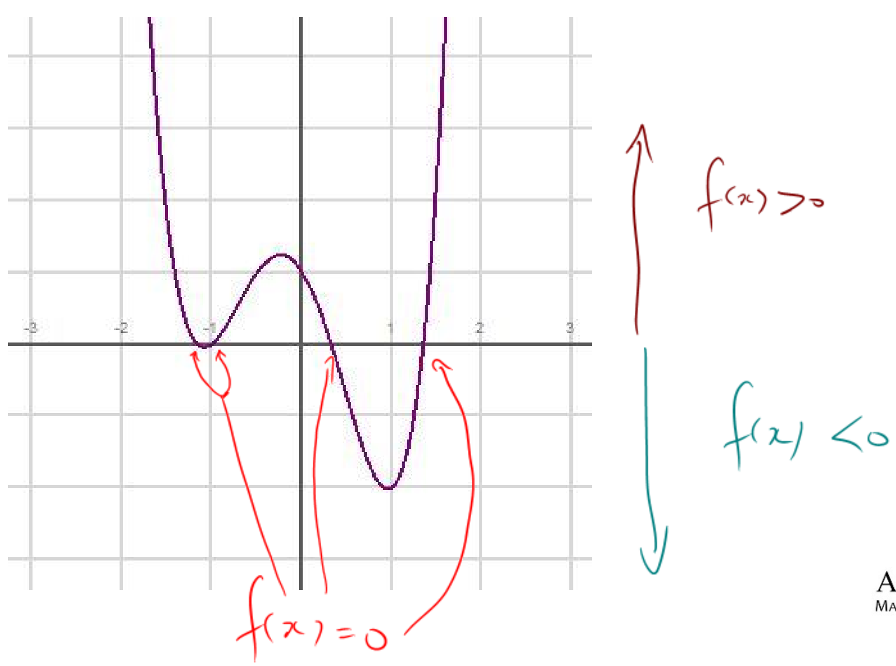
Advanced Functions

Fall 2017
Course Notes

Chapter 3 – Polynomial Equations and Inequalities

We will learn

- *how to find solutions to polynomial equations using tech and using algebraic techniques*
- *how to solve polynomial inequalities with and without tech*
- *how to apply the techniques and concepts to solve problems in volving polynomial models*



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Chapter 3 – Polynomial Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 4)

Thurs Work Period.

TBA....

Combine with unit 4.

3.1 Solving Polynomial Equations

Before embarking on this wonderful journey, it seems to me that it would be prudent to make some (seemingly silly) opening statements.

Seemingly Silly Opening Statements

1) Polynomial equations ARE NOT polynomial functions!
2) Solving any equation MEANS finding a **SOLUTION** (if a solution exists)!
3) Solving a polynomial equation is ALWAYS equivalent to finding the zeros of some polynomial function!

Example 3.1.1 (***back to Grade 9***)

Solve the linear equation

$$3(x-5)+2=5x+6$$

$3x-15+2=5x+6$

$3x-13=5x+6$ $-3x$ -6

$-19=2x$ $\therefore x=\frac{-19}{2}$

Example 3.1.2 (***remember grade 11?***) or 10?

Solve the quadratic equation

$$5x(x-1)+7=2x^2+9$$

Goal:

"stuff" $=0$

$5x^2-5x+7=\boxed{2x^2+9}$

$3x^2-5x-2=0$ $(\times)\ -6$ $(+)\ -5$ $-6, 1$

$\underbrace{3x^2-6x}_{3x}\ \underbrace{+1x-2}_{+1}=0$

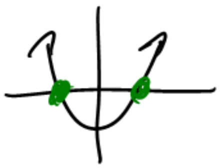
$(3x+1)(x-2)=0$

$x=-1/3$ and $x=2$

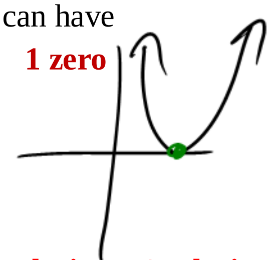
Geometrically speaking, solving a quadratic **equation** is equivalent to finding the zeros of a quadratic **function**.
Solving the **equation** in **Example 3.1.2** means the same thing as finding the zeros of the **function**

