Advanced Functions

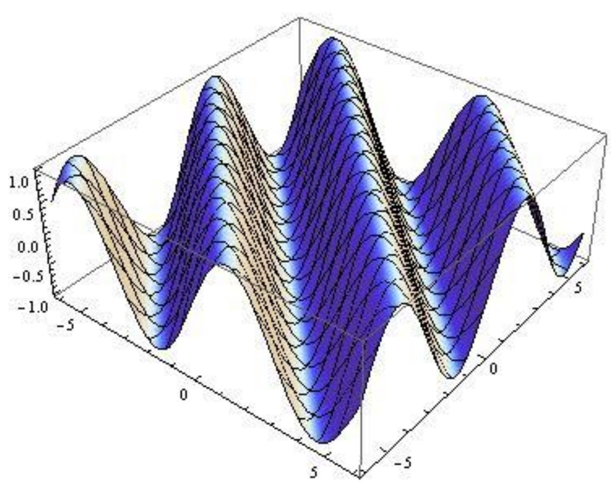
Course Notes

Unit 5 – Trigonometric Functions

Doing Trig with REAL Numbers

We will learn

- *about Radian Measure and its relationship to Degree Measure*
- *how to use Radian Measure with Trigonometric Functions*
- *about the connection between trigonometric ratios and the graphs of trigonometric functions*
- *how to apply our understanding of trigonometric functions to model and solve real world problems*



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Chapter 5 – Trigonometric Functions

*Contents with suggested problems from the Nelson Textbook (**Chapter 6**)*

You are welcome to ask for help, from myself or your peers, with any of the following problems. They will be handed in on the day of the Unit Test as a homework check.

5.1 Radian Measure and Arc Length

Pg. 321 #2edfh, 3 – 9

5.2 Trigonometric Ratios and Special Triangles (Part 1)

Pg. 330 #1b – f, 2bcd, 3

5.3 Trigonometric Ratios and Special Triangles (Part 2 – Exact Values)

Pg. 330 – 331 #5, 7, 9

5.4 Trigonometric Ratios and Special Triangles (Pt 3 – Getting the Angles)

Pg. 331 #6, 11, 16

5.7 Applications of Trigonometric Functions

Pg. 360 – 362 #4, 6, 9, 10

5.1 Radian Measure and Arc Length

Learning Goal: We are learning to use radian measure to represent the size of an angle.

Radian Measure

We are familiar with measuring angles using "degrees", and now we will turn to another measure for angles: **Radians**.

Before getting to the notion of radians, we need to learn some notation.

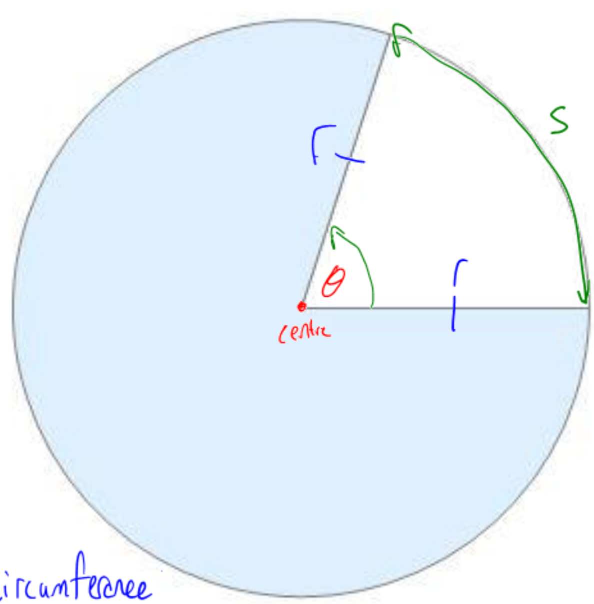
Picture

θ = central angle

r = radius

s = arc length subtended by the central angle

→ part of the circumference



There is a relationship between θ, r, and s.

$$\boxed{s = r\theta}$$

must be in radians

arc length formula

Definition 5.1.1

In a circle of radius r, a central angle θ subtending an arc of length $s = r$ measures 1 radian.

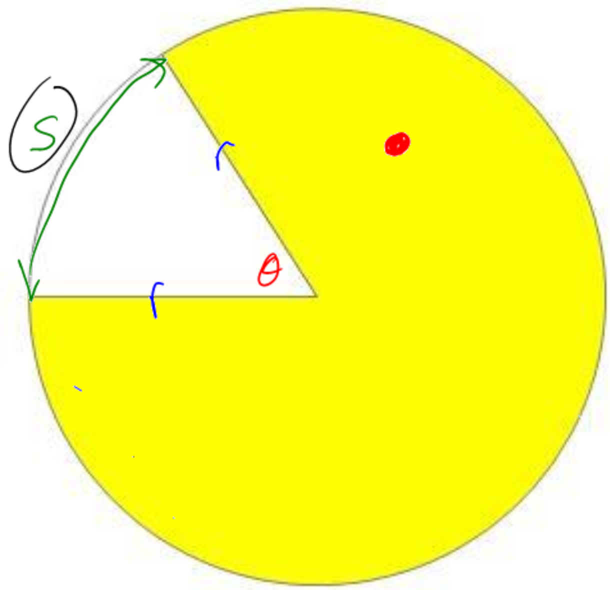
Picture

If $s=r$, $\theta = 1$ radian

$$s = r\theta$$

$$\frac{r}{r} = \frac{r\theta}{r}$$

$$1^{\text{rad.}} = \theta$$



Now, let's connect radians and degrees.

