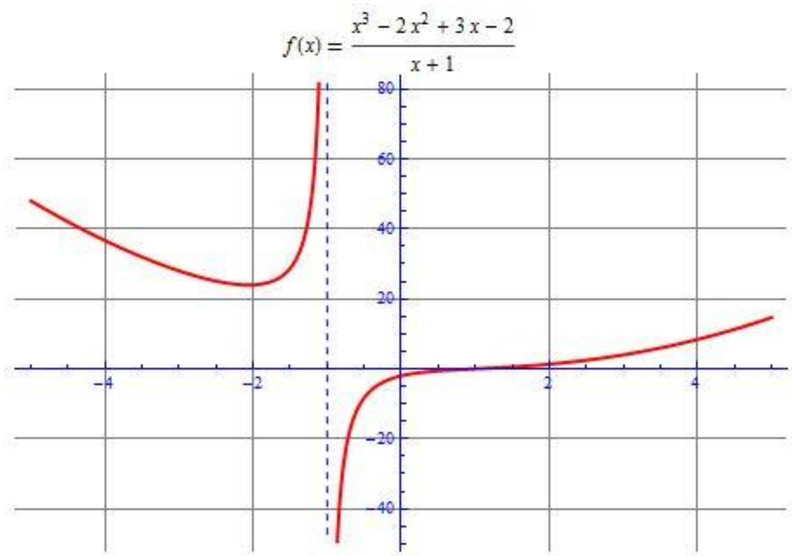
Advanced Functions

Course Notes

Unit 4 – Rational Functions, Equations and Inequalities

We are learning to

- *sketch the graphs of simple rational functions*
- *solve rational equations and inequalities with and without tech*
- *apply the techniques and concepts to solve problems involving rational models*



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Unit 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

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4.1 Rational Functions, Domain and Asymptotes

Learning Goal: We are learning to identify the asymptotes of rational functions.

Definition 4.1.1

A **Rational Function** is of the form

$f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ and both $p(x)$ and $q(x)$ are polynomial functions

e.g. $f(x) = \dfrac{3x^2 - 5x + 1}{2x - 1}$ is a rational function

$g(x) = \dfrac{\sqrt{2x+5}}{3x-2}$ not because $\sqrt{2x+5}$ is not a polynomial

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Domain

Definition 4.1.2

Given a rational function $f(x) = \dfrac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f = \{x \in \mathbb{R} \mid q(x) \neq 0\}$

↳ zeros of $q(x)$.

Example 4.1.1

Determine the natural domain of $f(x) = \dfrac{x^2 - 4}{x - 3}$.

$x = 3$

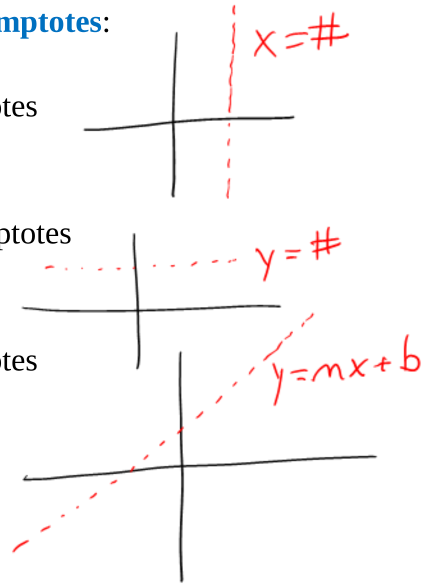
$D_f = \{x \in \mathbb{R} \mid x \neq 3\}$

$x \in (-\infty, 3) \cup (3, \infty)$

Asymptotes

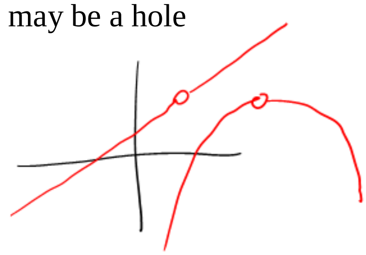
There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes
2) Horizontal Asymptotes
3) Oblique Asymptotes



Vertical Asymptotes

A rational function $f(x)=\dfrac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x)=0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.



Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x)=\dfrac{5x}{x^2-x-6}$

Always factor everything.

$$f(x)=\frac{5x}{(x-3)(x+2)}$$

$x=3$ $\quad x=-2$

If the factor stays, it is a V.A.

$$x\in(-\infty,-2)\cup(-2,3)\cup(3,\infty)$$

b) $h(x) = \frac{x+3}{x^2-9}$

$h(x) = \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x-3)}$

$x=3$ V.A.

$x=-3$ Hole because it disappeared.

