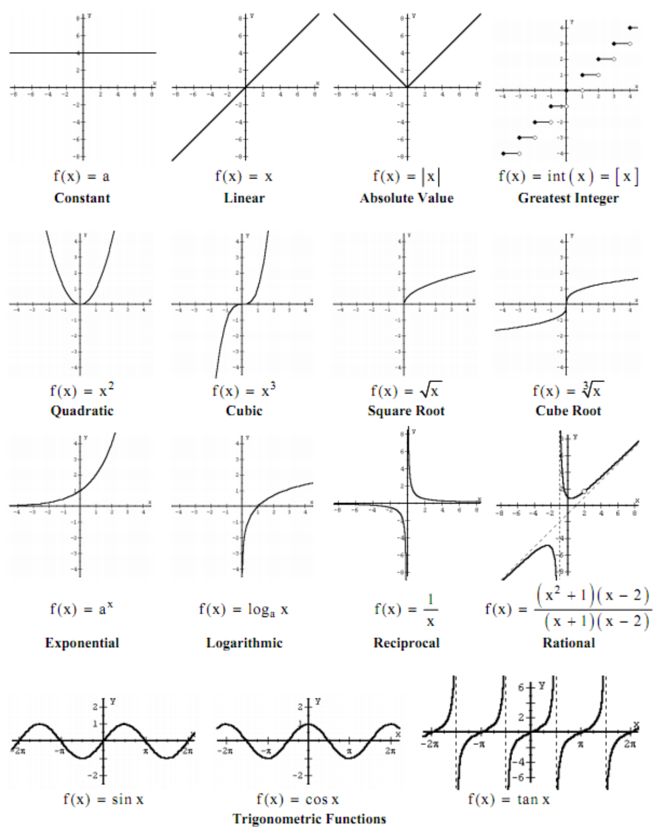
Advanced Functions

Fall 2017
Course Notes

Chapter 1 – Functions

PARENT FUNCTIONS



A∞Ω
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Chapter 1 – Introduction to Functions

Contents with suggested problems from the Nelson Textbook

1.1 Functions

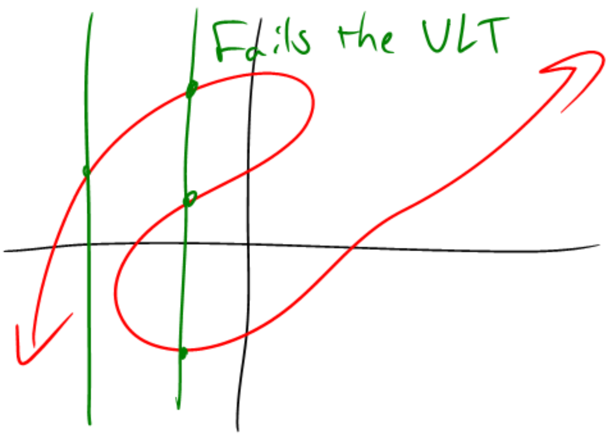
There are some people who argue that mathematics has just two basic building blocks: Numbers and Operations. This course is concerned with functions which can be considered number generators. A function takes a given number, and using mathematical operations generates another number. We will be examining the **relationship** between the given numbers, and the generated numbers for various functions.

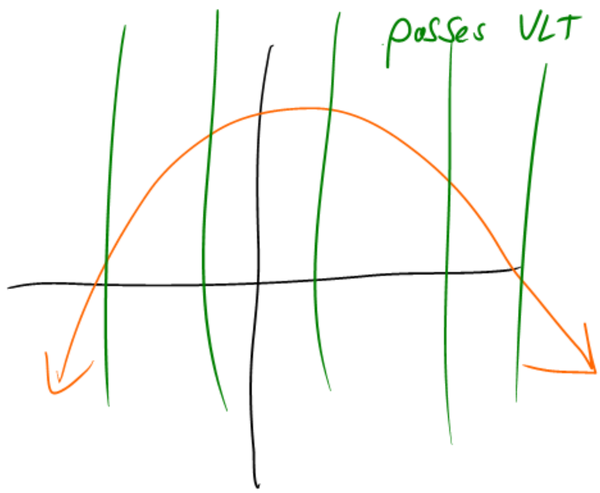
Definition 1.1.1

A **Function** is an algebraic rule which assigns exactly one element in a set called the range to each element in a set called the domain.
→ each "x" value produces one "y" value.

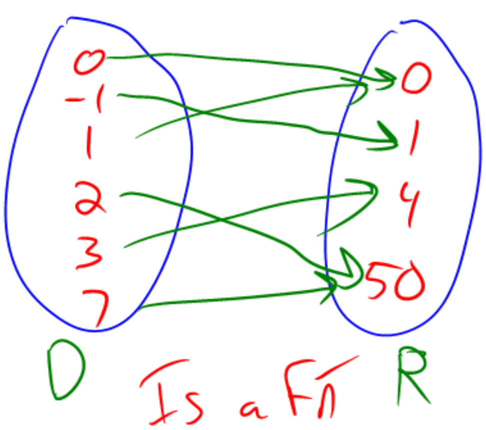
Pictures

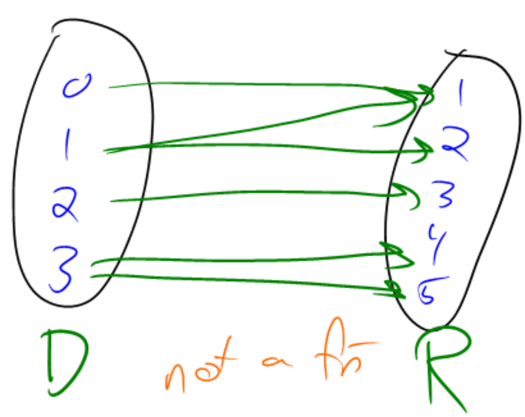
Vertical Line Test





Arrow Diagrams





$f(x) = \sqrt{x-1}$

Definition 1.1.2

Domain of a Function: A set of "x" values which "make sense" when plugged into $f(x)$.

Range of a Function
the set of functional values which are calculated or derived from the domain.

Function Notation

We use the notation $f(x)$ to "name" a function. This notation is powerful because it contains both the domain and the range. For example we might write $f(2)$, which shows that the domain value is $x = 2$, and that the range value (which we must calculate) is denoted $f(2)$.

Definition 1.1.3

The **Graph** of a function is a set of points $(x, f(x))$ and denoted $f(x) = \{(x, f(x)) \mid x \in D_f\}$

Example 1.1.1

Given the graph of the function $f(x) = \{(3,4),(2,-1),(7,8),(4,2),(5,4)\}$ determine:

a) $D_f = \{3, 2, 7, 4, 5\}$

b) $R_f = \{4, -1, 8, 2\}$

c) Is $f(x)$ a function?
Yes it is.

Example 1.1.2

Consider the sketch of the graph of $g(x)$, and determine:

a) $D_g = \{x \in \mathbb{R} \mid -4 \leq x \leq 5\}$

b) $R_g = \{g(x) \in \mathbb{R} \mid -2 \leq g(x) \leq 3\}$

c) Is $g(x)$ a function?
not a fn, fails VLT.

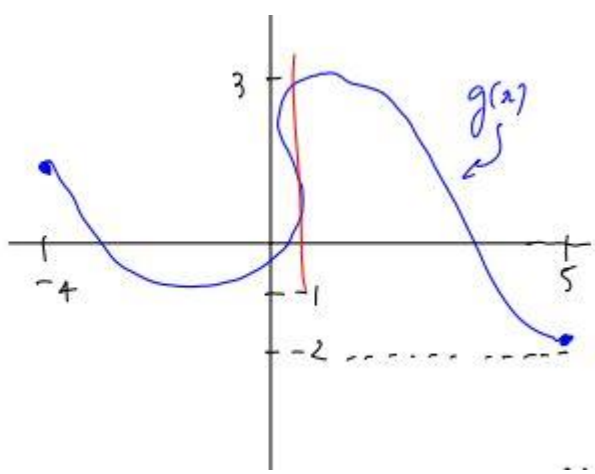


Figure 1.1.2

> *Note: In the above examples we have seen functions (and non-functions which we call relations) depicted graphically and numerically. We now turn to algebraic representations of functions. It is much more difficult to determine domain and range for functions given in an algebraic form, but the algebraic form is incredibly useful!*

Example 1.1.3

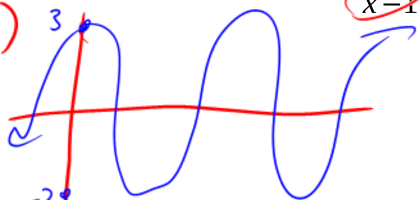
State the domain and range of the functions given in algebraic form.

a) $f(x) = 3\cos(2x)$

b) $g(t) = (t-2)^2 + 1$

c) $h(x) = \dfrac{2}{x-1}$

a)

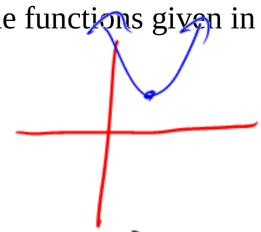


$D: \{x \in \mathbb{R}\}$ $\quad x \in (-\infty, \infty)$

$R: \{f(x) \in \mathbb{R} \mid -3 \le f(x) \le 3\}$

$f(x) \in [-3, 3]$

b)



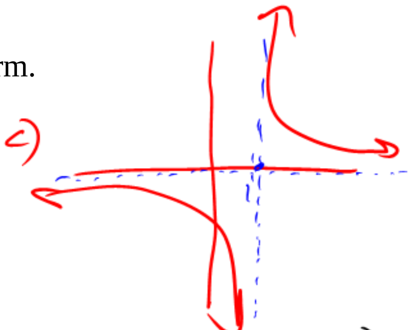
$D: \{t \in \mathbb{R}\}$

$R: \{g(t) \in \mathbb{R} \mid g(t) \ge 1\}$

$D: t \in (-\infty, \infty)$

$R: g(t) \in [1, \infty)$

c)



$D: \{x \in \mathbb{R} \mid x \ne 1\}$

$R: \{h(x) \in \mathbb{R} \mid h(x) \ne 0\}$

$D: x \in (-\infty, 1) \cup (1, \infty)$

union

Notations for Domain and Range

use example 2.

Interval Notation	Set Notation	Pseudo-set Notation
$D_g: x \in [-4, 5]$	$D: \{x \in \mathbb{R} \mid -4 \le x \le 5\}$	$D_g: -4 \le x \le 5$
$R_g: g(x) \in [-2, 3]$	$R: \{g(x) \in \mathbb{R} \mid -2 \le g(x) \le 3\}$	$R_g: -2 \le g(x) \le 3$

→ real numbers are assumed.

