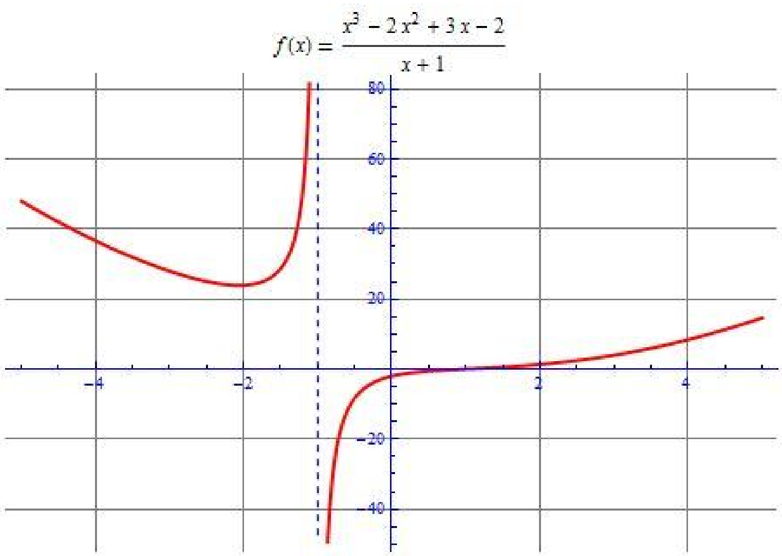
Advanced Functions

Fall 2017
Course Notes

Unit 4 – Rational Functions, Equations and Inequalities

We will learn

- *how to sketch the graphs of simple rational functions*
- *how to solve rational equations and inequalities with and without tech*
- *how to apply the techniques and concepts to solve problems involving rational models*



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Math@TD

Chapter 4 – Rational Functions, Equations and Inequalities

Contents with suggested problems from the Nelson Textbook (Chapter 5)

4.1 Introduction to Rational Functions and Asymptotes

Pg. 262 #1 - 3

4.2 Graphs of Rational Functions

Pg. 272 #1, 2 (Don't use any tables of values!), 4 – 6, 9, 10

4.4 Solving Rational Equations

Pg. 285 - 287 #2, 5 – 7def, 9, 12, 13

4.5 Solving Rational Inequalities

Pg. 295 - 297 #1, 3, 4 – 6 (def), 9, 11

use chapter 3 techiques.

Test October 19, 2017 (Thursday).

4.1 Rational Functions, Domain and Asymptotes

Definition 4.1.1

A **Rational Function** is of the form

$f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ and both $p(x)$ and $q(x)$ are polynomials.

e.g. $f(x) = \dfrac{3x^2 - 5x + 1}{2x - 1}$ is a rational fn.

$g(x) = \dfrac{\sqrt{2x+5}}{3x-2}$ not rational fn because $\sqrt{2x+5}$ is not a polynomial

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Domain

Definition 4.1.2

Given a rational function $f(x) = \dfrac{p(x)}{q(x)}$, then the **natural domain** of $f(x)$ is given by

$D_f = \{x \in \mathbb{R} \mid q(x) \neq 0\}$

the zeros of $q(x)$

Example 4.1.1

Determine the natural domain of $f(x) = \dfrac{x^2 - 4}{x - 3}$.

3 makes this not work

$D_f : \{x \in \mathbb{R} \mid x \neq 3\}$

$x \in (-\infty, 3) \cup (3, \infty)$

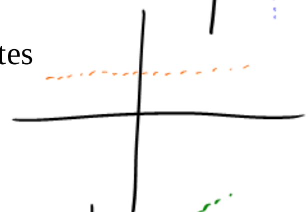
Asymptotes

There are 3 possible types of **asymptotes**:

1) Vertical Asymptotes

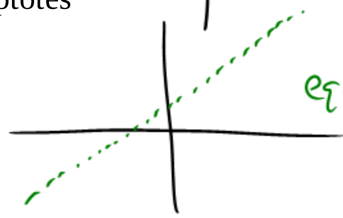
A vertical line with equation $x = \#$

2) Horizontal Asymptotes



equation is $y = \#$

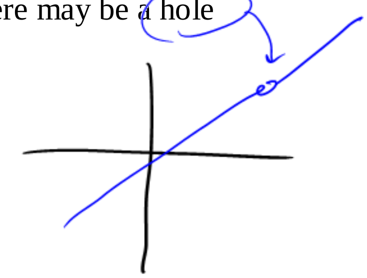
3) Oblique Asymptotes



equation is $y = mx + b$

Vertical Asymptotes

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ **MIGHT** have a V.A. when $q(x) = 0$, but there may be a hole discontinuity instead. A quick bit of algebra will dispense the mystery.