Note further that quadratic **functions** can have

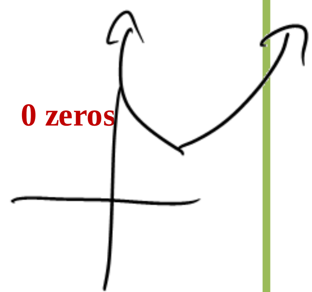
2 zeros



1 zero



0 zeros



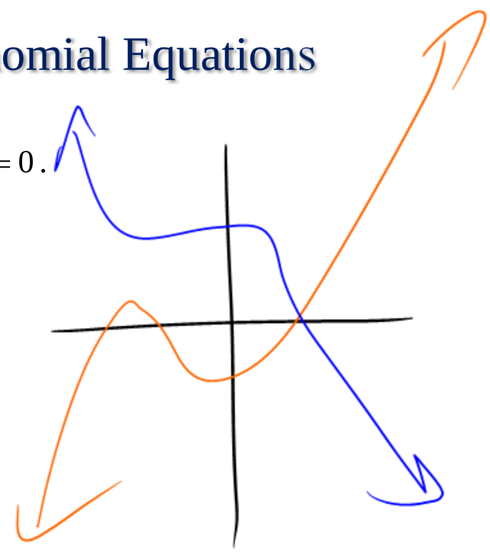
Thus quadratic **equations** can have **2 solutions**, **1 solution** or **no solutions**!

Comments about Higher Order Polynomial Equations

Consider the cubic **EQUATION** $x^3+2x^2-5+1=0$.
Q. How many zeros can this equation have?
Ans.

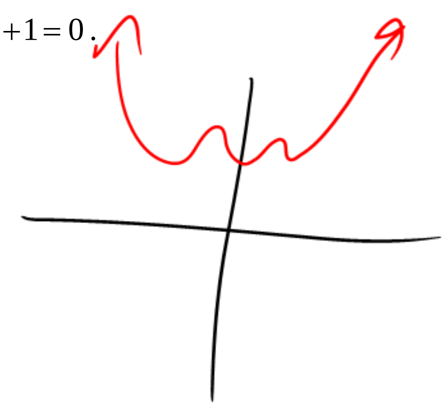
1, 2, or 3

Must have at least one.



Consider the quartic equation $4x^4-3x^3+5x^2-3x+1=0$.
Q. How many zeros can this equation have?
Ans.

0, 1, 2, 3, or 4



Example 3.1.3

Solve the polynomial equation by factoring:

$$4x^3 - 3x = 1$$

> ***Note***: Solving Polynomial Equations requires writing the equation in **Standard Form**, which is: "polynomial = 0"

$f(x) = 4x^3 - 3x \boxed{-1} = 0$

$\hookrightarrow \pm 1$

Try $f(1) = 4(1)^3 - 3(1) - 1$

$= 0 \quad \therefore \ x-1$ is a factor!

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ + & \downarrow & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \\ & x^2 & x & x^0 & \end{array}$$

$\therefore \ 4x^3 - 3x - 1 = (x-1)(4x^2 + 4x + 1)$

$= (x-1)(2x+1)^2$

$\therefore \ x = 1$ and $x = \frac{-1}{2}$

Example 3.1.4

Solve the equation by factoring:

$$12x^4 + 16x^3 - 11x = 13x^2 - 6$$

$f(x) = 12x^4 + 16x^3 - 13x^2 - 11x \boxed{+6} = 0 \longrightarrow \pm 1, \pm 2, \pm 3, \pm 6$

$f(-1) = 12(-1)^4 + 16(-1)^3 - 13(-1)^2 - 11(-1) + 6$

$= 12 - 16 - 13 + 11 + 6 = 0! \quad \therefore \ (x+1)$ is a factor

$$\begin{array}{r|rrrrr} -1 & 12 & 16 & -13 & -11 & 6 \\ + & \downarrow & -12 & -4 & 17 & -6 \\ \hline & 12 & 4 & -17 & 6 & 0 \end{array}$$

$\therefore \ (x+1)(12x^3 + 4x^2 - 17x + 6)$

Let $g(x) = 12x^3 + 4x^2 - 17x + 6$

$g(\pm 1) \neq 0, \ g(\pm 2) \neq 0, \ g(\pm 3) \neq 0, \ g(\pm 6) \neq 0$

Rational Zero Test

Factors of 12

± 1
± 2
± 3
± 4
± 6
± 12

Consider $\overset{a}{12}x^3 + 4x^2 - 17x + \overset{b}{6} = 0$.

We now, when using the factor theorem, will "test for zeros" using 2 steps:

1) Test for integer zeros using factors of the constant term.

   $\pm 1, \pm 2, \pm 3, \pm 6$

2) Test for rational zeros, where we consider $x = \frac{b}{a}$

   $\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}$

The possible rational zeros are:

24 possible factors !!