[= $\ge$ or $\le$

(= $>$ or $<$

Class/Homework for Section 1.1

Read Example 3 on Page 9
Pg. 11 – 13 #1 – 3, 5, 6, 7b-f, 9, 10, 12

1.2 Properties of Functions

Recall that we define the graph of a function to be the SET of Ordered Pairs:

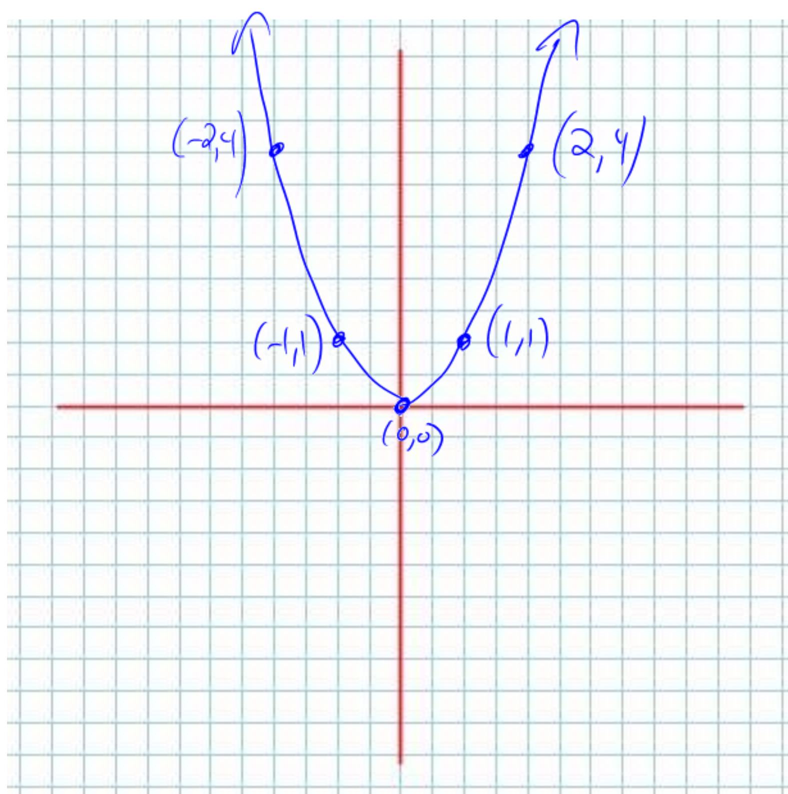
graph: $\{(x, f(x)) \mid x \in D_f\}$

We can visualize the graph of a function by plotting its ordered pairs on the Cartesian axes.

Example 1.2.1

e.g. $f(x) = x^2$ has the graph $\{(x, x^2) \mid x \in \mathbb{R}\}$

and looks like



Characteristics of a Function's Graph

Over the course we will be studying Polynomial, Rational, Trigonometric, Exponential and Logarithmic Functions. For now we are focussed on Polynomial and Rational Functions, but for each type of function we will try and understand various functional (fnal) behaviours (or characteristics).

The characteristics (behaviours) we are primarily interested in studying are:

1. Domain and Range.
2. Axis Intercepts
3. Intervals of Increase/Decrease → "chunks" of the domain
4. Odd and Even fns (symmetry)
5. Continuity vs discontinuity
6. Function end behaviors.

> Note: *Generally a geometric point of view will just mean that we'll look at pictures, but Geometry is actually* ***much*** *deeper than that!*

Intervals of Increase and Decrease

↳ only concerns the domain

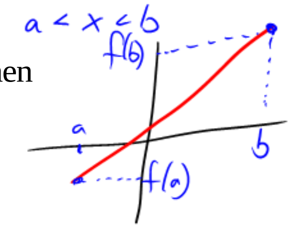
We will examine (when possible) functional behaviour from both algebraic and geometric points of view.

Definition 1.2.1

these are x-values ↓ → $a < x < b$

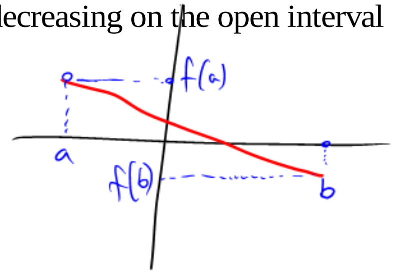
A function $f(x)$ is said to be increasing on the open interval (a,b) when

$f(a) < f(b)$



A function $f(x)$ is said to be decreasing on the open interval (a,b) when

$f(a) > f(b)$



Note the difference between open and closed intevals:

An open interval does not contan endpoints

(a, b)

A closed interval

does contan the endpoints

$[a, b]$

Example 1.2.2

Consider the function $f(x)$, represented graphically:

Determine where $f(x)$ is increasing and decreasing.

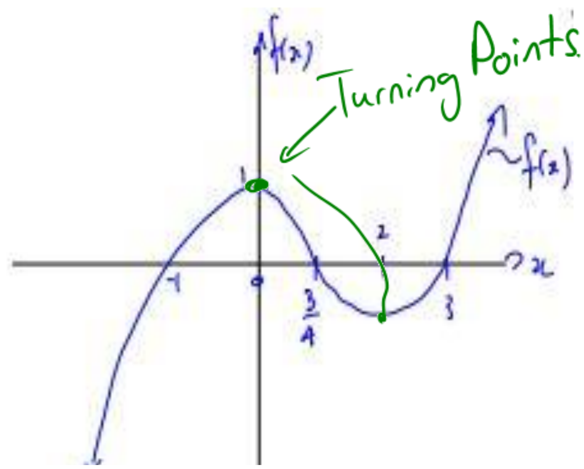
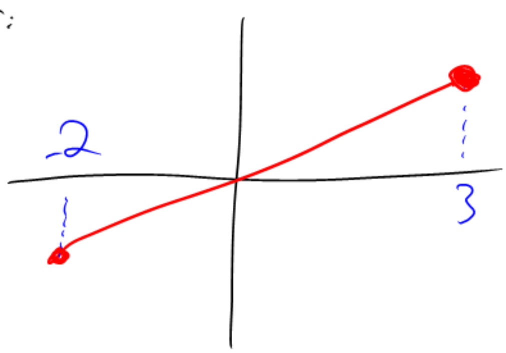


Figure 1.2.2

Increasing: $(-\infty, 0) \cup (2, \infty)$

Decreasing: $(0, 2)$

Consider:



Increasing on $[-2, 3]$

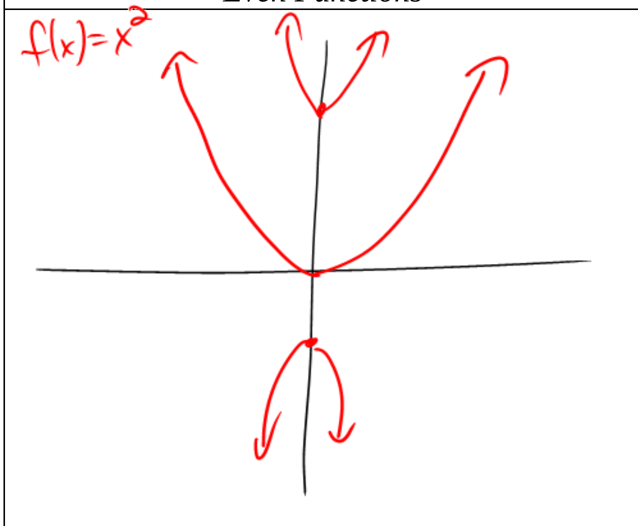
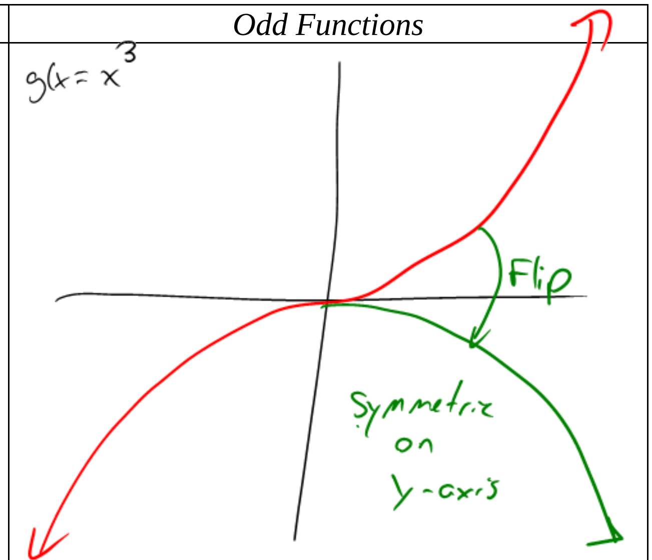
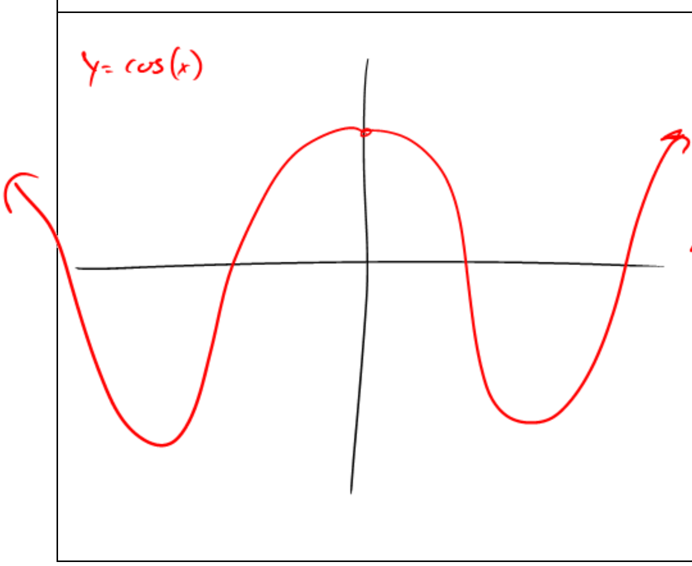
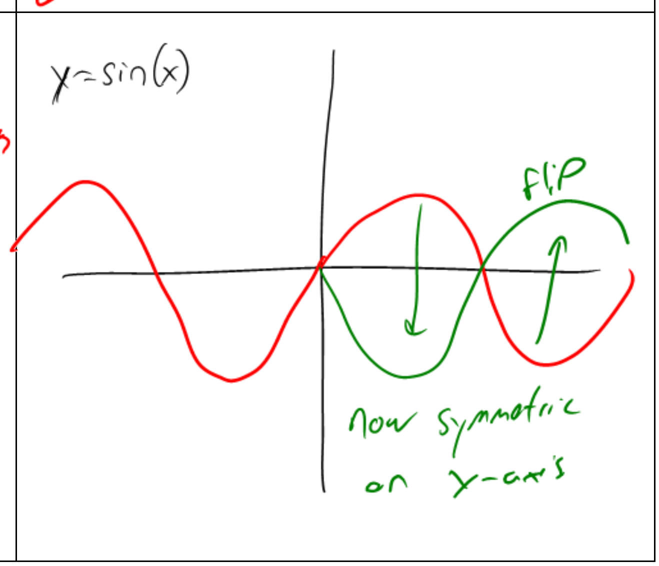
Odd vs. Even Functions

Note: This functional behaviour deals with SYMMETRY rather than the "power(s)" that you might see in various terms of the function.