Note: The circumference of a circle is given by $C = 2\pi r$

So, for a central angle of 360°, in a circle of radius $r = 1$, then

arc length is the whole circle.

$$s = C$$

$$s = 2\pi r$$

$$\cancel{r}\theta = 2\pi \cancel{r}$$

$$\theta = 2\pi \text{ radian}$$

$$360^{\circ} = 2\pi \text{ radians}$$

$$\boxed{180^{\circ} = \pi \text{ radians}}$$ our conversion factor.

$$1^\circ = \frac{\pi}{180} \text{ rads}$$

Example 5.1.1

Convert the following to radians:

a) $\cancel{30}^{\circ}\left(\frac{\pi}{\cancel{180}}\right)$ $= \frac{\pi}{6}$ rad.

b) $\cancel{45}^{\circ}\left(\frac{\pi}{\cancel{180}}\right)$ $= \frac{\pi}{4}$ rad

c) $\cancel{120}^{\circ}\left(\frac{\pi}{\cancel{180}}\right)$ $= \frac{2\pi}{3}$ rad

$$\frac{120\pi}{180}$$

d) $\cancel{315}^{\circ}\left(\frac{\pi}{\cancel{180}}\right)$ $= \frac{7\pi}{4}$ or $\frac{7}{4}\pi$

e) $161.3^{\circ}\left(\frac{\cancel{\pi}}{180}\right)$ (3.14) $= 2.81$ rad

Example 5.1.2

Convert the following to degrees (round to two decimal places where necessary)

$$\frac{180}{\pi} = 1 \text{ rad}$$

a) $\frac{7\cancel{\pi}}{12}$ ~~rad~~ $\left(\frac{180}{\cancel{\pi}}\right)$ $= 105^{\circ}$

b) $\frac{10\cancel{\pi}}{9}$ ~~rad~~ $\left(\frac{180}{\cancel{\pi}}\right)$ $= 200^{\circ}$

c) 2.5 ~~rad~~ $\left(\frac{180}{\cancel{\pi}}\right)$ (3.14) $= 143.31^{\circ}$

d) $\frac{\pi}{2}$ rad $= 90^{\circ}$

e) $-\frac{\cancel{\pi}}{3}$ ~~rad~~ $\left(\frac{180}{\cancel{\pi}}\right)$ $= -60^{\circ}$

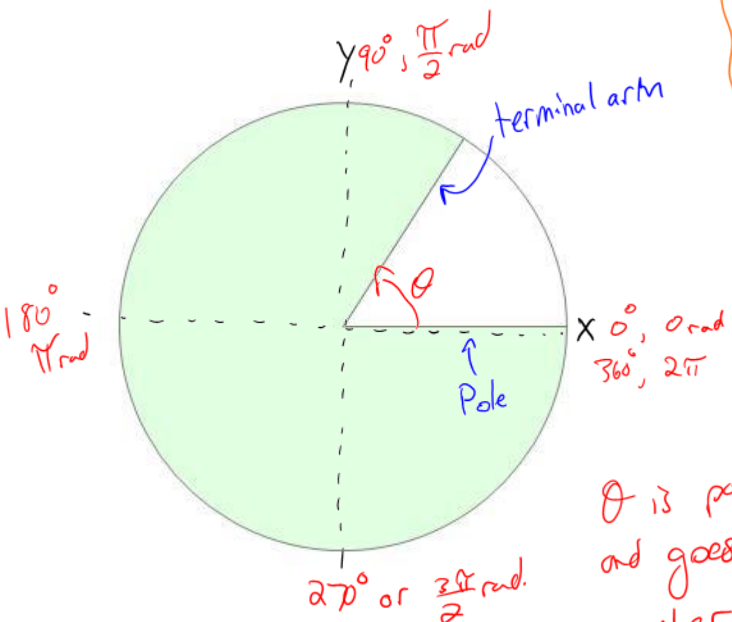
Q. What the rip is a negative degree?

Angles of Rotation

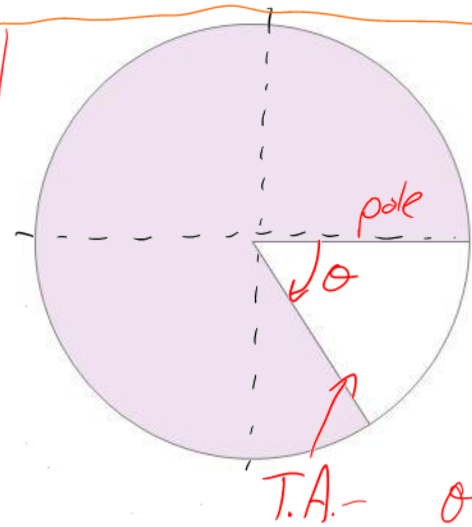
The sign on an angle can be thought of as the direction of rotation (around a circle).