$h(x) = \frac{1}{x-3}$

$x \in (-\infty,-3) \cup (-3,3) \cup (3,\infty)$

c) $g(x) = \frac{x^2-4}{x+2}$

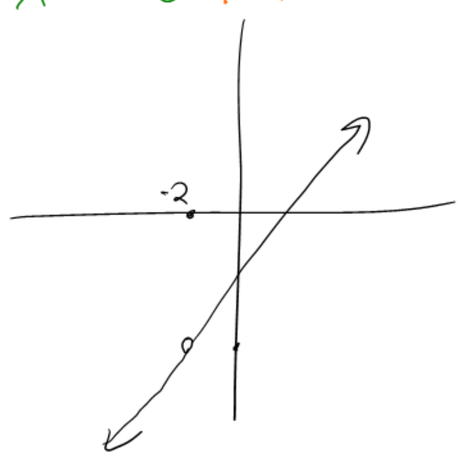
Factor Everything.

$g(x) = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}}$

$x=-2$ Hole.

$g(x) = x-2$

$x \in (-\infty,-2) \cup (-2,\infty)$



Horizontal Asymptotes

Here we are concerned with the end behavior of the rational function

i.e. We are asking, given a rational function $f(x)=\dfrac{p(x)}{q(x)}$, how is $f(x)$ behaving as $x \to \pm\infty$.

↳ SUPER BIG NUMBERS.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $\overset{\text{top}}{p(x)}$ $>$ Order of $\overset{\text{bottom}}{q(x)}$

   e.g. $f(x)=\dfrac{x^3-2}{x^2+1}$ order 3 / order 2

When the order of $p(x)$ is bigger than the order of $q(x)$ there is $\boxed{\text{NO}}$ horizontal asymptote

2) Order of numerator $=$ Order of denominator

   e.g. $f(x)=\dfrac{2x^2-3x+1}{3x^2+4x-5}$

If x is MASSIVE, the stuff behind the leading terms are inconsequential/irrelevant.

what's left is $\dfrac{2\cancel{x^2}}{3\cancel{x^2}}=$ $\boxed{\dfrac{2}{3}=y}$ is the horizontal asymptote

e.g. Determine the horizontal asymptote of $g(x)=\dfrac{3x-4x^5}{5x^5+2x-1}$

H.A. is $y=\dfrac{-4}{5}$

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x)=\dfrac{x^2-5x+6}{x^5+7}$

$\dfrac{(100)^2}{(100)^5} = \dfrac{10,000}{10,000,000,000} = \dfrac{1}{\text{Really Big \#}} =$ close to zero

H.A. is $y = 0$.

Oblique Asymptotes

These occur when the order of $p(x)$ is exactly one bigger than $q(x)$

e.g. $f(x)=\dfrac{x^2-2x+3}{x+1}$

With Oblique Asymptotes we are still dealing with end behaviors

O.A. have the form $y = mx+b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: by polynomial division!!

The O.A. is the quotient.

$g(x) = \dfrac{x^5-2x^3+3x^2-1}{x^4 \sim\sim}$

$$\begin{array}{r|l} & \boxed{x + \#} \ \text{OA} \\ x^4 \sim\sim & x^5 \sim\sim \end{array}$$

$f(x) = \dfrac{x^2-2x+3}{x+1}$

$$\begin{array}{r|rrr} -1 & 1 & -2 & 3 \\ & & -1 & 3 \\ \hline & 1 & -3 & 6 \end{array}$$

The O.A. is $y = 1x - 3$

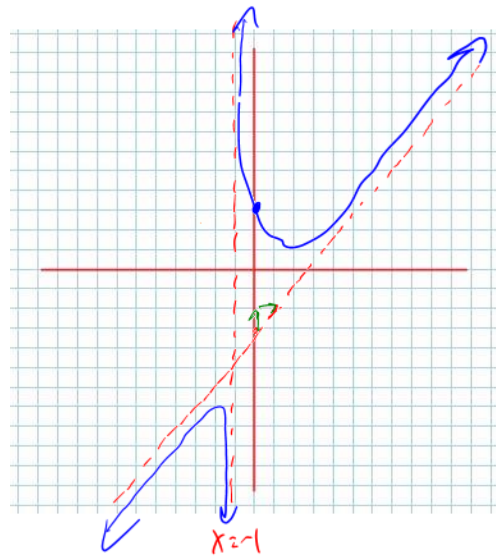
(Rough) Sketch of $f(x) = \dfrac{x^2 - 2x + 3}{x + 1}$