Basic Definitions:

- Even Functions are symmetric around the "y-axis"
- Odd Functions are symmetric around the origin (0,0)

Graphical point of view:

Even Functions	*Odd Functions*
	
	

Algebraically we will consider definitions for Even and Odd Functions:

Definition 1.2.2

A function $f(x)$ is **even** if $f(x) = f(-x)$ for every $x \in D_f$

A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for every $x \in D_f$

vertical flip

Example 1.2.3

a) Show $f(x) = 3x^4 + 2x^2 + 5$ is even .

Consider $f(-x) = 3(-x)^4 + 2(-x)^2 + 5$

$= 3x^4 + 2x^2 + 5$

$= f(x)$

$\therefore f(x)$ is even.

b) Show $g(x) = 5x^3 - 2x$ is odd.

We want $-(5x^3 - 2x)$

Consider $g(-x) = 5(-x)^3 - 2(-x)$

$= -5x^3 + 2x$

Factor out the negative

$= -(5x^3 - 2x)$

$= -g(x) \quad \therefore g(x)$ is odd.

c) Are i) $f(t) = 5t^3 - 2t + 1$ and

ii) $h(x) = \dfrac{3x^3 - 2x}{x^2 - 1}$ odd or even?

$$\frac{-2}{3} = -\frac{2}{3}$$

i) Consider $f(-t) = 5(-t)^3 - 2(-t) + 1$

$f(-t) = -5t^3 + 2t + 1 \Rightarrow f(-t) \neq f(t)$ ∴ not even

$= -(5t^3 - 2t - 1) \Rightarrow f(-t) \neq -f(t)$ ∴ not odd

ii) Consider $h(-x) = \dfrac{3(-x)^3 - 2(-x)}{(-x)^2 - 1}$

$= \dfrac{-3x^3 + 2x}{x^2 - 1}$

$\dfrac{-(3x^3 - 2x)}{x^2 - 1}$

$-\left(\dfrac{3x^3 - 2x}{x^2 - 1}\right) = -h(x)$ ∴ odd.

$h(-x) \neq h(x)$ ∴ not even.

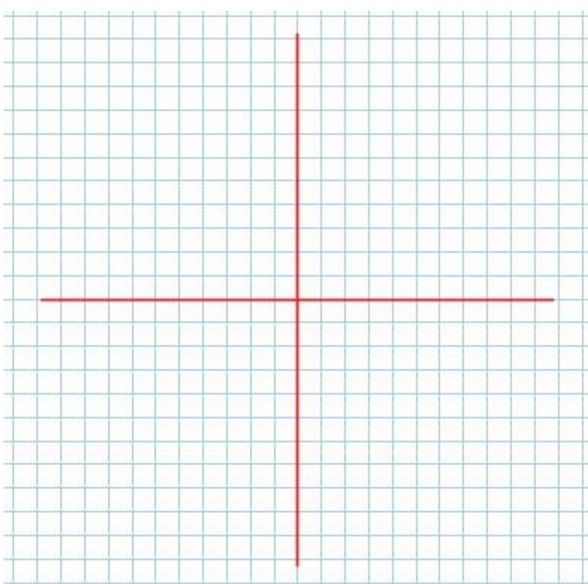
Continuity

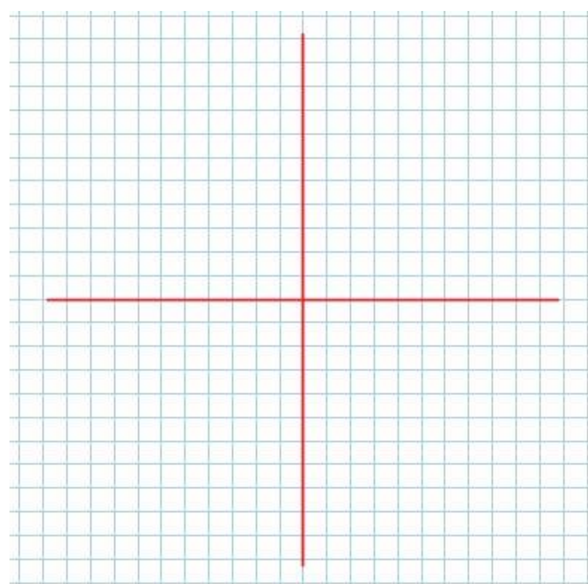
For the time being we will consider a (quite) rough definition of what it means for a function to be continuous. In fact, we will see that understanding what it means for a function to be discontinuous may be more helpful for now. In the course ***Calculus and Vectors***, a formal, algebraic definition of continuity will be considered.

Rough Definition

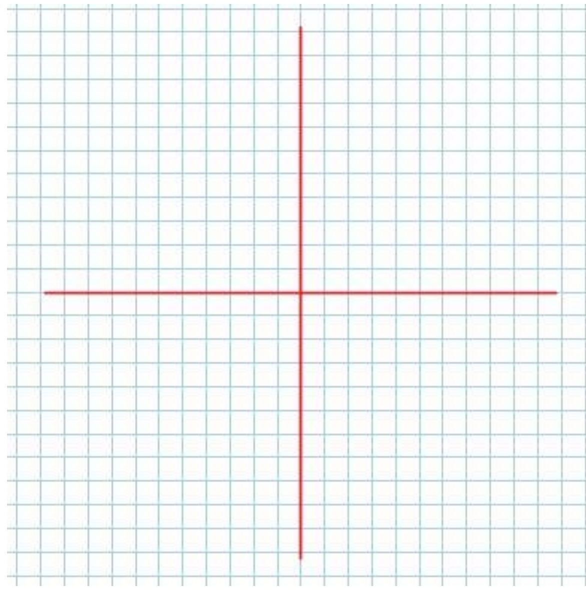
A function $f(x)$ is **continuous** (cts) on its domain D_f if

Pictures

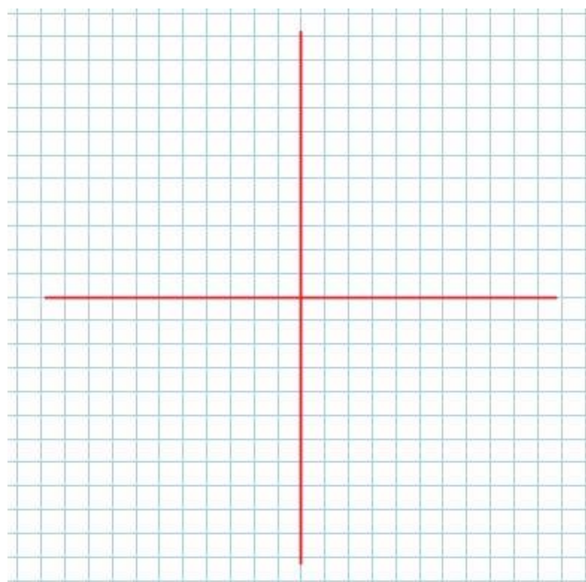




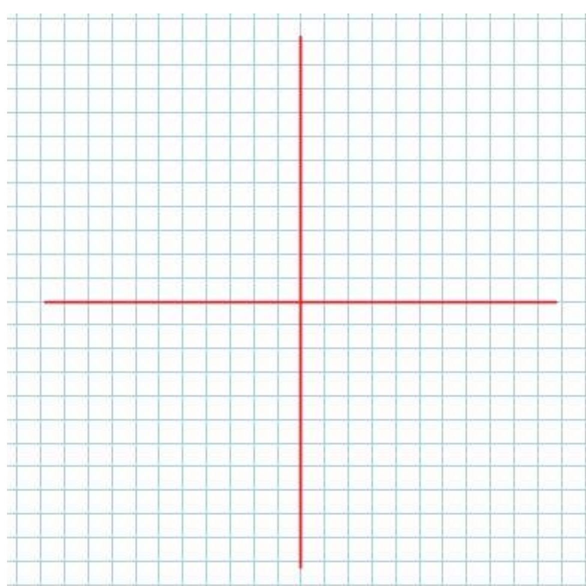
There are 3 types of **dis**continuities:



1)



2)



3)

End Behaviour of Functions

Here we are concerned with how the function is behaving as x gets

As x gets (which we write $x \to \infty$, or $x \to -\infty$)

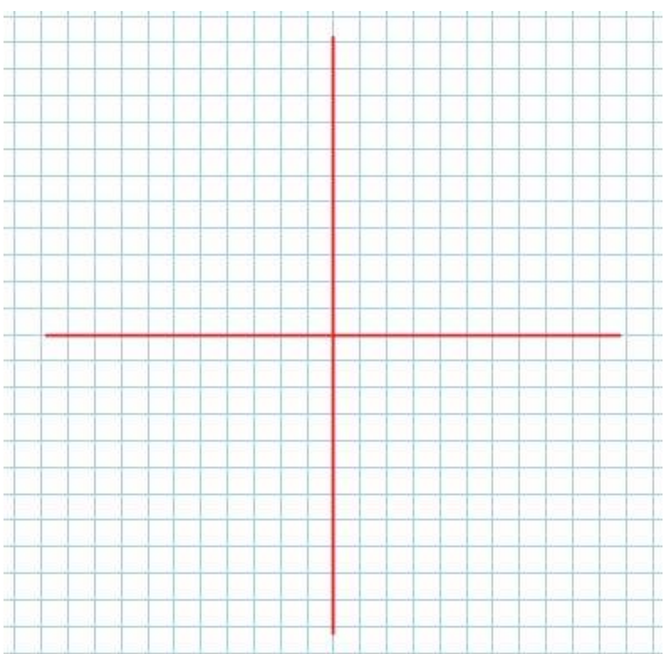
the functional values (for whatever function we are studying) can do one of three things:

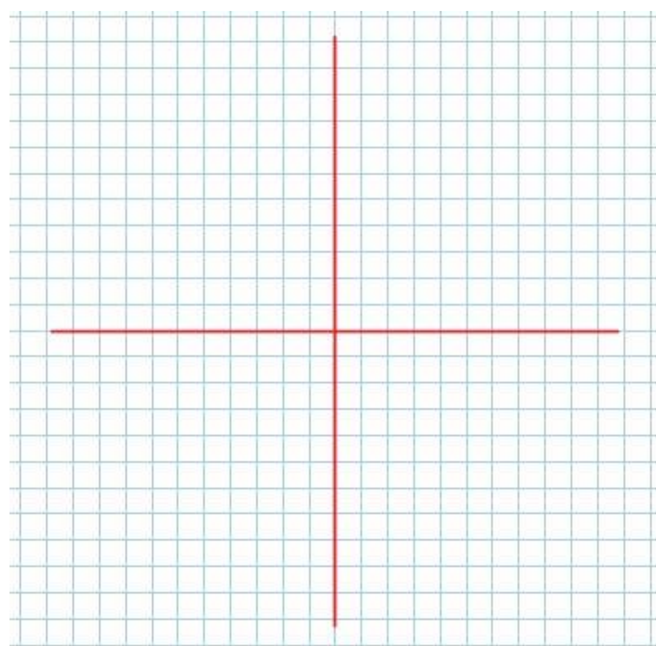
1) e.g. as $x \to \pm\infty$, $f(x) \to$

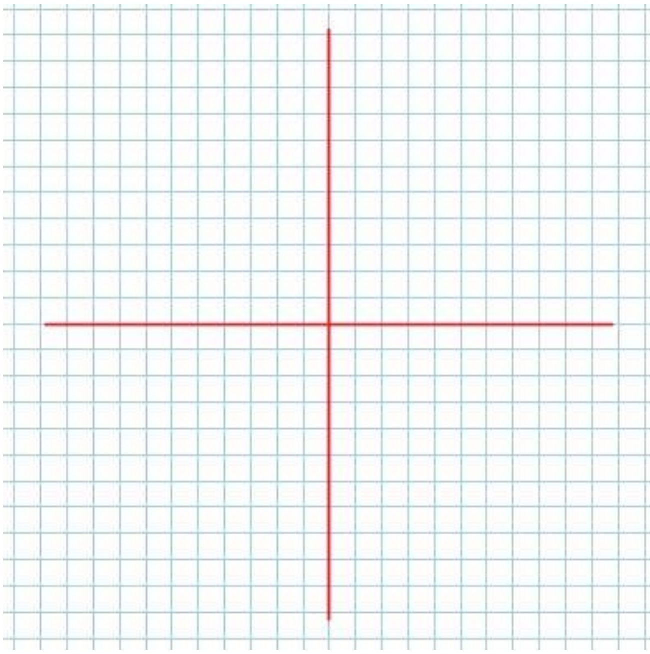
2)

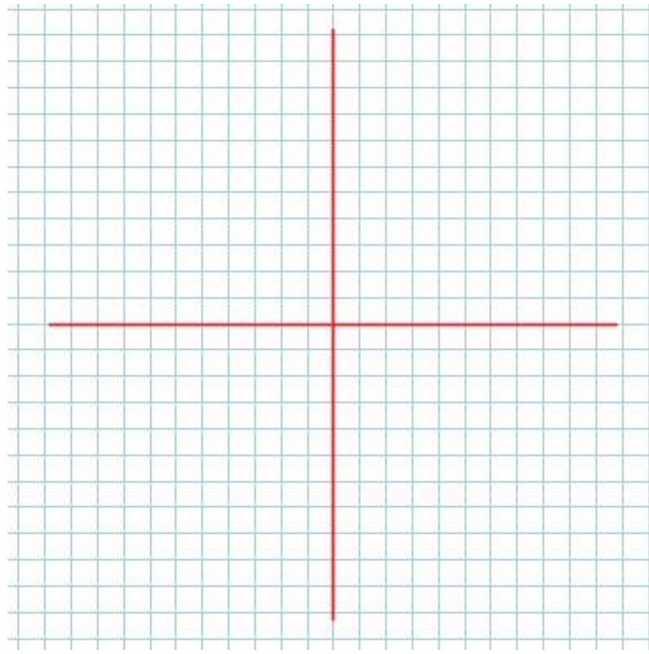
3)

Pictures:









Class/Homework for Section 1.2

Pg. 23 – 24 #5, 7 - 11

1.3 Transformations of Functions

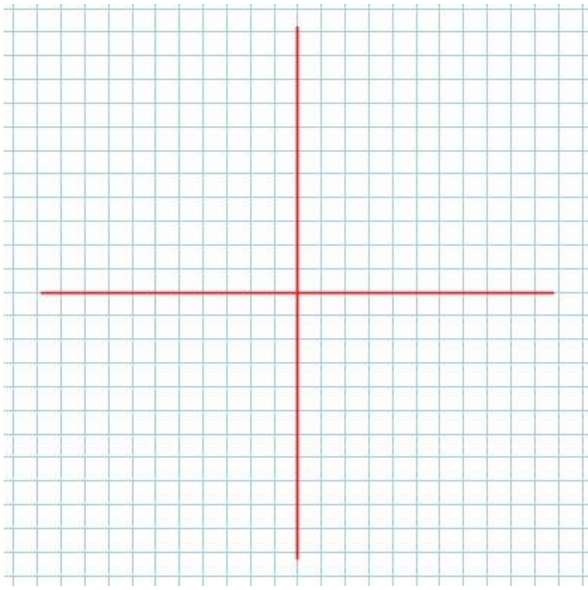
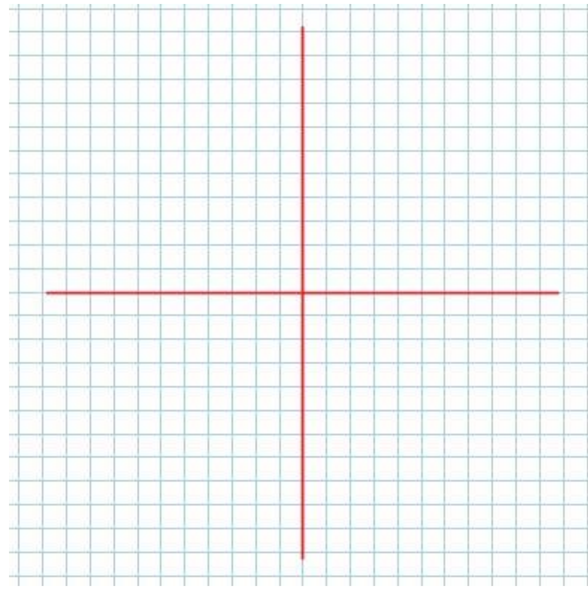
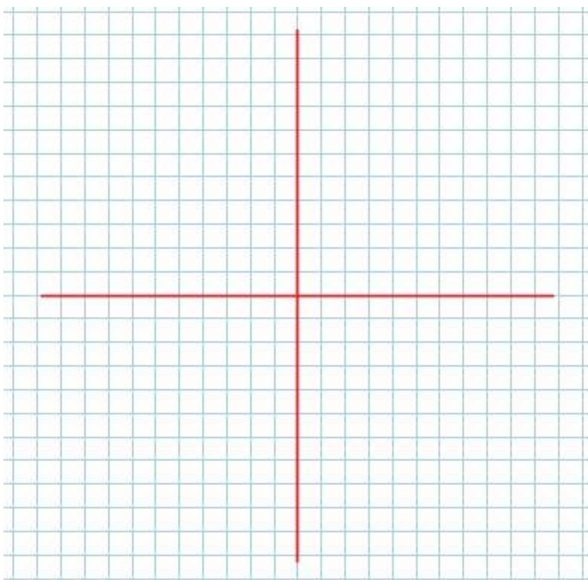
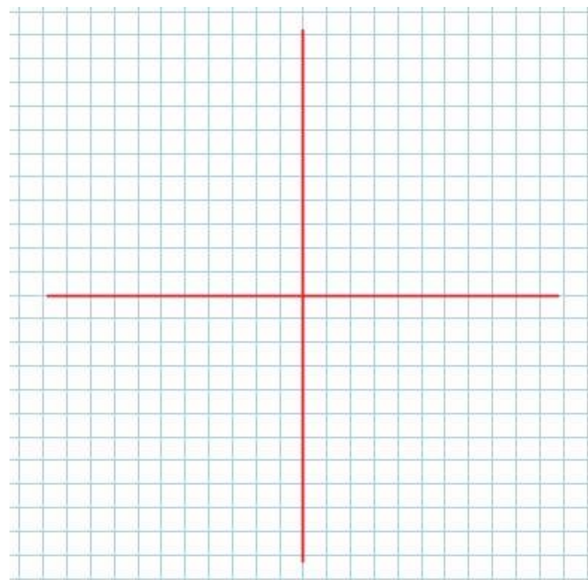
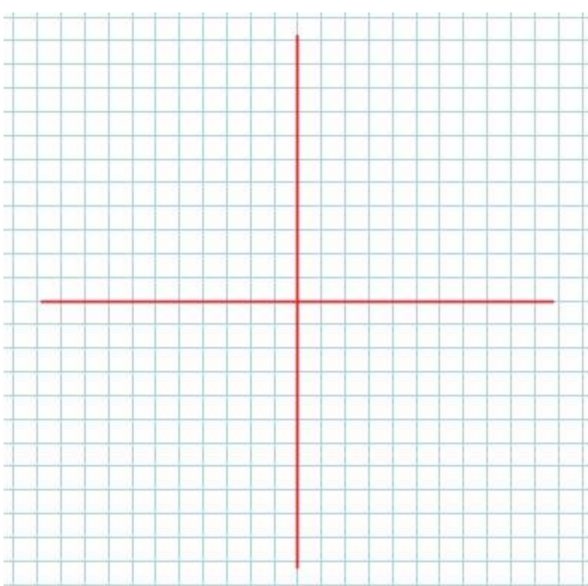
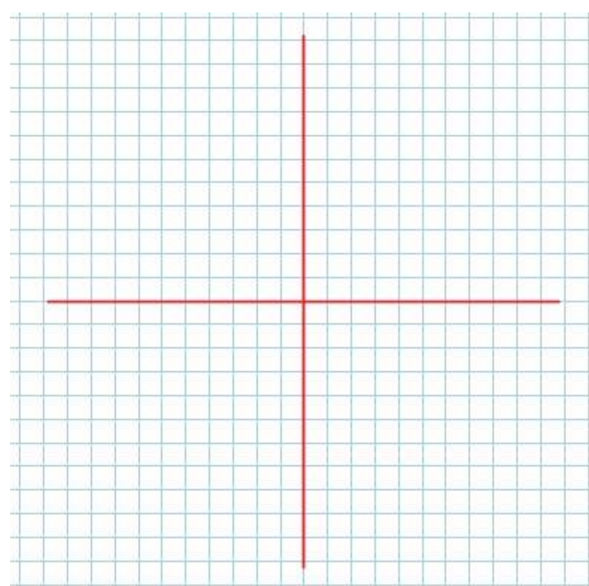
This section is pure review of material from Grade 11. If you've forgotten certain aspects of the concepts, ask for help. Recall that there are three basic transformations of functions. You've probably heard of Flips, Stretches and Shifts. More formal mathematical terms would be Reflections, Dilations and Translations, respectively. Recall also that transformations can occur both vertically and horizontally.

