therefore\ 4x^4+3x^2-2x+1=(x+1)(4x^3-4x^2+7x-9)+10$$

Example 2.4.5

Divide using your choice of method (*and you choose synthetic division…amen*)

$$(2x^3 - 9x^2 + x + 12) \div (2x - 3)$$

$\hookrightarrow x = \frac{3}{2}$

$$\begin{array}{c|cccc} \frac{3}{2} & 2 & -9 & 1 & 12 \\ & \downarrow & 3 & -9 & -12 \\ \hline & 2 & -6 & -8 & 0 \\ & \downarrow & \downarrow & \downarrow & \\ & 1 & -3 & -4 & \end{array}$$

Divide by the denomination

$$\therefore 2x^3 - 9x^2 + x + 12 = (2x-3)(x^2 - 3x - 4)$$
$$= (2x-3)(x+1)(x-4)$$

Example 2.4.6

Is $3x-1$ a factor of the function $f(x) = 6x - x^3 + 2 + 3x^4$?

$x = \frac{1}{3}$

$3x^4 - x^3 + 0x^2 + 6x + 2$

$$\begin{array}{c|ccccc} \frac{1}{3} & 3 & -1 & 0 & 6 & 2 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & \underbrace{3 \quad 0 \quad 0 \quad 6} & & & & \boxed{4} \\ & 1 & 0 & 0 & 2 & \end{array}$$

$x^3 + 2$

has a remainder, $\therefore$ not a factor.

Example 2.4.7 (OK…this is a lot of examples!)

Consider again (from **Example 2.4.6**) $f(x) = 3x^4 - x^3 + 6x + 2$, and calculate $f\left(\frac{1}{3}\right)$.

$$f\left(\frac{1}{3}\right) = \frac{3}{1}\left(\frac{1^4}{3^4}\right)^4 - \left(\frac{1}{3}\right)^3 + 6\left(\frac{1}{3}\right) + 2$$

$$= \frac{\cancel{3}}{1}\left(\frac{1}{\cancel{81}_{27}}\right) - \left(\frac{1}{27}\right) + 2 + 2$$

$$= \frac{1}{27} - \frac{1}{27} + 2 + 2 = 4$$

WHOAH!! This is the same remainder when dividing by $3x-1$.

Example 2.4.8

Consider **Example 2.4.5**. Let $g(x) = 2x^3 - 9x^2 + x + 12$, and calculate $g\left(\frac{3}{2}\right)$.

$$g\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$$

$$= \frac{\cancel{2}}{1}\left(\frac{27}{\cancel{48}}\right) - 9\left(\frac{9}{4}\right) + \frac{3^{\times 2}}{2_{\times 2}} + \frac{12^{\times 4}}{1_{\times 4}}$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{6}{4} + \frac{48}{4}$$

$$= \frac{0}{4} = 0!!!$$

The Remainder Theorem

Given a polynomial function, $f(x)$, divided by a linear binomial, $x-k$, then the remainder of the division is the value $f(k)$

Proof of the Remainder Theorem

Consider $f(x) \div (x-k)$

Then: $f(x) = (x-\boxed{k})(\overset{\text{quotient}}{q(x)}) + \overset{\text{remainder}}{r}$ } division statement.

$f(k) = \cancel{(\underbrace{k-k}_{=0})(q(k))} + r$

$f(k) = r$ □

Example 2.4.9

Determine the remainder of $\dfrac{5x^4-3x^3-50}{x-\boxed{2}}$. $= f(x)$

WAIT!!!! We MUST have a FUNCTION

$f(2) = 5(2)^4 - 3(2)^3 - 50$

$= 80 - 24 - 50$

$= 6.$

$$\begin{array}{c|ccccc} 2 & 5 & -3 & 0 & 0 & -50 \\ & & 10 & 14 & 28 & 56 \\ \hline & 5 & 7 & 14 & 28 & \boxed{6} \end{array}$$

Success Criteria:

- I can appreciate that synthetic division is "da bomb"
- I can use synthetic division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after synthetic division, there is no remainder (The Remainder Theorem)

2.5 The Factor Theorem

(Factors have been FOUND)

Learning Goal: We are learning the connections between a polynomial function and its remainder when divided by a binomial

The Factor Theorem

Given a polynomial function, $f(x)$, then $x-a$ is a factor of $f(x)$ if and only if $f(a)=0$.

Example 2.5.1

Use the Factor Theorem to factor x^3+2x^2-5x-6.

$f(x)=x^3+2x^2-5x-6$

Test the possible factor of 6

$\pm 1, \pm 2, \pm 3, \pm 6$

Try $x=1$ or $(x-1)$

$f(1)=1^3+2(1)^2-5(1)-6$

$=1+2-5-6$

$=-8\neq 0$

not a factor

Try $x=-1$ or $(x+1)$

$f(-1)=(-1)^3+2(-1)^2-5(-1)-6$

$=-1+2+5-6$

$=$ ☺ Is a factor!