Back to **Example 3.1.4**

For $g(x) = 12x^3 + 4x^2 - 17x + 6$ the possible rational zeros are:

$$g\left(\tfrac{1}{2}\right) = 12\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{1}{2}\right)^2 - 17\left(\tfrac{1}{2}\right) + 6$$

$$= 12\left(\tfrac{1}{8}\right) + \underbrace{\boxed{4\left(\tfrac{1}{4}\right)}}_{=1} - \frac{17}{2} + \frac{6}{1}$$

$$= \frac{3}{2} + \frac{2}{2} - \frac{17}{2} + \frac{12}{2} = 0 \text{ !!}$$

$\therefore (2x-1)$ is a factor

$$\begin{array}{c|cccc} \boxed{\tfrac{1}{2}} & 12 & 4 & -17 & 6 \\ & \downarrow & 6 & 5 & -6 \\ \hline & 12 & 10 & -12 & 0 \end{array}$$

$\div 2 \quad 6x^2 \quad 5x \quad -6$

$\otimes\ -36$
$\oplus\ 5$
$-4, 9$

$$\therefore (2x-1)(6x^2+5x-6)$$

$$(2x-1)(\underbrace{6x^2-4x}_{2x} \underbrace{+9x-6}_{+3})$$

$$(2x-1)(2x+3)(3x-2)$$

$$x = \frac{1}{2},\ x = \frac{-3}{2},\ x = \frac{2}{3}$$

Example 3.1.5

$\pm 1, \pm 3$

$\pm 1, \pm 2$

$\pm \frac{1}{3}, \pm \frac{2}{3}$

Solve the equation $3x^3 - 4x + 2 = 0$.

$$f(-1) = 3(-1)^3 - 4(-1) + 2$$
$$= -3 + 4 + 2$$
$$= 6$$

As it turns out!

$f(\pm 1) \neq 0$

$f(\pm 2) \neq 0$

$f(\pm \frac{1}{3}) \neq 0$

$f(\pm \frac{2}{3}) \neq 0$

$\therefore$ $f(x)$ does not factor!

However, this is a cubic fn and MUST have at least one solution

$x = -1.352$ by graphing.

Class/Homework for Section 3.1

Pg. 204 – 206 #1, 2, 6 – 8, 10 – 12, 14, 15
Note: for #14a) you may need to ask about "domain restrictions"

3.2 Linear Inequalities

easier

$>, \geq, <, \leq$

harder.

Once again, it seems a good idea to begin with a couple of opening statements.

Absolutely Non-Silly Opening Statements

1) The **algebra** of inequalities is the **SAME** as the algebra on equality (i.e. solving equations), with two exceptions:

   a) If you multiply or divide **by a negative**, then you must switch the direction of the inequality.

   b) We can have 2 sided inequalities – e.g.

   $3 \leq x \leq 5$ or $-3 > x > 8$

2) The **Solution Set** of inequalities is infinite.

$$\frac{-2x}{-2} < \frac{4}{-2}$$

$$x \boxed{>} -2$$

$$-2x \overset{+2x}{} \overset{-4}{<} \overset{-4}{4} \; {+2x}$$

$$\frac{-4}{2} < \frac{2x}{2}$$

$$-2 < x$$

Example 3.2.1

Solve the (linear) inequality $3x \overset{+2}{-2} > \overset{+2}{4}$.

$$\frac{3x}{3} > \frac{6}{3}$$

$$x > 2$$

① Graphing:

② Set Notation:

$\{x \in \mathbb{R} \mid x > 2\}$

③ Interval Notation

$x \in (2, \infty)$

Do the "math" everywhere.

Example 3.2.2

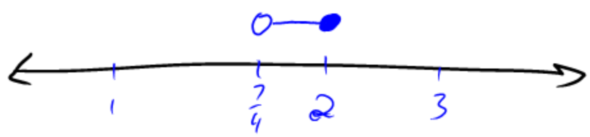
Solve the two sided inequality $-2 > -4x + 5 \ge -3$.

$$-2 \underset{-5}{} > -4x + 5 \underset{-5}{} \ge -3 \underset{-5}{}$$

$$\frac{-7}{-4} > \frac{-4x}{-4} \ge \frac{-8}{-4}$$

$$\frac{7}{4} < x \le 2$$

① Graph



② Interval:

$x \in \left(\frac{7}{4}, 2\right]$

Example 3.2.3

Solve $5 \le 3(x-2) - 4(x+3) \le 12$

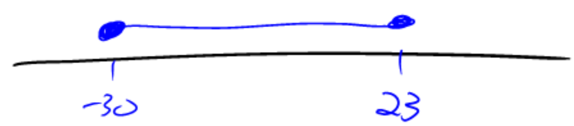
$$5 \le 3x - 6 - 4x - 12 \le 12$$

$$5 \underset{+18}{} \le -x - 18 \underset{+18}{} \le 12 \underset{+18}{}$$

$$\frac{23}{-1} \le \frac{-x}{-1} \le \frac{30}{-1}$$

$$-23 \ge x \ge -30$$

$$-30 \le x \le -23$$



Example 3.2.4

Write the following sketch of a solution set in interval and set notation:

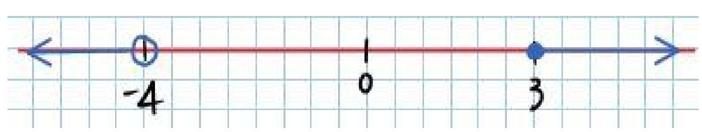


Figure 3.2.4

Set: $\{x \in \mathbb{R} \mid -4 > x \ge 3\}$

Interval: $x \in (-\infty, -4) \cup [3, \infty)$

Graphical Views of (non-linear) Polynomial Inequalities

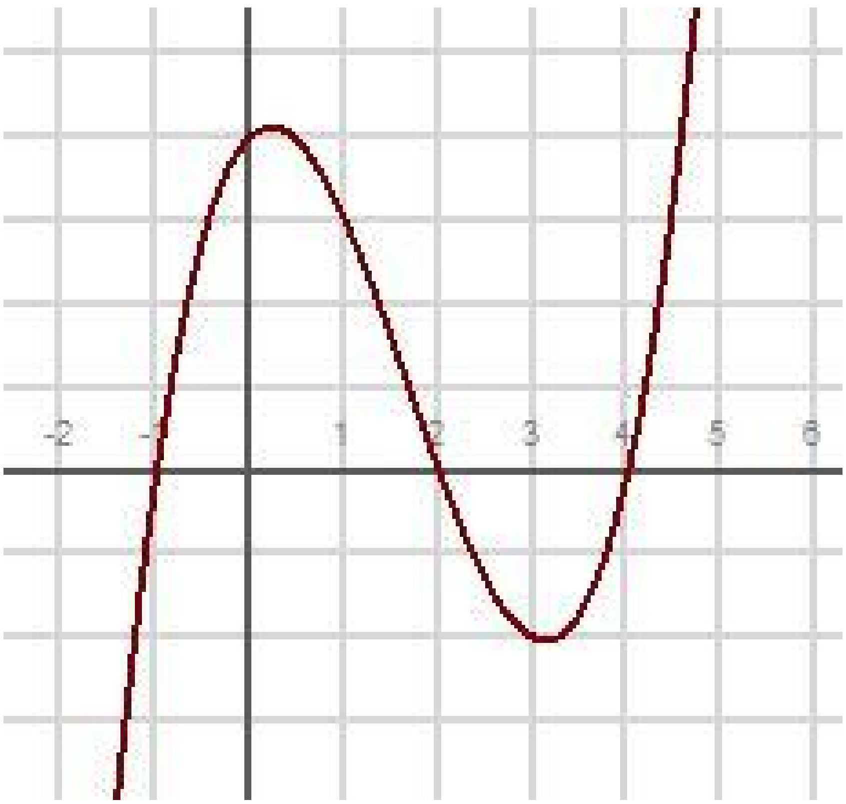
(*the Algebra is tough…*)

Example 3.2.5
Consider the sketch of the graph of some mystery cubic function.

Q. When (or better WHERE) are *the functional values* positive?

x-values/domain.

> 0



→ what x-values give positive "answers"

∴ we need intervals.