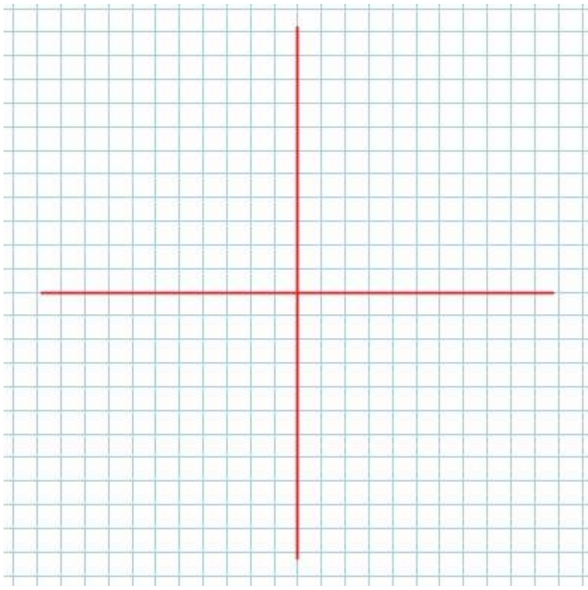
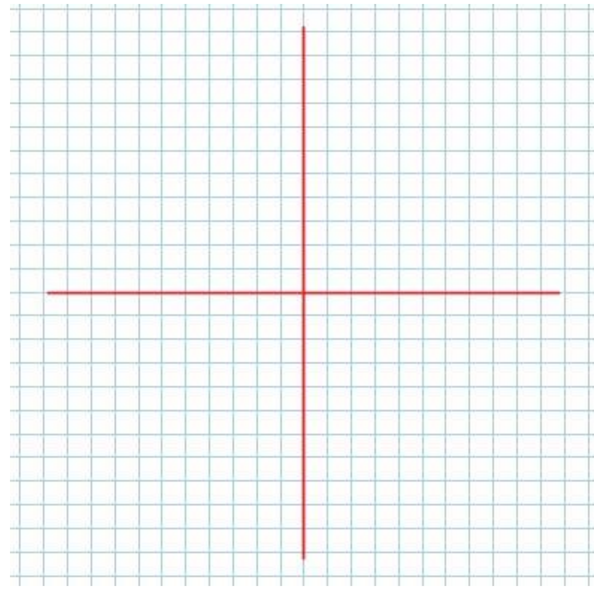
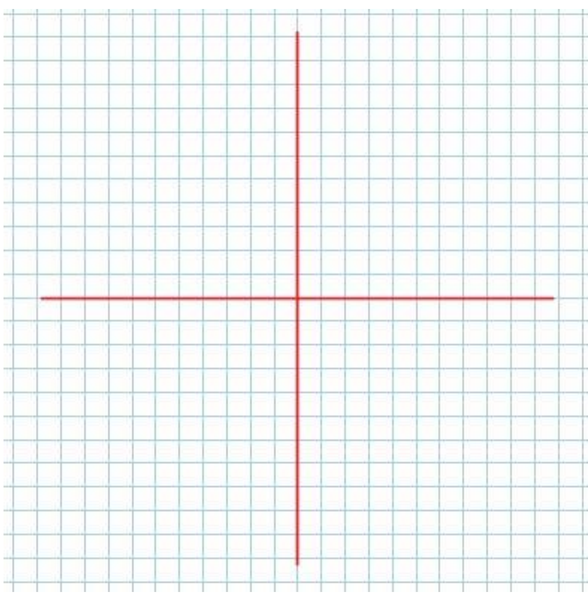
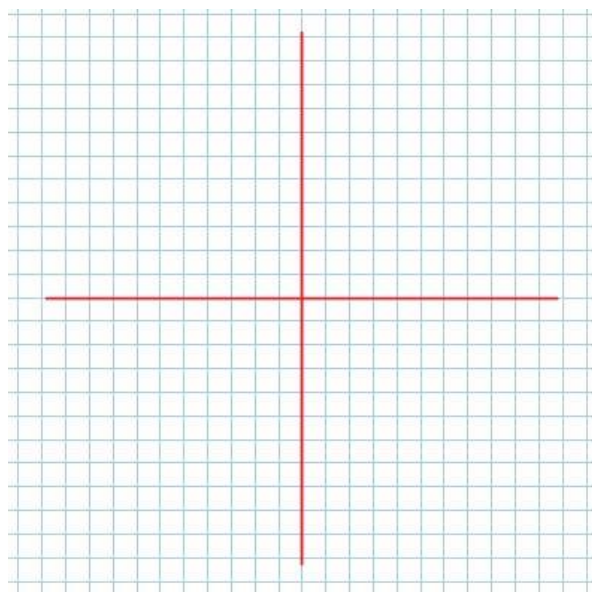
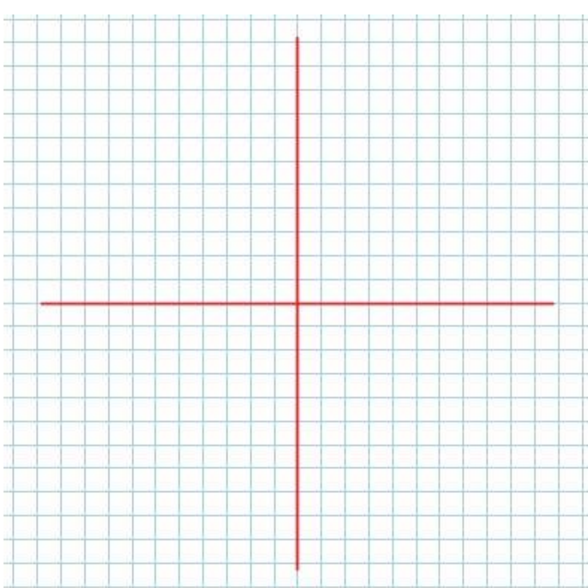
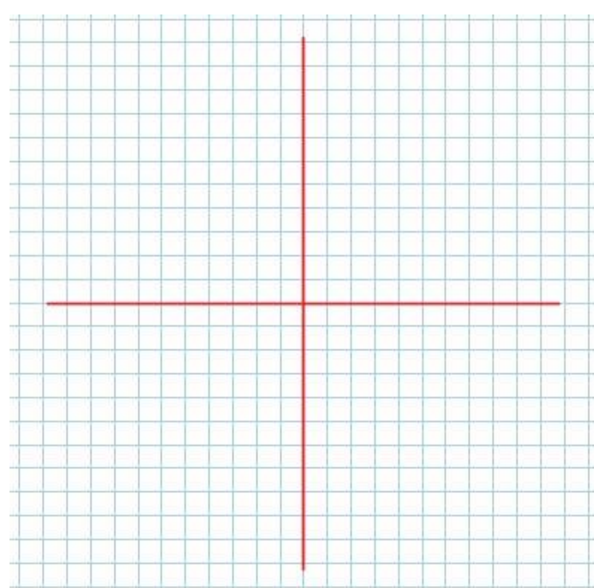
Definition 1.3.1

Given a function $f(x)$, then we denote transformations to $f(x)$ as

Class/Homework for Section 1.3

Complete the table on the Transformations Review Worksheet, and make sketches of all base and transformed functions. Hand in sketches for six of the functions.

1.4 Inverses of Functions

The inestimable William Groot has a saying:

An Inverse *Relation* is an UNDO

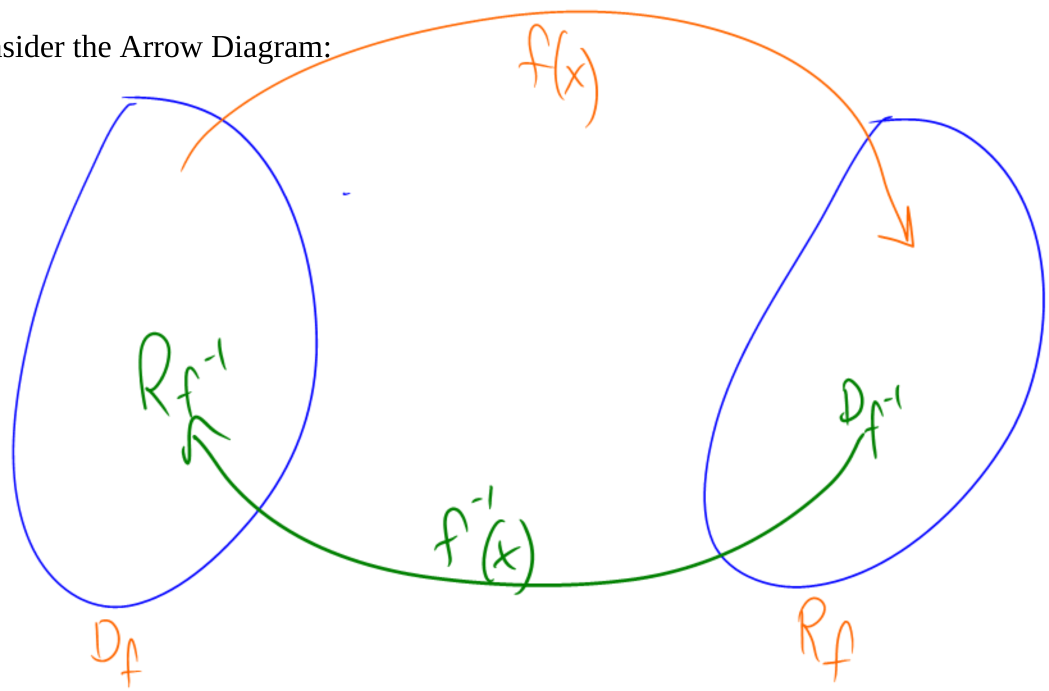
Definition 1.4.1

A **relation** is simply an algebraic relationship between domain values and range values.