V.A. $x = -1$

H.A. none

O.A. $y = \overset{m}{\frac{1}{1}}x \overset{b}{-3}$

y-int: $f(0) = 3$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x) = \dfrac{x+2}{x^2 + 3x + 2}$ order 1, order 2

$$f(x) = \frac{\cancel{x+2}}{(x+1)\cancel{(x+2)}}$$

$$f(x) = \frac{1}{x+1}$$

V.A.	$x = -1$
Hole	$x = -2$
H.A.	$y = 0$
O.A.	none

b) $g(x) = \dfrac{4x^2 - 25}{1x^2 - 9}$

$$g(x) = \frac{(2x-5)(2x+5)}{(x-3)(x+3)}$$

V.A.	$x = -3, 3$
Hole	None
H.A.	$y = \frac{4}{1} = 4$
O.A.	None

c) $h(x)=\dfrac{x^2+0x+0}{x+3}$

$$\begin{array}{r|rrr} -3 & 1 & 0 & 0 \\ & & -3 & 9 \\ \hline & 1 & -3 & 9 \end{array}$$

V.A	$x=-3$
Hole	None
H.A.	None
O.A.	$y=x-3$

Example 4.1.4

$(x+3)$

Determine an equation for a function with a vertical asymptote at $x=-3$, and a horizontal asymptote at $y=0$.

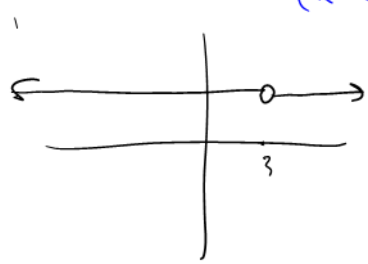
$$f(x)=\frac{1}{(x+3)}$$

Example 4.1.5

Determine an equation for a function with a hole discontinuity at $x=3$.

$(x-3)$

$$f(x)=\frac{(x-3)}{(x-3)}$$



Success Criteria:

- I can identify a hole when there is a common factor between *p(x)* and *q(x)*
- I can identify a vertical asymptote as the zeros of *q(x)*
- I can identify a horizontal asymptote by studying the degrees of *p(x)* and *q(x)*
- I can identify an oblique asymptote when the degree of *p(x)* is exactly 1 greater than *q(x)*

4.2 Graphs of Rational Functions

Learning Goal: We are learning to sketch the graphs of rational functions.

Note: In Advanced Functions we will only consider rational functions of the form
$$f(x)=\frac{ax+b}{cx+d}$$

Rational Functions of the form $f(x)=\frac{ax+b}{cx+d}$ will have:

1) One Vertical Asymptote

Comes from the denominator

$cx+d=0, \quad x=\frac{-d}{c}$

2) One Zero (unless $a=0$)

Comes from the numerator

$ax+b=0, \quad x=\frac{-b}{a} \qquad \left(\frac{-b}{a}, 0\right) \rightarrow$ x-intercept

3) Functional Intercept

y

$f(0)=\frac{\cancel{a(0)}+b}{\cancel{c(0)}+d}=\frac{b}{d} \qquad \left(0, \frac{b}{d}\right)$