Angles of rotations begin at the pole, and extend to the terminal arm

Pictures



θ is positive and goes in counter clockwise



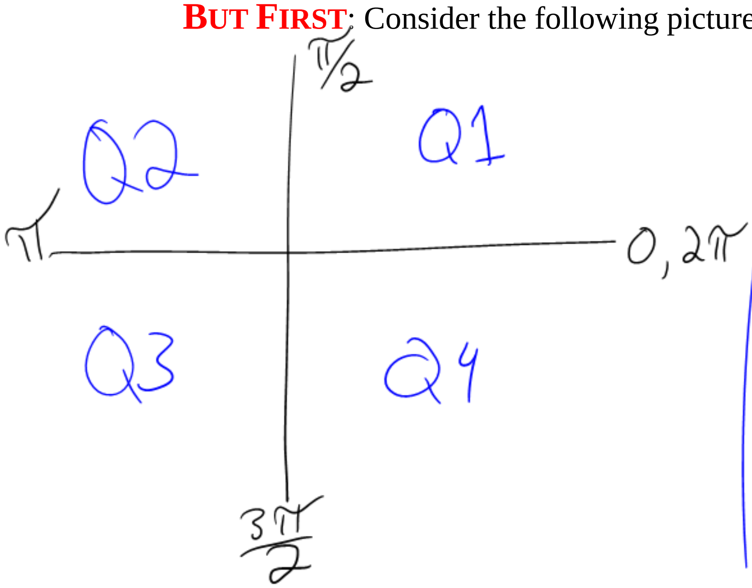
θ is negative and goes clock-wise

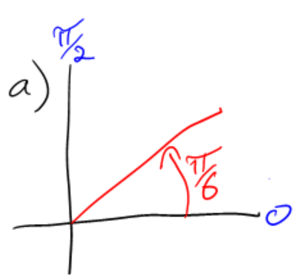
Example 5.1.3

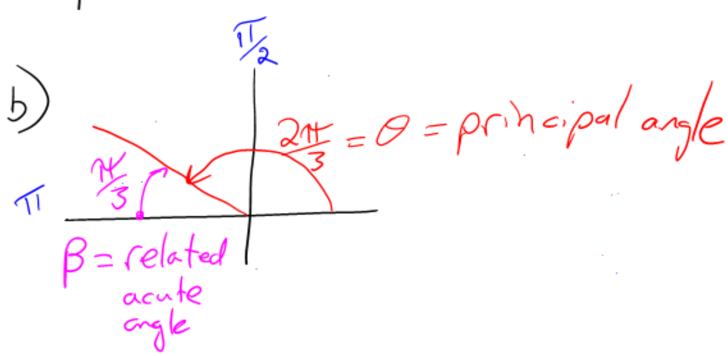
Sketch the following angles of rotation:

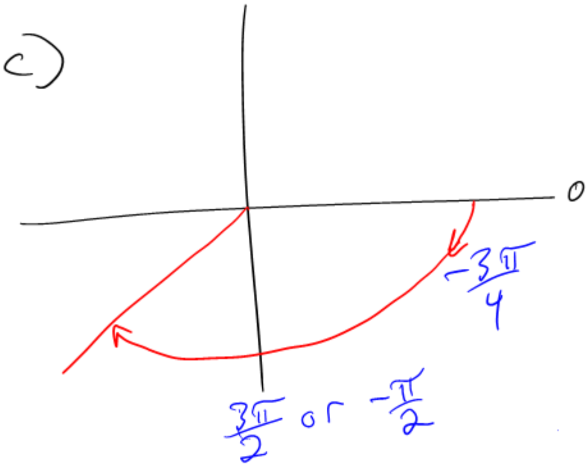
a) $\frac{\pi}{6}$ rad b) $\frac{2\pi}{3}$ rad c) $-\frac{3\pi}{4}$ rad d) $\frac{7\pi}{6}$

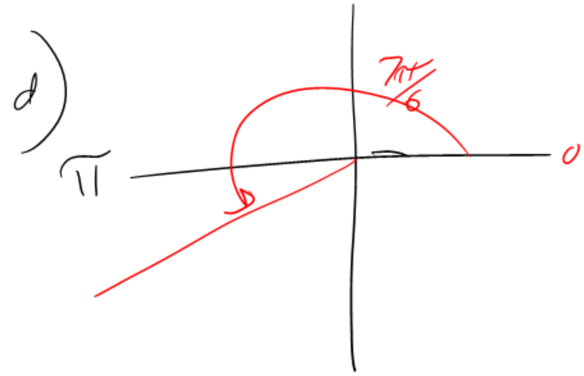
BUT FIRST: Consider the following picture:











Example 5.1.4

Determine the length of an arc, on a circle of radius $\overset{r}{5cm}$, subtended by an angle:

a) $\theta = 2.4$ rad

$S = r\theta$

$S = (5)(2.4)$

$S = 12cm$

b) $\theta = 120^{\circ}$

$120\left(\frac{\pi}{180}\right) = \frac{2\pi}{3}$

$S = r\theta$

$S = (5)\left(\frac{2\pi}{3}\right)$

$S = \frac{10\pi}{3} cm$

$S \approx 10.5cm.$

Success Criteria:

- I can understand that a radian is a real number
- I can convert from degrees to radians by multiplying by $\frac{\pi}{180^{o}}$
- I can convert from radians to degrees by multiplying by $\frac{180^{o}}{\pi}$
- principal angle, related acute, terminal arm, pole, angle of rotation

5.2 Trigonometric Ratios and Special Triangles

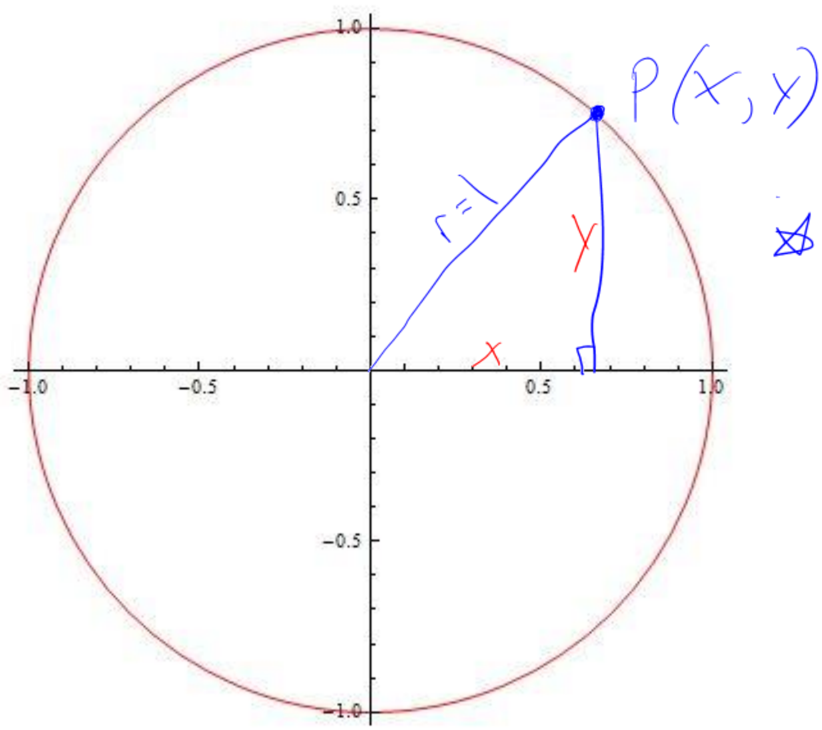
(*Part 1*)

-no calculator
-exact numbers

Learning Goal: We are learning to use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