Note: All functions are relations, but not all relations are functions

e.g. $x^2 + y^2 = 25$ is a relation, but it is not a function (it's a circle and so doesn't pass the VLT)

Consider the Arrow Diagram:



Big Concept

To determine the inverse of a function, switch x and y

Example 1.4.1

Given the graph of $f(x)$ determine: $D_f, R_f, f^{-1}(x), D_{f^{-1}}, R_{f^{-1}}$

$$f(x) = \{(2,3),(4,2),(5,6),(6,2)\}$$

$D_f : \{2, 4, 5, 6\}$

$R_f : \{2, 3, 6\}$

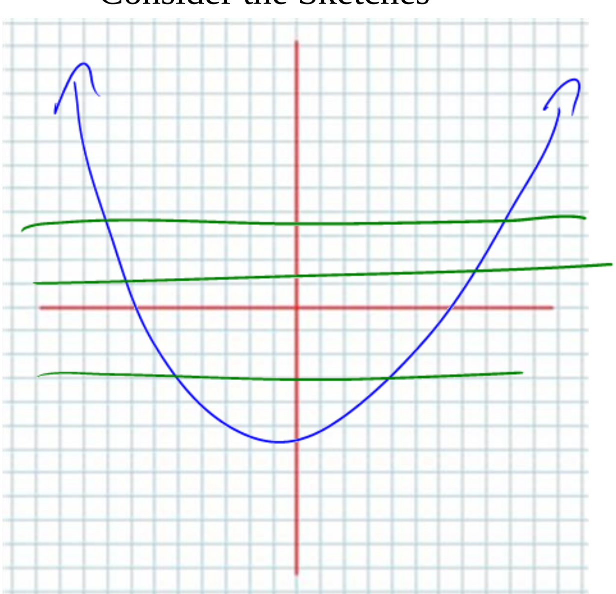
$f^{-1}(x) = \{(3,2), (2,4), (6,5), (2,6)\}$

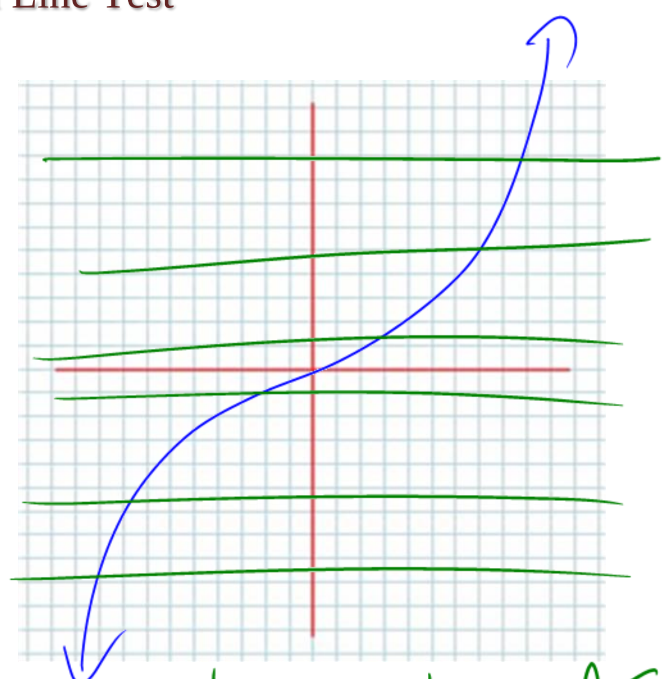
$D_{f^{-1}} = \{2, 3, 6\}$

$R_{f^{-1}} = \{2, 4, 5, 6\}$

Horizontal Line Test

Consider the Sketches





The inverse is not a function

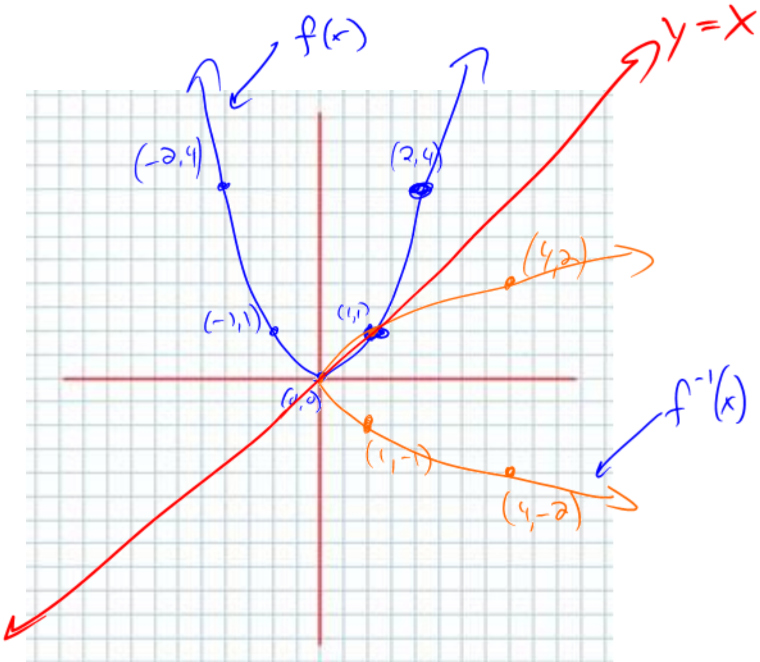
The would be a fn.

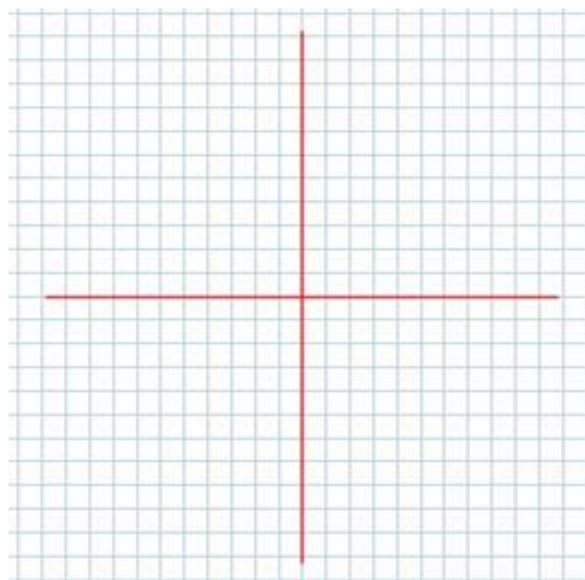
Determining the Inverse of a Function

We can determine the inverse of some given function in either of two ways: Graphically and Algebraically.

Note: Finding a function inverse graphically is not a very useful method, but it can be instructive.

Function Inverses Graphically





Flip the coordinates!

Everything is symmetric around the line $y=x$.

Is there a way to force $f^{-1}(x)$ to be a fn?

Yes! → Restricting the Domain

- do the inverse on only a portion of the domain.
- restrict up to and including the turning point.

for $f(x)=x^2$, use $(-\infty, 0]$ or $[0, \infty)$

Function Inverses Algebraically

Determining algebraic representations of inverse relations for given functions can be done in (at least) two ways:

1) Use algebra in a "brute force" manner (keeping in mind the Big Concept)
2) Use Transformations (keeping in mind "inverse operations")

Example 1.4.2

Determine the inverse of $f(x)=2\sqrt{\frac{1}{3}(x-1)}+2$.

State the domain and range of both the function and its inverse.

Here we will use "brute force".
Method:

1) Switch x and $f(x)$, and call "$f(x)$" ~~$f^{-1}(x)$~~. y
2) Solve for ~~$f^{-1}(x)$~~ y
3) Call y $f^{-1}(x)$.

$$x=2\sqrt{\tfrac{1}{3}(y-1)}+2$$

$$x-2=2\sqrt{\tfrac{1}{3}(y-1)}$$

$$\frac{x-2}{2}=\sqrt{\tfrac{1}{3}(y-1)}$$

$$\left(\frac{x-2}{2}\right)^2=\tfrac{1}{3}(y-1)$$

$$3\left(\frac{x-2}{2}\right)^2=y-1$$

$$3\left(\frac{x-2}{2}\right)^2+1=y$$

$$\therefore f^{-1}(x)=3\left(\frac{x-2}{2}\right)^2+1$$

$D_f: x\in[1,\infty)$	$D_{f^{-1}}: x\in(-\infty,\infty)$
$R_f: f(x)\in[2,\infty)$	$R_{f^{-1}}: f^{-1}(x)\in[1,\infty)$

Example 1.4.3

Using transformations determine the inverse of $f(x)=2\sqrt{\frac{1}{3}(x-1)}+2$.

H. Shift
V. Shift
H. stretch
V. Stretch

- Vertical becomes Horizontal

→ Flip everything / do the inverse operation.

→ Use the opposite fn

$$f^{-1}(x)=3\left(\tfrac{1}{2}(x-2)\right)^2+1$$

Example 1.4.4

Determine the inverse of $g(x) = -2(x-1)^2 + 3$.

Note that the natural domain of $g(x)$ is $(-\infty, \infty)$. However, $g(x)$ does not pass the HLT so its inverse is not a function. Determine a restricted domain for $g(x)$ so that $g^{-1}(x)$ ***is*** a function.

Determine $g^{-1}(x)$

-1	$+1$
$(\ )^2$	$\pm\sqrt{\ }$
$x\ -2$	$\div\ -2$
$+3$	-3

$$g^{-1}(x) = \pm\sqrt{\frac{x-3}{-2}} + 1$$

$$= \pm\sqrt{\frac{-1}{2}(x-3)} + 1$$

Domain of $(-\infty, 1]$

or $[1, \infty)$

Example 1.4.5

Given $f(x) = kx^2 - 3$ and given $f^{-1}(5) = 2$, find k.

$f^{-1}(x) \Rightarrow (5, 2)$

$f(x) \Rightarrow (2, 5)$

$f(2) = \boxed{k(2)^2 - 3 = 5}$

$4k = 8$

$k = 2$

$\therefore f(x) = 2x^2 - 3$

Class/Homework for Section 1.4

Pg. 43 – 45 #2 – 4, 7, 9, 12, 13, 15

1.5 Piecewise Defined Functions

Some aspects of "reality" exhibit different (as opposed to changing) $\boxed{\text{behaviors}}$

To capture those different behaviors mathematically may require using different $\boxed{\text{functions}}$ over different $\boxed{\text{pieces/intervals}}$ of the domain.

Absolute Value

Before discussing piecewise defined functions in general, we will first review the concept of *absolute value*.