$x \in (-1, 2) \cup (4, \infty)$

Figure 3.2.5

Example 3.2.6

Consider the sketch of the quartic $g(x)$, and determine where

a) $g(x) \leq 0$

b) $g(x) > 2$

c) $-1 \leq g(x) \leq 2$

a) $x \in [-4, 1] \cup [3]$

b) $x \in (-\infty, -5) \cup (5, \infty)$

c) $x \in [-5, -3] \cup [0, 5]$

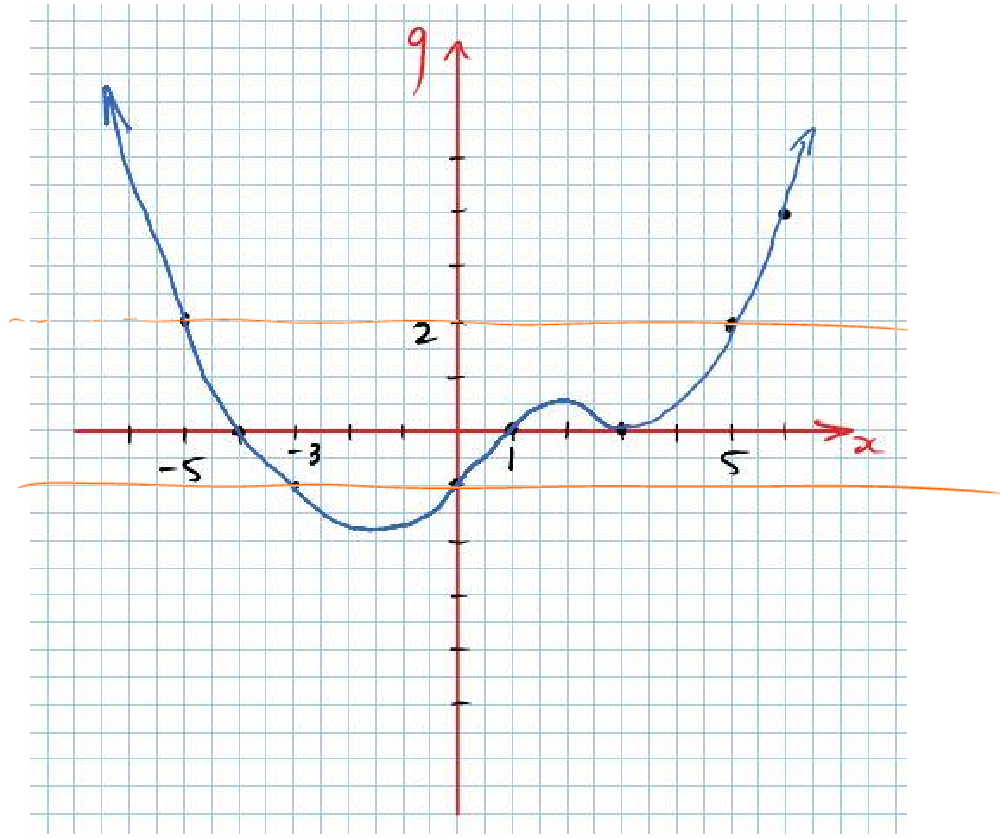


Figure 3.2.6

Class/Homework for Section 3.2

Pg. 213 – 215 #1, 2, 4, 5, 7, 9, 13

3.3 Solving Polynomial Inequalities

For this section, no opening statements are required….

Non-Required Opening Statement

Solving non-linear polynomial inequalities can be accomplished in two ways:

1) Graphically (sometimes called Geometrically)
2) Algebraically (which tends to be more useful)

Example 3.3.1

Solve $(2x-1)(x-2)(x+3) \geq 0$. **REMEMBER: *FACTORED FORM IS YOUR FRIEND***

$x = \frac{1}{2}$ $x = 2$ $x = -3$

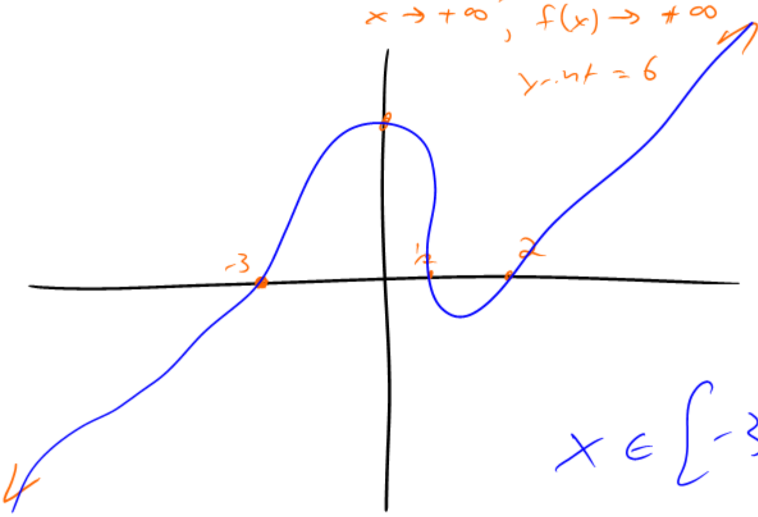
Graphically: L.T. $: 2x^3$

$x \to -\infty$, $f(x) \to -\infty$

$x \to +\infty$, $f(x) \to +\infty$

y-int $= 6$

Note: Solving an inequality graphically is rather easy, **BUT** Solving algebraically may require a lot more work.