Consider the circle of radius 1:

unit circle



☆ Always connect the terminal arm to the pole. or x-axis

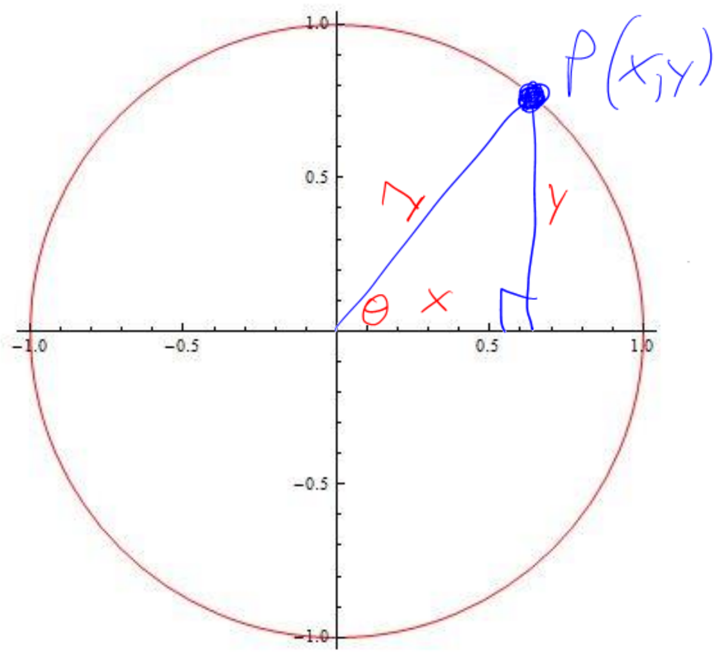
By Pythagorean Theorem:

$x^2 + y^2 = 1$

Recall the six main Trigonometric Ratios:

Primary Trig Ratios	Reciprocal Trig Ratios
$\sin\theta = \dfrac{opp}{hyp}$	$\dfrac{1}{\sin\theta} = \csc\theta = \dfrac{hyp}{opp}$
$\cos\theta = \dfrac{adj}{hyp}$	$\dfrac{1}{\cos\theta} = \sec\theta = \dfrac{hyp}{adj}$
always between 0 and 1.	greater than one.
$\tan\theta = \dfrac{opp}{adj}$	$\dfrac{1}{\tan\theta} = \cot\theta = \dfrac{adj}{opp}$

Consider again the circle of radius 1 (but now keeping in mind SOH CAH TOA)



$\sin\theta = \dfrac{y}{1} = y$

$\cos\theta = \dfrac{x}{1} = x$

$\tan\theta = \dfrac{y}{x}$

Note: $P(x,y)$ can now be represented by $P(\cos\theta, \sin\theta)$

The Pythagorean Identity

$$x^2 + y^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

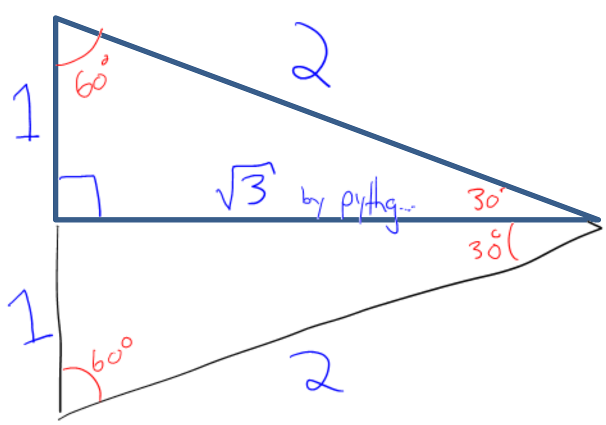
$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

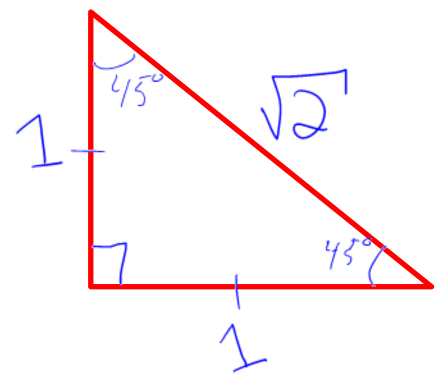
Special Triangles in Radians

Recall: We have two “Special Triangles”. In **degrees** they are:

Equilateral Triangle

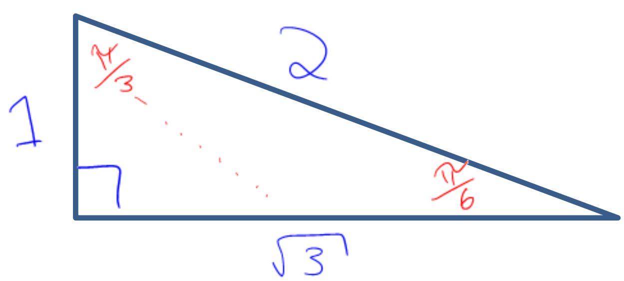
Isosceles Triangles.

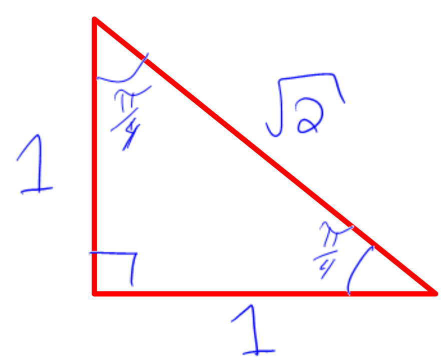




In radians we have

$$30°\left(\frac{\pi}{180°}\right) = \frac{\pi}{6} \quad 60°\left(\frac{\pi}{180°}\right) = \frac{\pi}{3} \quad 90°\left(\frac{\pi}{180}\right) = \frac{\pi}{2} \quad 45°\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$$