Definition 1.5.1

The absolute ***value*** of a number, *x*, is given by

$$|x| = \begin{cases} x, & x \geq 0 \quad [0, \infty) \\ -x, & x < 0 \quad (-\infty, 0] \end{cases}$$

e.g.'s

$|22| = 22$

$|-8| = 8$

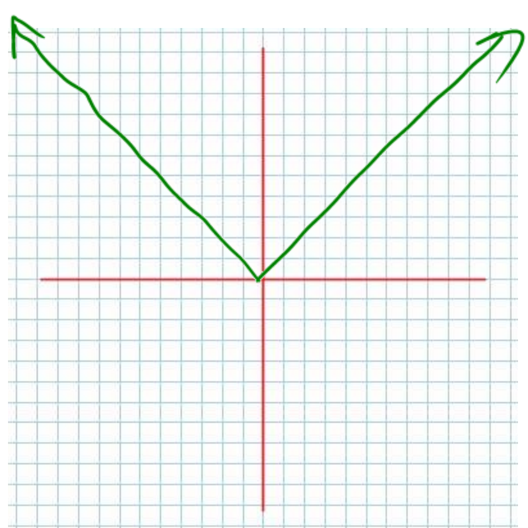
$|8-13| = 5$

Absolute Value Functions

We can define the function which returns the absolute value for any given number as

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Picture

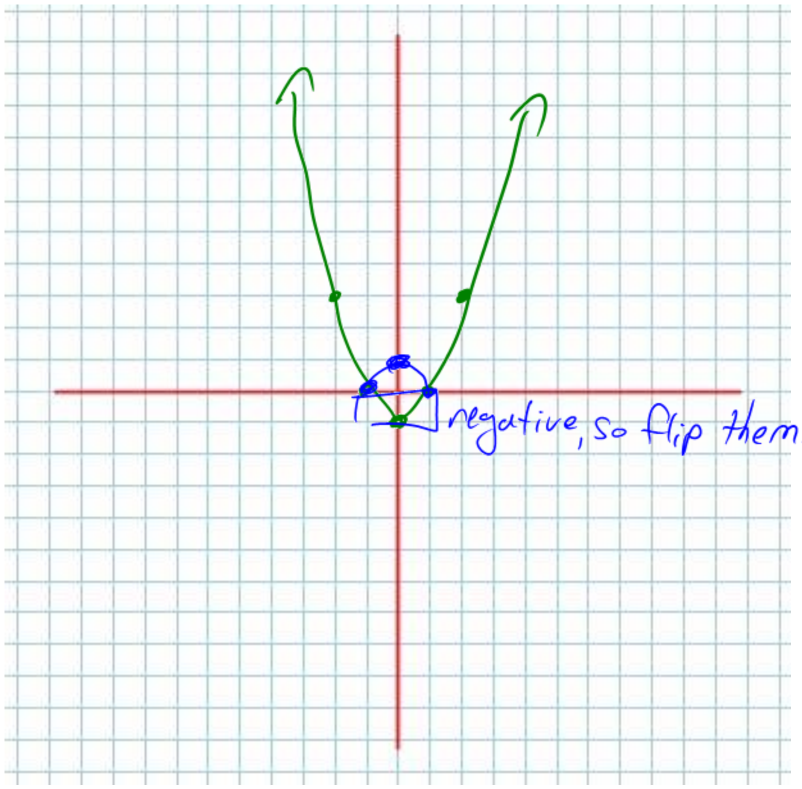


(Two behaviours!)

We can go further and define functions which return the absolute value for more complicated expressions.

e.g. Sketch $g(x)=\left|x^2-1\right|$ (note: g(x) takes the absolute value of the ***functional values*** for the "basic" function $f(x)=x^2-1$)

$$g(x)=\begin{cases} x^2-1, & (-\infty,-1] \\ -(x^2-1), & (-1,1) \\ x^2-1, & [1,\infty) \end{cases}$$

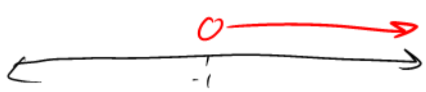


(Three functional behaviours)

Absolute Value and Domain Intervals (and Quadratic Inequalities)

e.g.'s Sketch the solution sets of the following inequalities:

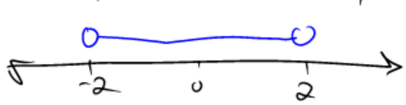
a) $x>-1$



b) $x\le 2$

c) $1<x\le 4$

d) $-2<x<2$



Note the symmetry in part d)! Sometimes it's useful to think of absolute value as

Using the above notion we can thus use absolute value to denote the interval $-2 < x < 2$ as $|x| < 2$

distance from the origin

e.g. Solve the quadratic equation

$x^2 = 4$

$x = \pm 2$ or $|x| = 2$

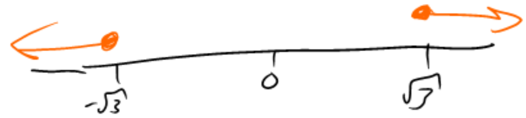
e.g. Solve the quadratic inequalities, and sketch the solution sets:

a) $x^2 < 4$

$|x| < 2$ or $-2 < x < 2$

b) $x^2 \geq 3$

$|x| \geq \sqrt{3}$ or $-\sqrt{3} \geq x \geq \sqrt{3}$

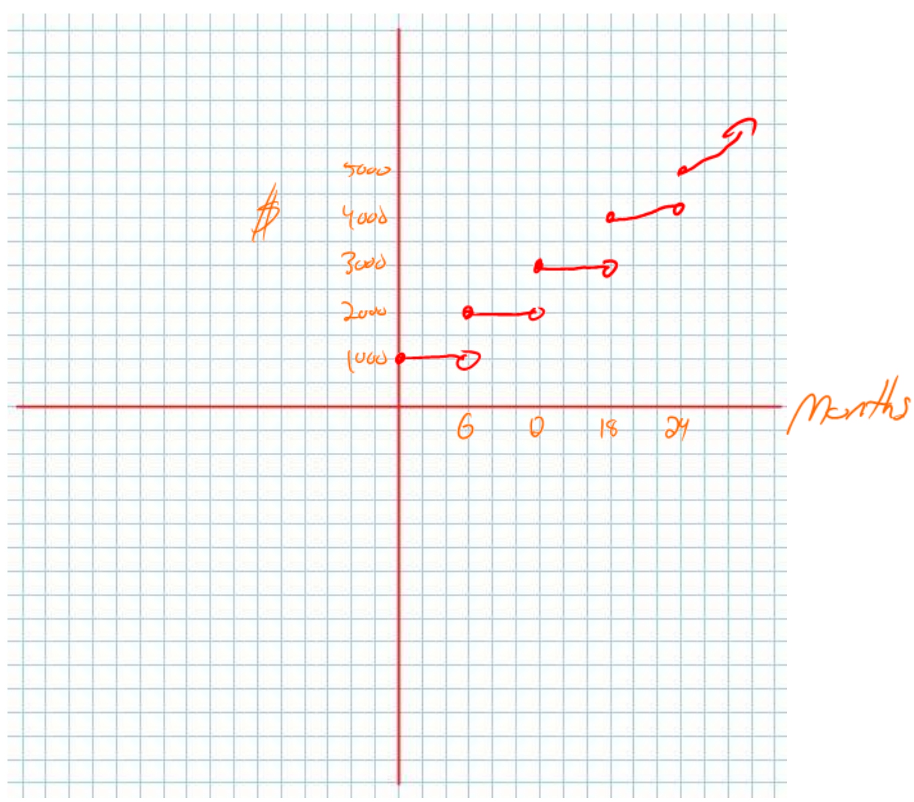


And now we return our attention to general **Piecewise Defined Functions**

Example 1.5.1

You are saving for university, and place $1000 into a sock every six months. After 18 months you wake up and put the money in your sock into an interest bearing bank account. You continue making deposits. Give a graphical representation of this situation.

What is the behaviour of the amount of money you have saved? How is the behaviour changing?

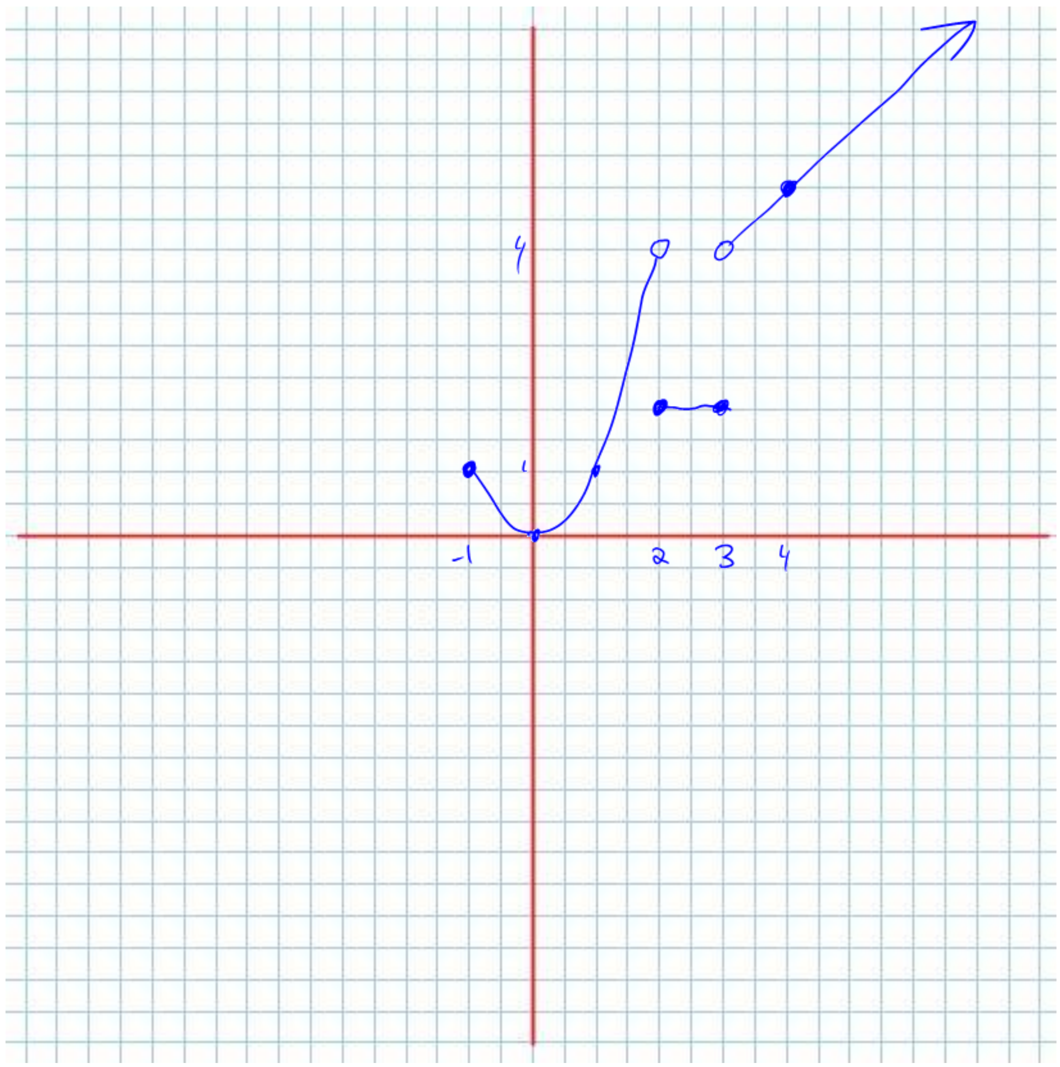


Example 1.5.2

Determine the graphical representation for:

$$f(x)=\begin{cases} x^2, & x\in[-1,2) \\ 2, & x\in[2,3] \\ x+1, & x\in(3,\infty) \end{cases}$$

Note the notation we use for piecewise defined functions. Each functional behaviour has a mathematical representation, defined over its own piece of the domain (just like the Absolute Value function we considered earlier.