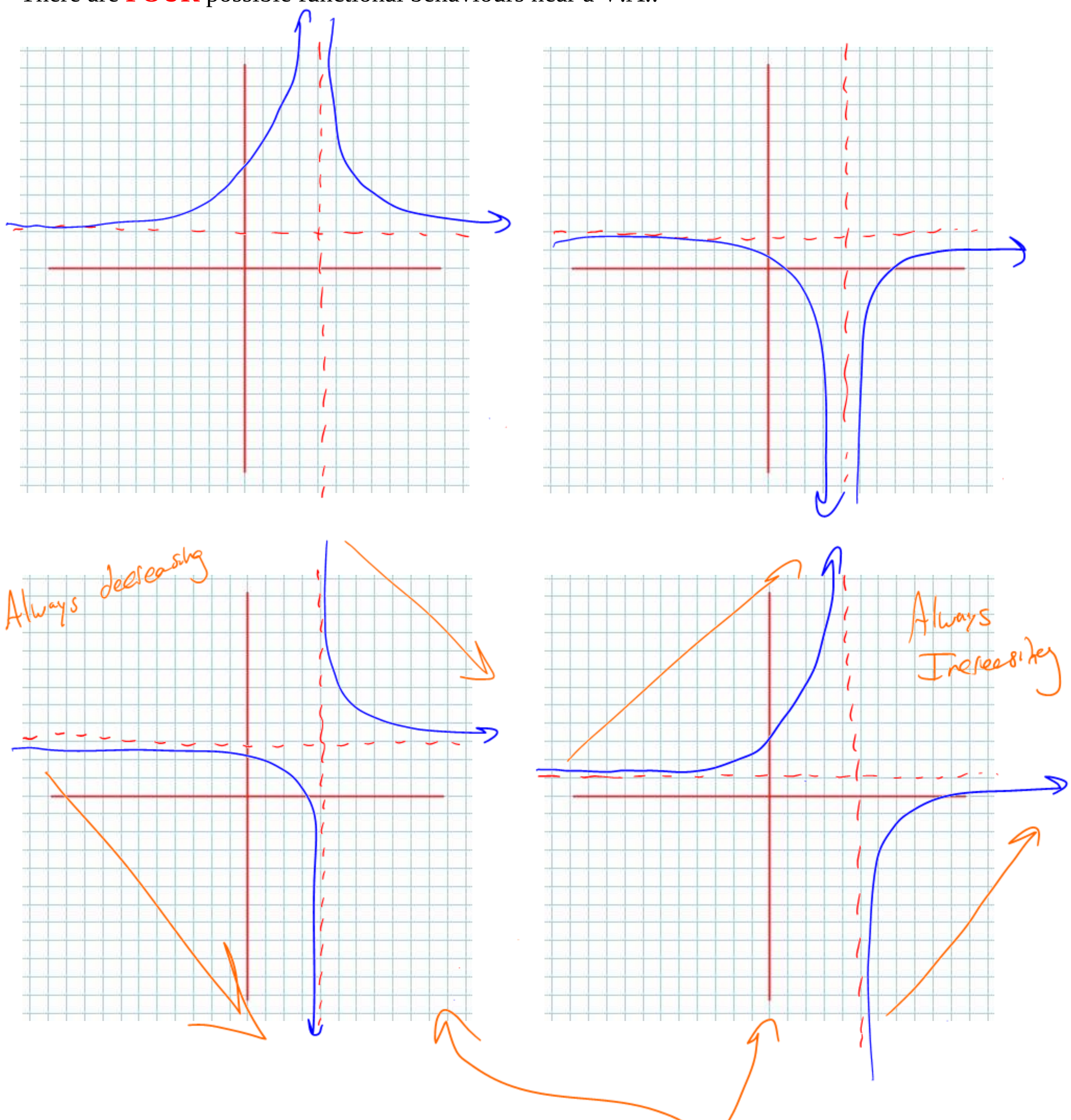
4) A Horizontal Asymptote

H.A. is $y=\frac{a}{c}$ If $a=0$, H.A. is $y=0$

5) These functions will always be either always increasing or always decreasing.

Functional Behaviour Near A Vertical Asymptote

There are **FOUR** possible functional behaviours near a V.A.:



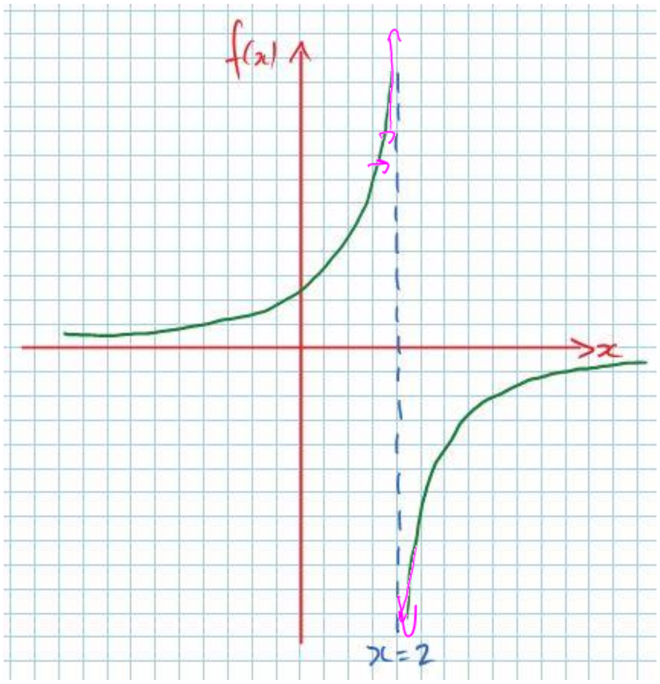
For functions of the form $f(x) = \dfrac{ax+b}{cx+d}$ we will see behaviours

The questions is, **how do we know which**?

We need to **analyze** the function **near the V.A**.

We need to become familiar with some **Notation.**

Consider some rational function with a sketch of its graph which looks like:



End behaviors:

$x \to -\infty$, $f(x) \to$

$x \to 2^-$ left side, $f(x) \to \infty$

$x \to 2^+$ right side, $f(x) \to -\infty$

Example 4.2.1

Determine the functional behaviour of $f(x) = \dfrac{2x+1}{x-3}$ near its V.A.