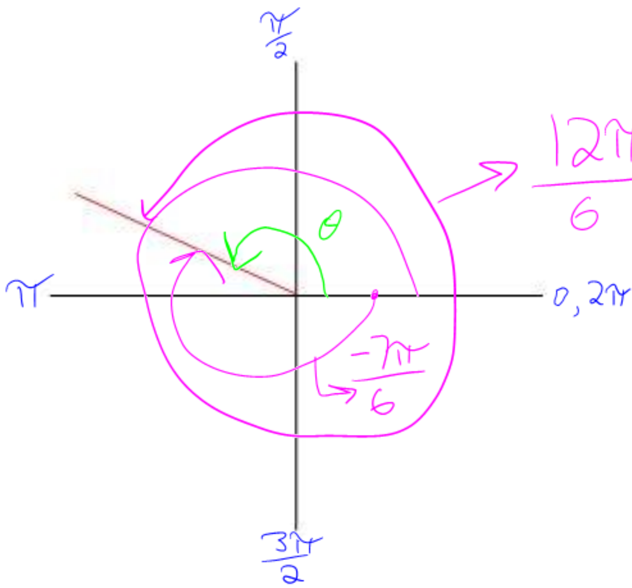


MEMORIZE THESE!

Angles of Rotations and Trig Ratios

Consider the following sketch of the angle of rotation $\theta = \frac{5\pi}{6}$:

$$\frac{2\pi}{1} = \frac{12\pi}{6}$$



$$\frac{12\pi}{6} + \frac{5\pi}{6} = \frac{17\pi}{6}$$

There are infinitely many angles of rotation for each terminal arm.

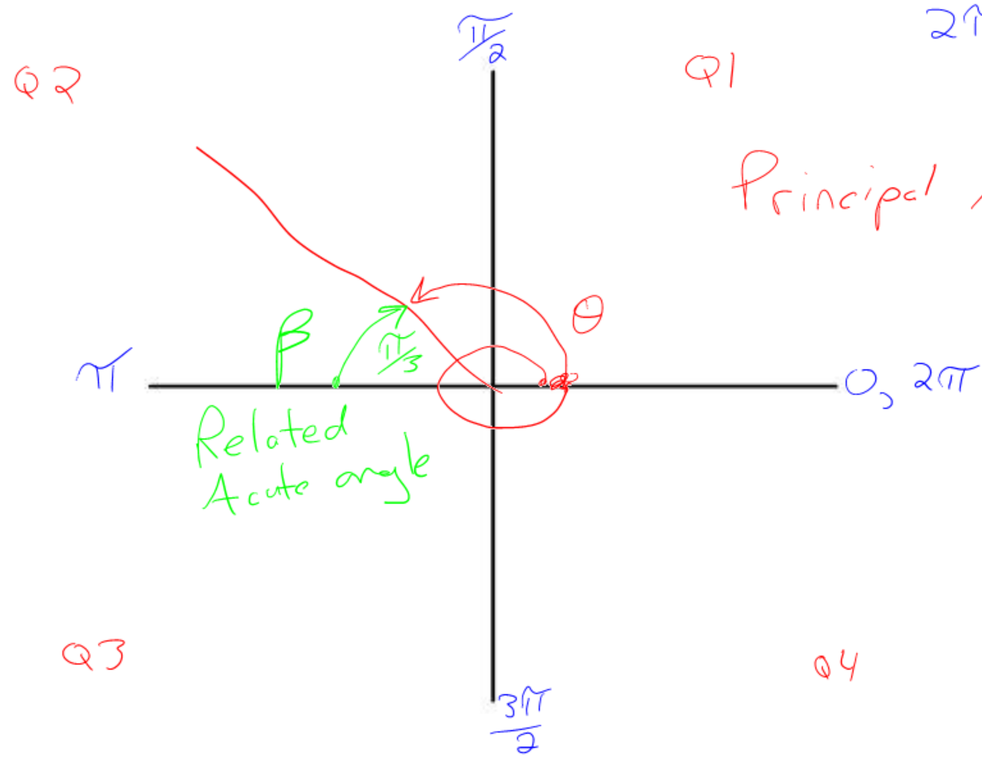
In this Example, we call $\theta = \frac{5\pi}{6}$ the **PRINCIPAL ANGLE**, or the angle in standard position. This means the smallest positive angle.

Note: All principal angles $\underbrace{\theta \in [0, 2\pi]}_{\text{one circle}}$

Example 5.2.1

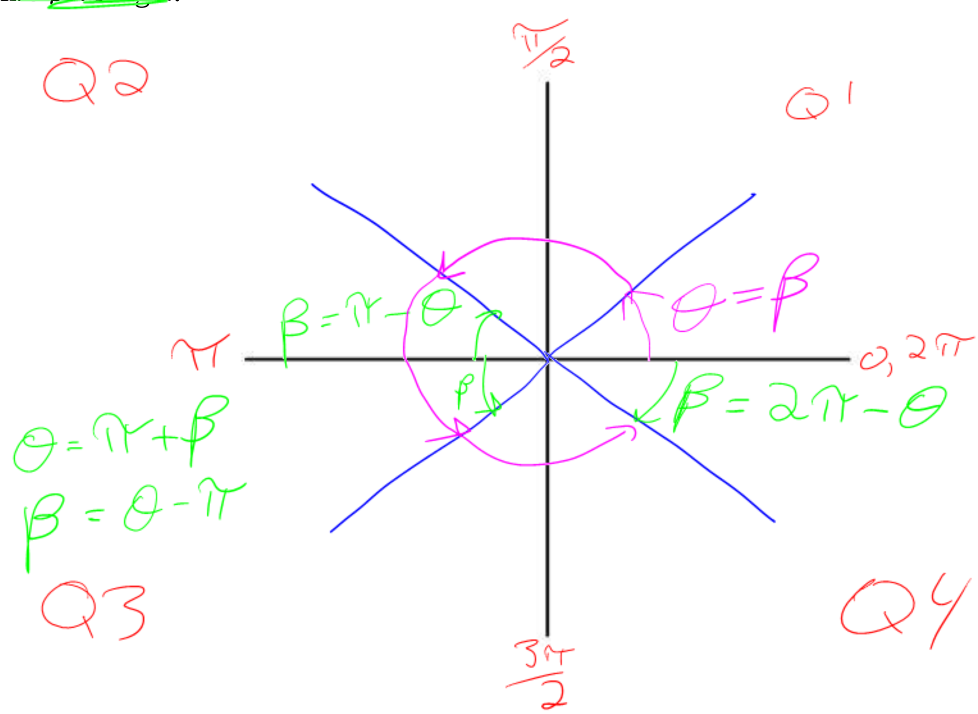
Sketch the angle of rotation $\theta = \dfrac{8\pi}{3}$ and determine the principal angle.