WAIT!!!! We need a FUNCTION

$f(x)=(x-\boxed{a})(x-\boxed{b})(x-\boxed{c})$

$(a)(b)(c)=-6$

$\therefore$ the factors must divide -6.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1x^2 & 1x & -6 & 0 \end{array}$$

$\therefore x^3+2x^2-5x-6=(x+1)(x^2+x-6)$

A M

$=(x+1)(x-2)(x+3)$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Example 2.5.2

Factor **fully** $x^4 - x^3 - 16x^2 + 4x + \boxed{48} = f(x)$

Try: $x = 2 \quad (x-2)$

$f(2) = (2)^4 - (2)^3 - 16(2)^2 + 4(2) + 48$

$= 16 - 8 - 64 + 8 + 48$

$= 0$ Factor!!

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -16 & 4 & 48 \\ & & 2 & 2 & -28 & -48 \\ \hline & 1 & 1 & -14 & -24 & 0 \end{array}$$

$\therefore (x-2)(\underbrace{x^3 + x^2 - 14x - 24}_{g(x)})$

Try: $x = 6 \quad (x-6)$

$g(6) = 6^3 + 6^2 - 14(6) - 24$

$\neq 0$ not a factor :(

Try $x = -2 \quad (x+2)$

$g(-2) = (-2)^3 + (-2)^2 - 14(-2) - 24$

$= -8 + 4 + 28 - 24$

$= 0$:)

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$\therefore x^4 - x^3 - 16x^2 + 4x + 48 = (x-2)(x+2)(x^2 - x - 12)$

$= (x-2)(x+2)(x+3)(x-4)$

Example 2.5.3 (*Pg 177 #6c in your text***)**

Factor fully $x^4+8x^3+4x^2-48x$

$$= x(\underbrace{x^3+8x^2+4x-48}_{f(x)})$$

Test $x=2$

$$g(2) = (2)^3 + 8(2)^2 + 4(2) - 48$$
$$= 8 + 32 + 8 - 48$$
$$= 0!$$

$$\begin{array}{r|rrrr} 2 & 1 & 8 & 4 & -48 \\ & & 2 & 20 & 48 \\ \hline & 1x^2 & 10 & 24 & 0 \end{array}$$

$$\therefore x^4+8x^3+4x^2-48x = x(x-2)(x^2+10x+24)$$
$$= x(x-2)(x+6)(x+4)$$

Example 2.5.4 (*Pg 177 #10*) $=f(x)$ $f(1)=10$

When ax^3-x^2+2x+b is divided by $x-1$ the remainder is 10. When it is divided by $x-2$ the remainder is 51. Find a and b.

$f(2)=51$

This problem is very instructive.

$$a(1)^3-(1)^2+2(1)+b=10$$
$$a \boxed{-1+2} + b = 10$$
$$a+b=9$$
$$a+(51-8a)=9$$
$$-7a=-42$$
$$\boxed{a=6 \quad b=3}$$

$$a(2)^3-(2)^2+2(2)+b=51$$
$$8a-4+4+b=51$$
$$8a+b=51$$
$$\boxed{b=51-8a}$$

$$a+\cancel{b}=9$$
$$-(8a+\cancel{b}=51)$$
$$-7a=-42$$
$$a=6$$

Success Criteria:

- I can use test values to find the factors of a polynomial function
- I can factor a polynomial of degree three or greater by using the factor theorem
- I can recognize when a polynomial function is not factorable

2.6 Factoring Sums and Differences of Cubes

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Knowing how to factor a sum or difference of cubes is a simple matter of remembering patterns.

Learning Goal: We are learning to factor a sum or difference of cubes.

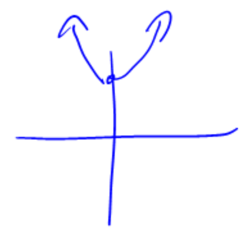
Example 2.6.1 (*Recalling the pattern for factoring a Difference of Squares*)

Factor $\sqrt{4x^2} - \sqrt{25}$

$= (2x+5)(2x-5)$

Note: Sums of Squares DO NOT factor!!

e.g. Simplify $x^2 + 4$



Differences of Cubes $8x^3 - 27$

Same
Opposite
Always
Positive

Pattern

$(cube_1 - cube_2) = (cuberoot_1 - cuberoot_2)(cuberoot_1^{\,2} + cuberoot_1 \times cuberoot_2 + cuberoot_2^{\,2})$

$8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)$

TWO POSITIVES and ONE NEGATIVE

Sums of Cubes (These DO factor!!) $8x^3 + 27$

Pattern

$(cube_1 + cube_2) = (cuberoot_1 + cuberoot_2)(cuberoot_1^{\,2} - cuberoot_1 \times cuberoot_2 + cuberoot_2^{\,2})$

$8x^3 + 27 = (2x + 3)\underbrace{(4x^2 - 6x + 9)}_{\text{not factorable.}}$

Same, opposite, always positive

Example 2.6.2

Factor $x^3 - 8$ $= (x-2)(x^2+2x+4)$

Example 2.6.3

Factor $27x^3 + 125y^3$ $= (3x+5y)(9x^2-15xy+25y^2)$

Example 2.6.4

Factor $1 - 64z^3$ $= (1-4z)(1+4z+16z^2)$

Example 2.6.5

Factor $1000x^3 + 27$ $= (10x+3)(100x^2-30x+9)$

Example 2.6.6

Factor $x^6 - 729$ $= (x^2-9)(x^4+9x^2+81)$

$(x^2)^3 = x^6$

$= (x-3)(x+3)(x^4+9x^2+81)$

Success Criteria:

- I can use patterns to factor a sum of cubes
- I can use patterns to factor a difference of cubes