$x = 3$

$x \to 3^-$, $f(x) \to -\infty$

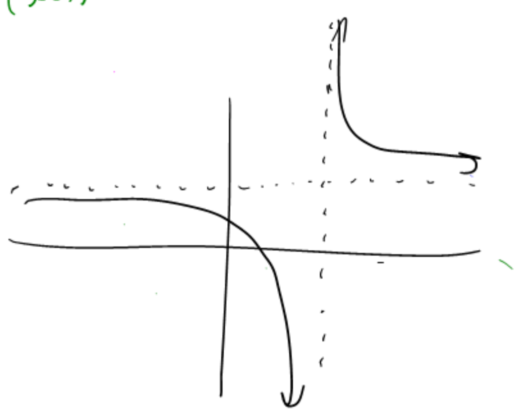
Test $x = 2.99$ and $x = 2.999$

$f(2.99) = -698$ $\quad$ $f(2.999) = -6998$

$x \to 3^+$, $f(x) \to \infty$

Test $x = 3.01$ and $x = 3.001$

$f(3.01) = 702$ $\quad$ $f(3.001) = 7002$



We now have the tools to sketch some graphs!

Example 4.2.2

Sketch the graph of the given function. State the domain, range, intervals of increase/decrease and where the function is positive and negative.

a) $f(x) = \dfrac{2x+1}{x-1}$

HA: $y = \frac{2}{1} = 2$

V.A.: $x = 1$

x-int: $x = \frac{-1}{2}$

y-int: $y = \frac{1}{-1} = -1$

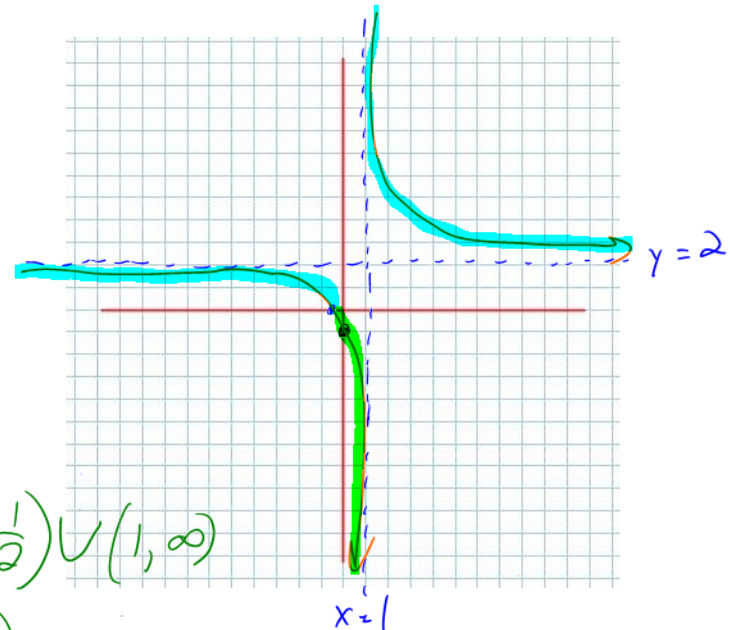
$D_f = (-\infty, 1) \cup (1, \infty)$

$R_f = (-\infty, 2) \cup (2, \infty)$

$f(x)$ is always decreasing

$f(x) > 0$ when $x \in (-\infty, -\frac{1}{2}) \cup (1, \infty)$

$f(x) < 0$ when $x \in (\frac{-1}{2}, 1)$



b) $g(x) = \dfrac{3x-2}{2x+5}$

H.A: $y = \frac{3}{2}$

V.A: $x = \frac{-5}{2}$

x-int: $x = \frac{2}{3}$

y-int: $y = \frac{-2}{5}$

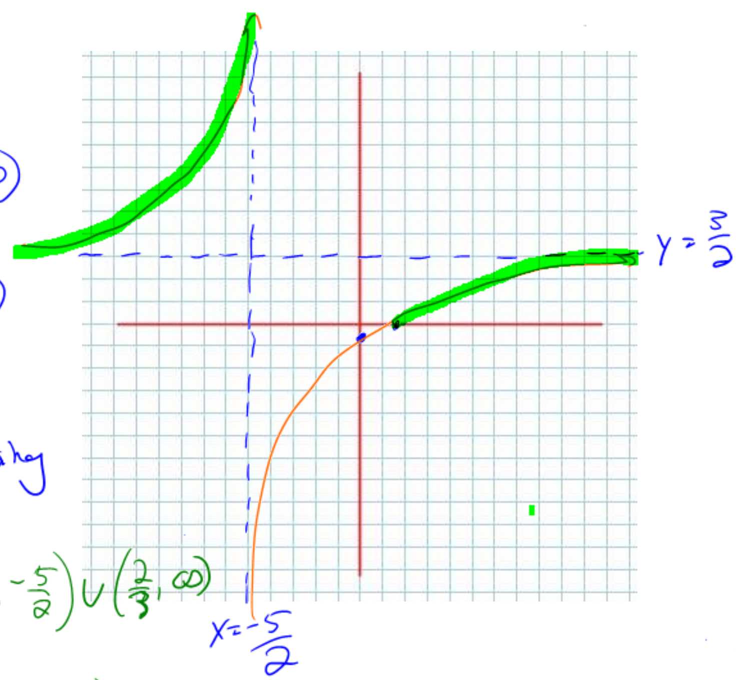
$D_g = (-\infty, \frac{-5}{2}) \cup (\frac{-5}{2}, \infty)$