Example 4.1.2

Determine the domain, and V.A., or hole discontinuities for:

a) $f(x) = \dfrac{5x}{x^2 - x - 6}$

$$f(x) = \frac{5x}{(x-3)(x+2)}$$

These factor stay ∴ V.A.

$$D_f = \{x \in \mathbb{R} \mid x \neq -2,\ x \neq 3\}$$

$$x \in (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$$

b) $h(x)=\dfrac{x+3}{x^2-9}$

$$h(x)=\frac{\cancel{x+3}}{\cancel{(x+3)}(x-3)}$$

$$h(x)=\frac{1}{(x-3)}$$

V.A. at $x=3$

Hole at $x=-3$ because it disappeared.

$$D_h:\{x\in\mathbb{R}\mid x\neq -3,\ x\neq 3\}$$

FACTOR EVERYTHING

c) $g(x)=\dfrac{x^2-4}{x+2}$

$$g(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

$$g(x)=x-2$$

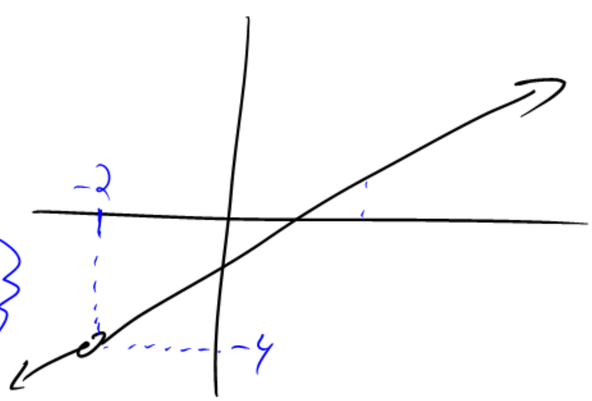
$x+2=0$

$x=-2$

$x=-2$ is a hole.

$$D_g:\{x\in\mathbb{R}\mid x\neq -2\}$$

$$R_g:\{g(x)\in\mathbb{R}\mid g(x)\neq -4\}$$



Horizontal Asymptotes

Here we are concerned with END BEHAVIORS of the rational fn.

i.e. We are asking, given a rational function $f(x)=\dfrac{p(x)}{q(x)}$, how is $f(x)$ behaving as

$x \to \pm\infty$.

Now, since $p(x)$ and $q(x)$ are both polynomials, they have an order (degree). We must consider **three possible situations regarding their order**:

1) Order of $\overset{\text{top}}{p(x)}$ $>$ Order of $\overset{\text{bottom}}{q(x)}$

   e.g. $f(x)=\dfrac{x^3-2}{x^2+1}$ Divide every term by the highest order term.

$$f(x)=\frac{\frac{x^3}{x^3}-\frac{2}{x^3}}{\frac{x^2}{x^3}+\frac{1}{x^3}} = \frac{1-\cancel{\frac{2}{x^3}}\to 0}{\cancel{\frac{x^2}{x^3}}\to 0+\cancel{\frac{1}{x^3}}\to 0} = \frac{1}{0}$$

"plug in ∞"

∴ No H.A.

2) Order of numerator $=$ Order of denominator

   e.g. $f(x)=\dfrac{2x^2-3x+1}{3x^2+4x-5}$

$$f(x)=\frac{\frac{2\cancel{x^2}}{\cancel{x^2}}-\cancel{\frac{3x}{x^2}}\to 0+\cancel{\frac{1}{x^2}}\to 0}{\frac{3x^2}{x^2}+\cancel{\frac{4x}{x^2}}\to 0-\cancel{\frac{5}{x^2}}\to 0} = \frac{2-0+0}{3+0-0} = \frac{2}{3}$$

∴ $y=\frac{2}{3}$ is the H.A.

e.g. Determine the horizontal asymptote of $g(x)=\dfrac{3x\boxed{-4}x^5}{\boxed{5}x^5+2x-1}$

∴ H.A. is $y=\frac{-4}{5}$

3) Order of numerator $p(x) <$ Order of denominator $q(x)$

e.g. $f(x) = \dfrac{x^2 - 5x + 6}{x^5 + 7}$

$$f(x) = \frac{\cancel{\frac{x^2}{x^5}}^{\to 0} - \cancel{\frac{5x}{x^5}}^{\to 0} + \cancel{\frac{6}{x^5}}^{\to 0}}{\frac{x^5}{x^5} + \cancel{\frac{7}{x^5}}^{\to 0}} = \frac{0}{1} = 0 \qquad \therefore \text{H.A. is } y = 0.$$

Oblique Asymptotes

These occur when the order of the top is <u>exactly one</u> greater than the order of the bottom.

e.g. $f(x) = \dfrac{x^2 - 2x + 3}{x - 1} \longrightarrow \dfrac{\text{order } 2}{\text{order } 1}$ ∴ it will have an O.A.

With Oblique Asymptotes we are still dealing with end behaviors.

O.A. have the form $y = mx + b$ (shocking, I know!) The question we have to face is this:

How do we find the line representing the O.A.?

Ans: By polynomial division!