$x \in [-3, \frac{1}{2}] \cup [2, \infty)$

Example 3.3.1 (Continued)

Solve $(2x-1)(x-2)(x+3)\boxed{\geq 0}$ $+$

Algebraically

For this technique we will construct an "**Interval Chart**", which can also be thought of as a "**table of signs**" (and wonders?)

Note: It is often helpful to remember that in mathematics we are dealing with **NUMBERS**.

Numbers have signs: Positive or Negative

e.g. $(x-2)$ is a **NUMBER** whose sign switches from +'ve to –'ve at $x=2$ (*i.e. the sign switches at the zero of the factor*)

The Interval Chart looks like:

Intervals	Split the Domain $(-\infty,\infty)$ at all **ZEROS** of the Factors		
Test Values	Choose a Domain	value inside each	Interval
Sign on 1st Factor	$+$		
Sign on 2nd Factor	$+$		
Sign on 3rd Factor	$-$		
Sign on the Product of Factors	Find the Intervals $-$	with the **sign** we	want to answer the question

$x=-3, \quad x=\frac{1}{2}, \quad x=2$

For our problem above, our chart will look like:

Intervals	$(-\infty,-3)$	$(-3,\frac{1}{2})$	$(\frac{1}{2},2)$	$(2,\infty)$
Test Values	-4	0	1	3
$x+3$	$-$	$+$	$+$	$+$
$2x-1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
Product	$-$	$+$	$-$	$+$

$\therefore x \in \left[-3, \frac{1}{2}\right] \cup [2, \infty)$

$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Example 3.3.2

Solve algebraically $4x^4 + 16x^3 + x^2 - 39x - 18 < 0$.

Wait a second….where is your friend and mine…

Factored Form

Let $f(x) = 4x^4 + 16x^3 + x^2 - 39x - 18$

Try $f(-1) = 4(-1)^4 + 16(-1)^3 + (-1)^2 - 39(-1) - 18$

$= 4 - 16 + 1 + 39 - 18$

$\neq 0$

Try $f(-2) = 4(-2)^4 + 16(-2)^3 + (-2)^2 - 39(-2) - 18$

$= 0$ Trust me :)

$$\begin{array}{r|rrrrr} -2 & 4 & 16 & 1 & -39 & -18 \\ & \downarrow & -8 & -16 & 30 & 18 \\ \hline & 4 & 8 & -15 & -9 & 0 \end{array}$$

$\therefore (x+2)(4x^3 + 8x^2 - 15x - 9) < 0$

$\pm 1, \pm 3, \pm 9$

Let $g(x) = 4x^3 + 8x^2 - 15x - 9$

Try $g(-3) = 4(-3)^3 + 8(-3)^2 - 15(-3) - 9$

$= -108 + 72 + 45 - 9$

$= 0$

$$\begin{array}{r|rrrr} -3 & 4 & 8 & -15 & -9 \\ & \downarrow & -12 & 12 & 9 \\ \hline & 4 & -4 & -3 & 0 \end{array}$$

$\therefore (x+2)(x+3)(4x^2 - 4x - 3) < 0$

$(x+2)(x+3)(\underbrace{4x^2 - 6x}_{2x} + \underbrace{2x - 3}_{+1}) < 0$

$= (x+2)(x+3)(2x+1)(2x-3) < 0$

$x = -2 \quad x = -3 \quad x = \frac{-1}{2} \quad x = \frac{3}{2}$

Intervals	$(-\infty, -3)$	$(-3, -2)$	$(-2, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{3}{2})$	$(\frac{3}{2}, \infty)$
T.V.	-5	-2.5	-1	0	2
$x+3$	$-$	$+$	$+$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$2x+1$	$-$	$-$	$-$	$+$	$+$
$2x-3$	$-$	$-$	$-$	$-$	$+$
Product	$+$	$-$	$+$	$-$	$+$

$\therefore \; 4x^4 + 16x^3 + x^2 - 39x - 18 < 0$

when $x \in (-3, -2) \cup (-\frac{1}{2}, \frac{3}{2})$

Class/Homework for Section 3.3

Pg. 225 – 228 #2, 5 – 7, 10 – 13