$R_g = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

$g(x)$ is always increasing

$g(x) > 0$ when $x \in (-\infty, -\frac{5}{2}) \cup (\frac{2}{3}, \infty)$

$g(x) < 0$ when $x \in (\frac{-5}{2}, \frac{2}{3})$



Example 4.2.3

Consider question #9 on page 274: $I(t) = \dfrac{15t + 25}{t}$

Success Criteria:

- I can identify the horizontal asymptote as $\frac{a}{c}$
- I can identify the vertical asymptote as $-\frac{d}{c}$
- I can identify the y-intercept as $\frac{b}{d}$
- I can identify the x-intercept as $-\frac{b}{a}$

4.4 Solving Rational Equations

Learning Goal: We are learning to solve rational equations. Think rationally!

Solving a Rational Equation is **VERY MUCH** like solving a Polynomial Equation. Thus, this stuff is so much fun it should be illegal. But it isn't illegal unless you break a rule of algebra. Math Safe!

KEY (*this is a major key for you music buffs*)

Multiplying by the Multiplicative Inverse of the Common Denominator is wonderful to use **WHEN YOU HAVE** something like:

$$\text{RATIONAL}_1 + \text{RATIONAL}_2 = \text{RATIONAL}_3$$

e.g.

$$\left(\frac{3}{x-2} + \frac{3}{2} = \frac{4(x+5)}{x}\right)\frac{(x-2)(x)(2)}{1} \qquad \text{CD: } (x-2)(2)(x)$$

$x \neq 2$ $\qquad$ $x \neq 0$

$$\frac{(x-2)(x)(2)}{1}\left(\frac{3}{x-2}\right) + \frac{(x-2)(x)(2)}{1}\left(\frac{3}{2}\right) = \frac{(x-2)(x)(2)}{1}\left(\frac{4(x+5)}{x}\right)$$

$$6x + 3(x)(x-2) = 8(x-2)(x+5)$$

Expand, polynomial $=0$, then solve.

Make Sure To Keep **RESTRICTIONS ON X** In Mind

This means that restrictions cannot be solutions.

Example 4.4.1

a) Solve $\left(\frac{x}{5} = \frac{9}{18}\right)(5)(18)$

no restrictions

CD: (5)(18)

$$\frac{18x}{18} = \frac{45}{18}$$

$$x = \frac{5}{2}$$

b) Solve $\left(\frac{1}{x} - \frac{5x}{3} = \frac{2}{5}\right)(3)(5)(x)$

RESTRICTIONS

$x \neq 0$

CD: (3)(5)(x)

$$15 - 25x^2 = 6x$$

$$0 = 25x^2 + 6x - 15 \quad \text{D.N.F}$$

use Quad. Formula or Technology

$$x = -0.9 \quad \text{and} \quad x = 0.66$$

✓ ✓

c) Solve $\dfrac{3}{x}+\dfrac{4}{x+1}=\dfrac{2}{1}$ $(x)(x+1)$

RESTRICTIONS

$x \neq 0$

$x \neq -1$

CD: $(x)(x+1)$

$3(x+1)+4x=2x(x+1)$

$3x+3+4x=2x^2+2x$

$7x+3=2x^2+2x$

$-7x \quad -3$

$0=2x^2-5x-3$

m: -6 A: -5

$-6, +1$

$(2x-6)(2x+1)$

$(x-3)(2x+1)$

$0=(x-3)(2x+1)$

$x=3$ ✓

$x=\frac{-1}{2}$ ✓

d) Solve $\dfrac{10}{x^2-2x}+\dfrac{4}{x}=\dfrac{5}{x-2}$ $(x)(x-2)$

$x(x-2)$

RESTRICTIONS

$x \neq 0$

$x \neq 2$

CD: $(x)(x-2)$

$10+4(x-2)=5x$

$10+4x-8=5x$

$-4x \quad -4x$

$2=x$

However, $x \neq 2$ ∴ there are no solutions to this equation.