Example 1.5.3

Determine a possible algebraic representation which describes the given functional behaviour.

$$g(x) = \begin{cases} \sqrt{x+1}, & [-1, 3) \\ 1, & [3, 5] \end{cases}$$

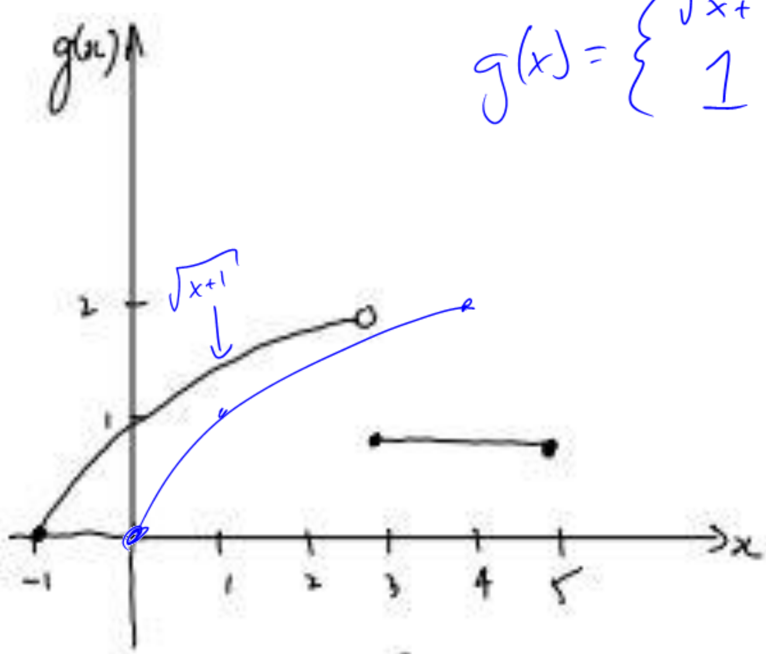


Figure 1.5.3

Class/Homework for Section 1.5

(Abs. Value.) Pg. 16 #2, 4 – 8 (think about transformations!), 10
(Piecewise) Pg. 51 – 53 #1 – 5, 7 – 9

1.6 Combinations of Functions

By now you should have a pretty good sense of how to combine numbers.

e.g. $3+5,\ 7\div 4,$ or $3.9\times\dfrac{4}{3}$ etc.

Functions can be thought of as number generators, and if numbers can be combined, then in the same way (using the basic algebraic operations) we should be able to combine functions too.

A **BIG QUESTION** to ask is:

A **BIGGER QUESTION** to ask is:

Going back to basic graphs of functions may prove helpful in understanding what's happening here.

Example 1.6.1

Given the functions

$$f(x) = \{(-1,2),\ (0,5),\ (1,\pi),\ (2,-4)\}$$

$$g(x) = \left\{(-2,1),\ (-1,-1),\ (0,5),\ \left(1,\frac{2\pi}{3}\right)\right\}$$

determine:

a) $f(x) + g(x)$

b) $\mathrm{g(x)} \div \mathrm{f(x)}$

c) $g(x) \times \left(-f(x)\right)$

Note how the domain of the combined functions is determined by the domains of the original functions!

Definition 1.6.1

Given the functions $f(x)$ and $g(x)$ with domains D_f and D_g respectively, then the domain of the combined function $(f * g)(x)$ is given by:

(Note: The operation $"*"$ could mean any of the basic algebraic operations)

Example 1.6.2

Given the sketches of the functions $f(x)$ and $g(x)$ determine graphically (giving both a rough sketch and a sample (at least 3 points) of the graph):

a) $f(x)-g(x)$

b) $f(x)\times g(x)$

c) $\left(g(x)\right)^2$

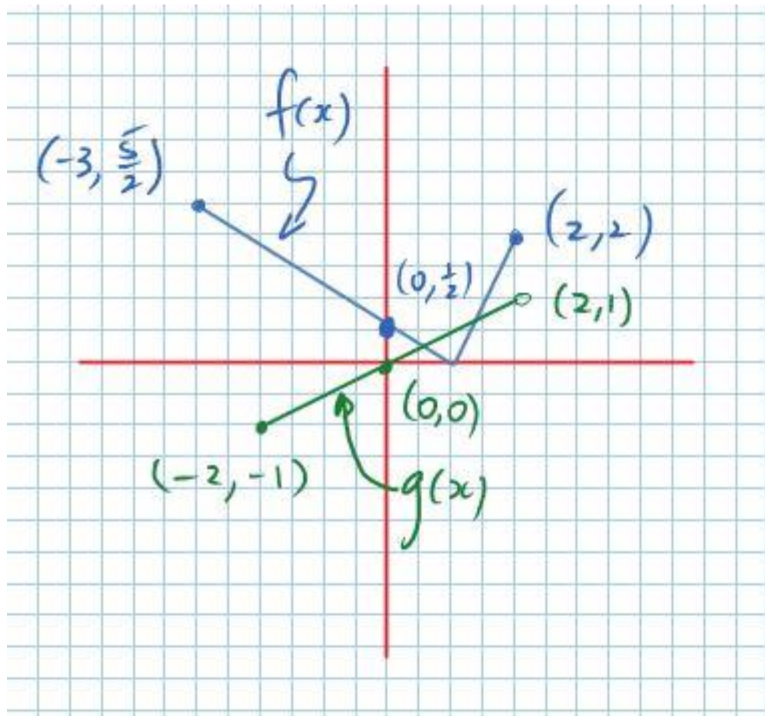
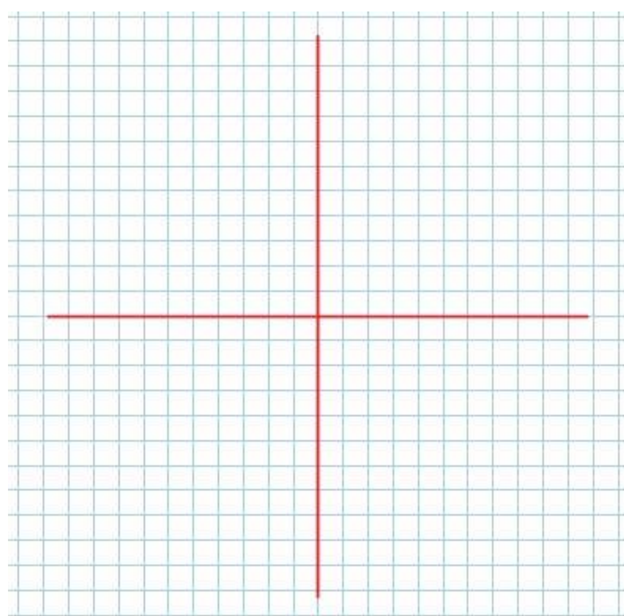
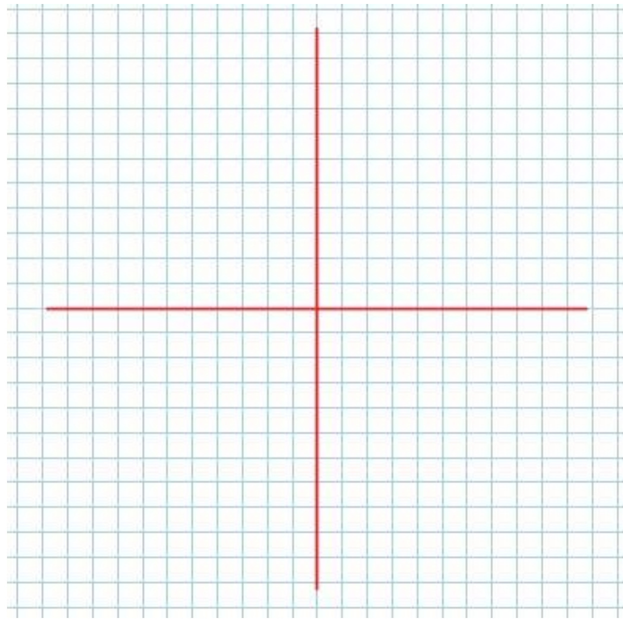


Figure 1.6.2

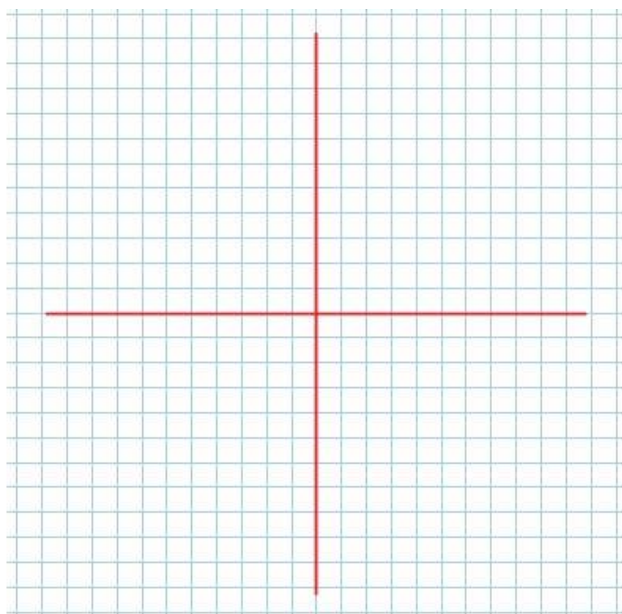
a)



b)



c)



Class/Homework for Section 1.6

Pg. 56 – 57 #1, 2a, 3b, 7