$2\pi = \dfrac{6\pi}{3}$



Principal Angle $= \dfrac{2\pi}{3}$

~~Principal Angle~~:



θ = principal

β = related Acute

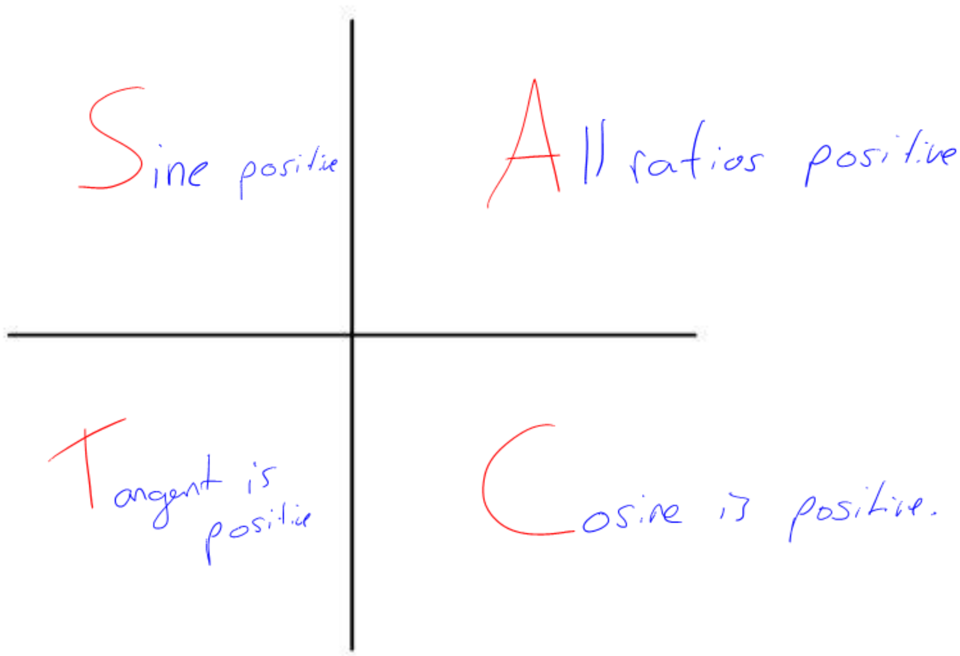
We now have enough tools to calculate the trigonometric ratios of any angle!

For any given angle θ (in radians from here on) we will:

1) Draw θ in standard position (i.e. draw the principal angle for θ)
2) Determine the related acute angle (between the terminal arm and the polar axis)
3) Use the related acute angle and the CAST RULE (and SOH CAH TOA) to determine the trig ratio in question

Recall the CAST RULE

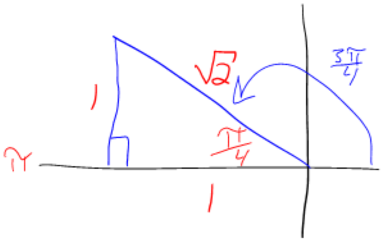
Note: The CAST RULE determines the sign (+ or -) of the trig ratio



Example 5.2.2 Q2

Determine the trig ratio $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

sine is positive in Q2



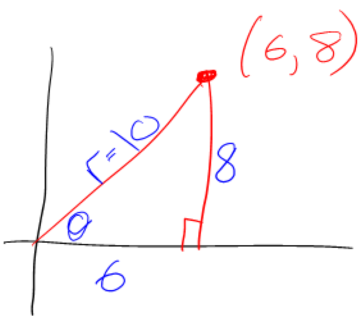
Example 5.2.3 Q1

The point $(6,8)$ lies on the terminal arm (of length *r*) of an angle of rotation. Sketch the angle of rotation.

Determine: a) the value of *r*

b) the primary trig ratios for the angle

c) the value of the angle of rotation in radians, to two decimal places



a) $6^2 + 8^2 = r^2$

$36 + 64 = r^2$

$100 = r^2$

$10 = r$

b) $\sin\theta = \frac{8}{10} = \frac{4}{5}$

$\cos\theta = \frac{6}{10} = \frac{3}{5}$

$\tan\theta = \frac{8}{6} = \frac{4}{3}$

$\sin 30 = 0.5$ degrees

$\sin 30 = -0.98$ rad

c) $\sin\theta = \frac{4}{5}$

$\theta = \sin^{-1}\left(\frac{4}{5}\right)$

$\theta = 0.93$ rad

Success Criteria

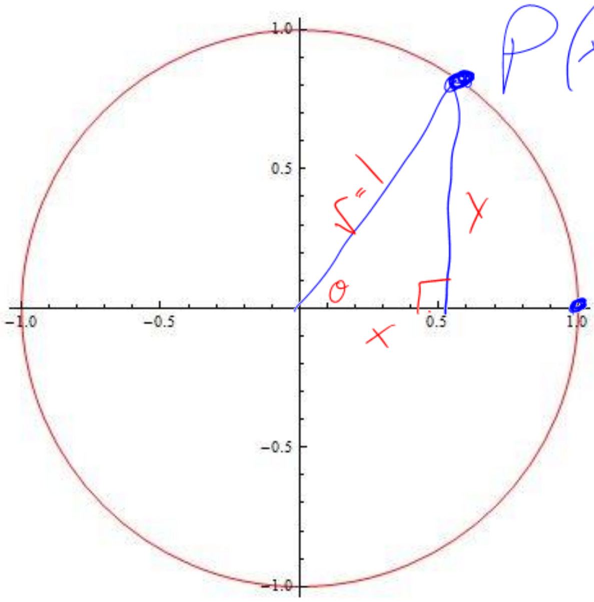
- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can write the trig ratios for any angle using "x, y, and r"
- I can use the CAST rule to determine where a ratio is positive or negative