e) Solve $\left(\frac{16x}{1} - \frac{5}{x+2} = \frac{15}{x-2} - \frac{60}{(x-2)(x+2)}\right)\frac{(x+2)(x-2)}{1}$

Restrictions: $x \neq -2$

$x \neq 2$

CD: $(x+2)(x-2)$

$$16x\underbrace{(x+2)(x-2)} - 5(x-2) = 15(x+2) - 60$$

$$16x(x^2-4) - 5x + 10 = 15x + 30 - 60$$

$$16x^3 - 64x - 5x + \overset{+30}{10} = 15x - 30$$

$-15x$

$$\frac{16x^3 - 84x + 40 = 0}{4}$$

$$\underbrace{4x^3 - 21x + 10}_{f(x)} = 0$$

$\to \pm 1, \pm 2, \pm 5, \pm 10$

Test: $f(2) = 4(2)^3 - 21(2) + 10$

$= 32 - 42 + 10$

$= 0 \quad \therefore (x-2)$ is a factor

$$\begin{array}{r|rrrr} 2 & 4 & 0 & -21 & 10 \\ & & 8 & 16 & -10 \\ \hline & 4 & 8 & -5 & 0 \end{array}$$

$\therefore (x-2)(4x^2 + 8x - 5) = 0$

$(x-2)(2x+5)(2x-1) = 0$

M: -20

A: 8

$10, -2$

$(4x+10)(4x-2)$

$(2x+5)(2x-1)$

$\cancel{x = 2}$ inadmissable

$x = \frac{-5}{2}$

$x = \frac{1}{2}$

Example 4.4.2

From your Text: Pg. 285 #10

The Turtledove Chocolate factory has two chocolate machines. Machine A takes *m* minutes to fill a case with chocolates, and machine B takes *m + 10* minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

Rate of Work

$\dfrac{\text{a job}}{\text{time}} = \dfrac{\text{Filling } \boxed{a} \text{ case}}{\text{minutes}}$ ($a = 1$)

Machine A $= \dfrac{1}{m}$ Machine B $= \dfrac{1}{m+10}$ Together $= \dfrac{1}{15}$

add.

$$\left(\frac{1}{m} + \frac{1}{m+10} = \frac{1}{15}\right)(m)(m+10)(15)$$

Restrictions: $m \neq 0$

$m \neq -10$

CD: $(m)(m+10)(15)$

$$15(m+10) + 15m = m(m+10)$$

$$15m + 150 + 15m = m^2 + 10m$$

$-30m$

$$0 = m^2 - 20m - 150$$

by Q.F. or tech:

$m = 25.8$

$\therefore$ Machine A takes 25.8 minutes to fill a case and Machine B takes 35.8 minutes

Success Criteria:

- I can recognize that the zeros of a rational function are the zeros of the numerator
- I can solve rational equations by multiplying each term by the lowest common denominator, then solving the resulting polynomial equation
- I can identify inadmissible solutions based on the context of the problem

4.5 Solving Rational Inequalities

Learning Goal: We are learning to solve rational inequalities using algebraic and graphical approaches.

The joy, wonder and peace these bring is really quite amazing

e.g. Solve $\left(\frac{x-2}{7} \geq 0\right)^{7}$

$x-2 \geq 0$

$x \geq 2$

Note: For Rational Inequalities, with a variable in the denominator, you **CANNOT** multiply by the multiplicative inverse of the common denominator!!!!

Why? If the factor, $x+3$, is negative, then the inequality would need to flip.

Example 4.5.1

Solve $\frac{x-2}{x+3} \geq 0$

We solve by using an Interval Chart

For the intervals, we split $(-\infty, \infty)$ at all zeros (**where the numerator is zero**), and all restrictions (**where the denominator is zero**) of the (SINGLE) rational expression. Keep in mind that it may take a good deal of algebraic manipulation to get a SINGLE rational expression…

zero at $x=2$ restriction of $x=-3$

Interval	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
Test values	-4	0	3
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
Ratio	$+$	$-$	$+$

$\therefore \frac{x-2}{x+3} \geq 0$ when

$x \in (-\infty, -3) \cup [2, \infty)$

Example 4.5.2

Solve $\frac{1}{x+5} < 5$

DO NOT CROSS MULTIPLY (or else)

- Get everything on one side
- Simplify into a single Rational Expression using a common denominator
- Interval Chart it up

$$\frac{1}{x+5} - \frac{5(x+5)}{1(x+5)} < 0$$

$$\frac{1-5(x+5)}{x+5} < 0$$

$$\frac{1-5x-25}{x+5} < 0$$

$$\frac{-5x-24}{x+5} < 0$$

zero at $x = \frac{24}{-5} = -4.8$

restriction at $x = -5$

Intervals	$(-\infty, -5)$	$(-5, -4.8)$	$(-4.8, \infty)$
T.V.	-6	-4.9	0
$-5x-24$	+	+	−
$x+5$	−	+	+
Ratio	−	+	−

$\therefore \frac{1}{x+5} < 5$ when $x \in (-\infty, -5) \cup (-4.8, \infty)$

Example 4.5.3

Solve $\dfrac{x^2+3x+2}{x^2-16} \geq 0$

FACTORED FORM IS YOUR FRIEND

$$\frac{(x+2)(x+1)}{(x-4)(x+4)} \geq 0$$

Zeros at $x = -2, -1$

Restrictions at $x = -4, 4$

Intervals	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
T.V.	-5	-3	-1.5	0	5
$x+2$	$-$	$-$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$	$+$	$+$
Ratio	$+$	$-$	$+$	$-$	$+$

$\therefore \dfrac{x^2+3x+2}{x^2-16} \geq 0$ when $x \in (-\infty, -4) \cup [-2, -1] \cup (4, \infty)$

this is bigger/higher order.

$3x+5 \geq 4x-8$

$-3x \quad +8 \qquad -3x \quad +8$

$13 \geq x$

$\frac{1}{x} = x^{-1}$

Example 4.5.4

Solve $\frac{3}{x+2} \leq x$

$0 \leq \frac{x(x+2)}{1(x+2)} - \frac{3}{x+2}$

$0 \leq \frac{x^2+2x-3}{x+2}$

$0 \leq \frac{(x+3)(x-1)}{x+2}$

$\frac{-x^2-2x+3}{x+2} \leq 0$

$\frac{-1(x^2+2x-3)}{x+2} \leq 0$

$\frac{\cancel{-1}(x+3)(x-1)}{\cancel{-1}\,x+2} \cancel{\leq} \geq 0$

zeros at $x=-3, 1$

restriction at $x=-2$

Intervals	$(-\infty,-3)$	$[-3,-2)$	$(-2,1]$	$[1,\infty)$
T.V.	-4	-2.5	0	2
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
Ratio	$-$	$+$	$-$	$+$

$\therefore \frac{3}{x+2} \leq x$ when $x \in [-3,-2) \cup [1,\infty)$

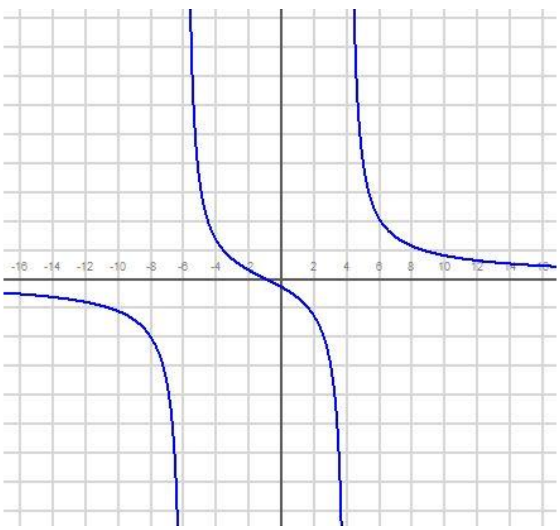
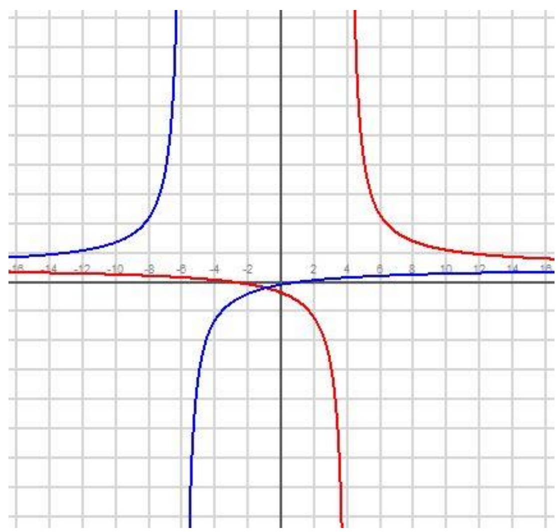
Example 4.5.5

From your Text: Pg. 296 #6a

Using **Graphing Tech**

Solve $\frac{x+3}{x-4} \geq \frac{x-1}{x+6}$

Note: There are **TWO** methods, both of which require a **FUNCTION** (let $f(x) = \ldots$ returns)

1) Get a Single Function (on one side of the inequality)	2) Use Two Functions (one for each side)
	

Success Criteria:

- I can recognize that an inequality has many possible intervals of solutions
- I can solve an inequality algebraically, using an interval chart
- I can solve an inequality graphically