5.3 Trigonometric Ratios and Special Triangles

(*Part 2 – Exact Values*)

Learning Goal: We are learning to use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

Recall the "Unit Circle" from yesterday:



$P(x, y) = P(\cos\theta, \sin\theta)$

$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \frac{y}{x}$

$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

With this circle (and without a calculator!) we can evaluate EXACTLY the trig ratios for the angles (in radians) $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ radians. Axis angles

(1, 0) 0, 2π

$\sin(0) = 0 = \sin(2\pi)$

$\cos(0) = 1 = \cos(2\pi)$

$\tan(0) = \frac{0}{1} = 0 = \tan(2\pi)$

π (−1, 0)

$\sin(\pi) = 0$

$\cos(\pi) = -1$

$\tan(\pi) = \frac{0}{-1} = 0$

(0, 1)

$\sin\left(\frac{\pi}{2}\right) = 1$

$\cos\left(\frac{\pi}{2}\right) = 0$

$\tan\left(\frac{\pi}{2}\right) = \frac{1}{0}$ = undefined

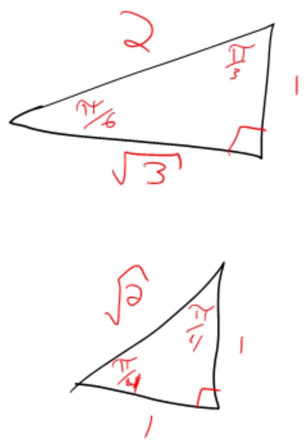
(0, −1) $\frac{3\pi}{2}$

$\sin\left(\frac{3\pi}{2}\right) = -1$

$\cos\left(\frac{3\pi}{2}\right) = 0$

$\tan\left(\frac{3\pi}{2}\right) = \frac{-1}{0}$ = undefined

Now, using **Special Triangles**, and **CAST** we can evaluate **EXACTLY** trig ratios for “special angles”.



Note: A trig ratio is a NUMBER.
Numbers have 2 qualities

1) Value
2) sign (+/−)

Thus a trig ratio has a Value

(which we e**valu**ate using the related acute angle and Special Triangles)

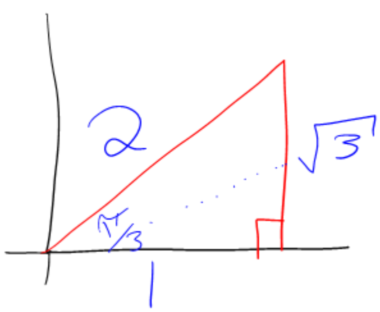
AND, a trig ratio has a sign from the CAST Rule.

Example 5.3.1

Determine **Exactly** (i.e. the **use of a calculator** means **MARKS OFF**)

Q1

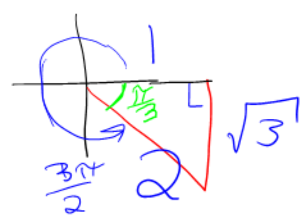
a) $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$



Sine is positive in Q1

Q4

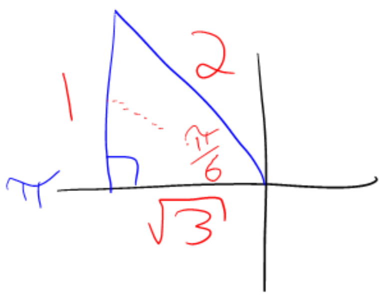
d) $\sec\left(\frac{5\pi}{3}\right) = \frac{2}{1} = 2$



$\boxed{\cos\frac{\pi}{3} = \frac{1}{2}}$

Q2

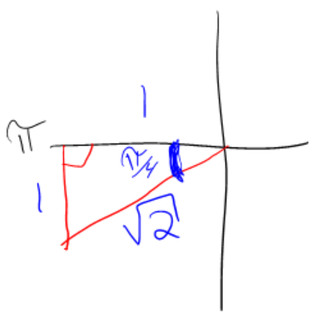
b) $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$



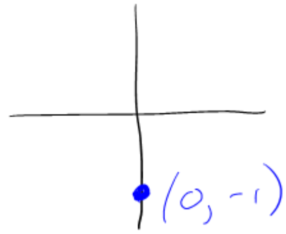
cosine is negative in Q2.

Q3

c) $\tan\left(\frac{5\pi}{4}\right) = \frac{1}{1} = 1$

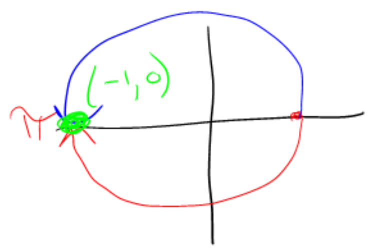


e) $\tan\left(\frac{3\pi}{2}\right) = \frac{y}{x} = \frac{-1}{0} = \text{undefined}$



f) $\csc(-\pi) = \csc(\pi) = \frac{1}{\sin(\pi)} = \frac{1}{0}$

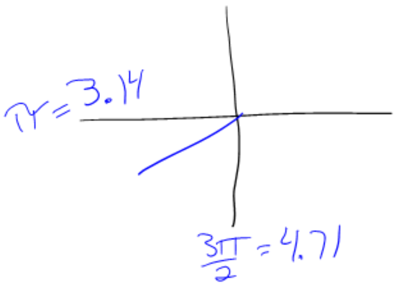
$= \text{undefined}$



Example 5.3.2

Given $\sin(4)$ determine:

a) The quadrant $\theta = 4$ is in. b) The sign of $\sin(4)$ (no calculators!)



In Q3.

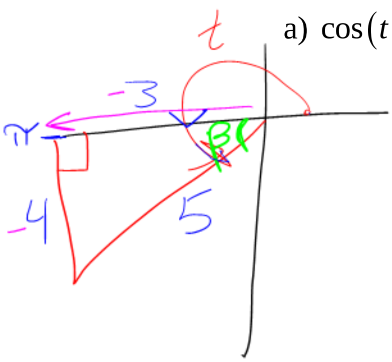
$\sin(4)$ would be negative because in Q3, only tan is positive.

Example 5.3.3

Q3

Given $\sin(t) = -\frac{4^{\,op}}{5_{\,hyp}}$, $\pi \le t \le \frac{3\pi}{2}$, determine

a) $\cos(t)$ b) $\tan(t)$ c) t in radians, rounded to three decimal places.



a) $\cos(t) = \frac{-3}{5}$ b) $\tan(t) = \frac{-4}{-3} = \frac{4}{3}$

c) Always use a positive ratio to get the β then convert to θ.

$\tan(\beta) = \frac{4}{3}$

$\beta = \tan^{-1}\left(\frac{4}{3}\right)$

$\beta = 0.927$

$t = \pi + \beta$

$t = 3.141 + 0.927$

$t = 4.068$ rad

Success Criteria

- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can identify the trig ratios of the 4 axis angles
- I can write the trig ratios for any angle using "x, y, and r"
- I can use the CAST rule to determine where a ratio is positive or negative

5.4 Trigonometric Ratios and Special Triangles

(*Part 3 – Getting the Angles*)

Learning Goal: We are learning to determine the exact values for both angles between 0 and 2π, given a particular trig ratio.

We have been looking at **evaluating exact values** for trigonometric ratios using special triangles and CAST, given an **angle of rotation**. We now turn our attention to the **inverse operation** – determining **angles of rotation** given a **trig ratio**.

Example 5.4.1

Determine exactly:

a) $\sin\left(\frac{\pi}{6}\right)$

b) $\sin\left(\frac{5\pi}{6}\right)$

Note:

Example 5.4.2

Determine BOTH angles of rotation, θ, for $0 \le \theta \le 2\pi$ given

a) $\sin(\theta) = \frac{\sqrt{3}}{2}$

Procedure

1) Determine the quadrants θ is in.
2) Draw the angles of rotation.
3) Determine the related acute angle φ and construct the appropriate special triangles.
4) Determine the angles of rotation.

b) $\cos(\theta) = -\frac{1}{\sqrt{2}}$

c) $\cot(\theta) = -\sqrt{3}$

d) $\sin(\theta) = -1$

Example 5.4.3

Determine θ where $\frac{3\pi}{2} \le \theta \le 2\pi$ for $\csc(\theta) = -2$

Example 5.4.4

Determine θ when $\cos\theta = -0.8213$.

Practice Problems (Homework)

Determine the angles of rotation, θ, for $0 \le \theta \le 2\pi$:

a) $\sin(\theta) = -\frac{\sqrt{3}}{2}$

b) $\sec(\theta) = \sqrt{2}$

c) $\tan(\theta) = \frac{1}{\sqrt{3}}$

d) $\cot(\theta) = -1$

e) $\csc(\theta) = \frac{2}{\sqrt{3}}$

f) $\cos(\theta) = 0$

g) $\sin(\theta) = 1$

h) $\sqrt{3}\cos(\theta) - 2\cos(\theta)\cdot\sin(\theta) = 0$

Success Criteria

- I can write the angles in the special triangles using both radians (Unit Circle) and degrees
- I can find the principal angle θ, by first finding the related acute angle β
- I can, given a trig ratio, determine the exact values for both angles between 0 and 2π
- I can identify the trig ratios of the 4 axis angles
- I can write the trig ratios for any angle using "x, y, and r"
- I can use the CAST rule to determine where a ratio is positive or negative

5.7 Applications of Trigonometric Functions

Learning Goal: We are learning to solve real-world problems that can be modeled with a trigonometric function.

For any phenomenon in the real world which has a periodically repeating behaviour, Trigonometric Functions can be used to describe and analyze that behaviour. There are a myriad of such phenomena. From the rise and fall of tides to computer gaming habits, Trigonometric Functions have a say.

Figure 5.7.1 A periodic rise and fall in online gamers

Example 5.7.1

From your text: Pg. 345 #9

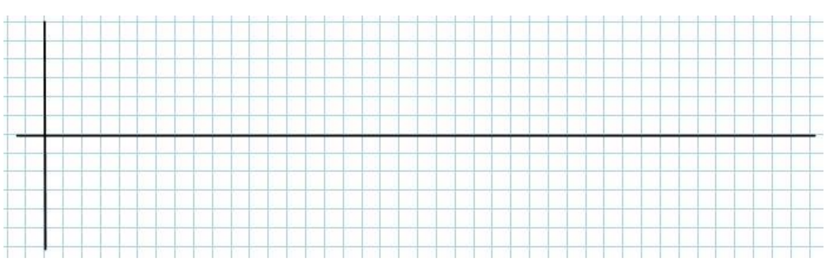
9. Each person's blood pressure is different, but there is a range o
A pressure values that is considered healthy. The function $P(t) = -20\cos\frac{5\pi}{3}t + 100$ models the blood pressure, p, in millimetres of mercury, at time t, in seconds, of a person at res

a) What is the period of the function? What does the period represent for an individual?

b) How many times does this person's heart beat each minute

c) Sketch the graph of $y = P(t)$ for $0 \leq t \leq 6$.

d) What is the range of the function? Explain the meaning of range in terms of a person's blood pressure.



Example 5.7.2

From your text Pg. 361 #7

7. A person who was listening to a siren reported that the frequency of the sound fluctuated with time, measured in seconds. The minimum frequency that the person heard was 500 Hz, and the maximum frequency was 1000 Hz. The maximum frequency occurred at $t = 0$ and $t = 15$. The person also reported that, in 15, she heard the maximum frequency 6 times (including the times at $t = 0$ and $t = 15$). What is the equation of the cosine function that describes the frequency of this siren?

Success Criteria:

- I can model a real-world situation using a trigonometric function
- I can use the trigonometric model to solve problems