The O.A. is the quotient.

$$f(x) = \frac{x^2 - 2x + 3}{x - 1}$$

$$\begin{array}{r|rrr}
1 & 1 & -2 & 3 \\
 & \downarrow & 1 & -1 \\
\hline
 & \underbrace{1 \quad -1}_{\text{O.A}} & & \boxed{2} \text{ remainder.}
\end{array}$$

$y = 1x - 1$

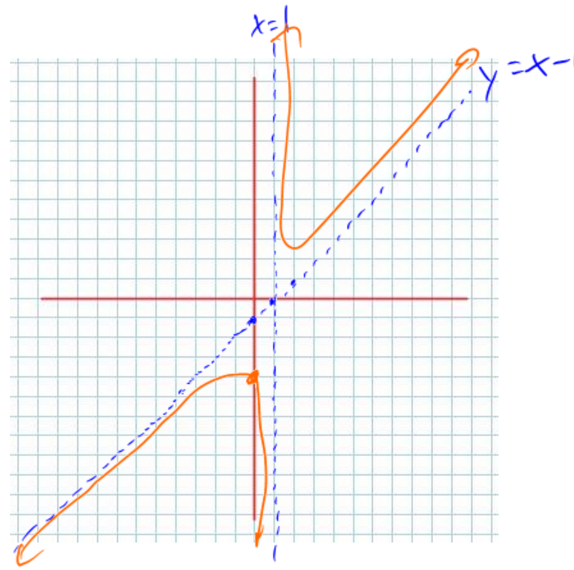
(Rough) Sketch of $f(x)=\dfrac{x^2-2x+3}{x-1}$

V.A. $x=1$

H.A. None or N.A.

O.A. $y=x-1$

y-int: $(0,-3)$



Example 4.1.3

Determine the equations of all asymptotes, and any hole discontinuities for:

a) $f(x)=\dfrac{x+2}{x^2+3x+2}$ $\begin{array}{l}\rightarrow \text{order } 1\\ \rightarrow \text{order } 2\end{array}$

$$f(x)=\frac{\cancel{x+2}}{(x+1)(\cancel{x+2})}$$

$$f(x)=\frac{1}{x+1}$$

denom

V.A	$x=-1$
Hole	$x=-2$
H.A.	$y=0$
O.A.	N.A.

b) $g(x)=\dfrac{\boxed{4}x^2-25}{\boxed{1}x^2-9}$

$$g(x)=\frac{(2x-5)(2x+5)}{(x+3)(x-3)}$$

V.A.	$x=3$, $x=-3$
Hole	N.A.
H.A.	$y=4$
O.A.	N.A.

c) $h(x) = \dfrac{x^2 + 0x + 0}{x+3}$

$$\begin{array}{r|rrr} -3 & 1 & 0 & 0 \\ & & -3 & 9 \\ \hline & 1x & -3 & 9 \end{array}$$

$\boxed{1x - 3}$

V.A.	$x = -3$
Hole	N.A.
H.A.	N.A.
O.A.	$y = x - 3$

Example 4.1.4

Determine an equation for a function with a vertical asymptote at $x = -3$, and a horizontal asymptote at $y = 0$.

bottom is bigger.

$x+3 \rightarrow$ denom.

$f(x) = \dfrac{8}{x+3}$ or $f(x) = \dfrac{3x^2 - 2x + 10000}{(x+3)(x-8)(x+5)}$

Example 4.1.5

Determine an equation for a function with a hole discontinuity at $x = 3$.

$f(x) = \dfrac{(x-3)(x^2 - 8x + 5)}{(x-3)(2x^{99} + 8x - 200000)}$

$x-3$ in top and bottom.

Class/Homework for Section 4.1

Pg. 262 #1 - 3

4.2 Graphs of Rational Functions

Note: In Advanced Functions we will only consider rational functions of the form

$$f(x)=\frac{ax+b}{cx+d}$$

Rational Functions of the form $f(x)=\frac{ax+b}{cx+d}$ will have: $cx+d=0$

1) One Vertical Asymptote

$$cx+d=0$$

$$x=\frac{-d}{c}, \text{ unless } c=0$$

2) One Zero (unless $a=0$, no zero)

$$\hookrightarrow 0=\frac{ax+b}{\cancel{cx+d}} \rightarrow x=\frac{-b}{a}$$

3) Functional Intercept

Y

$$f(0)=\frac{a(0)+b}{c(0)+d}=\frac{b}{d} \Rightarrow \left(0, \frac{b}{d}\right)$$

4) A Horizontal Asymptote

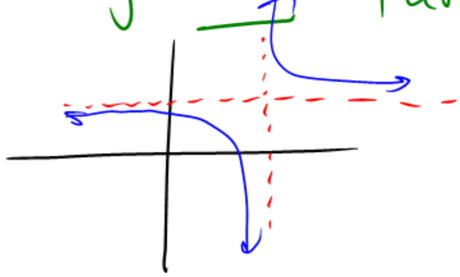
$$y=\frac{a}{c}$$

unless $a=0$, then the H.A is $y=0$.

5) These functions will always be either

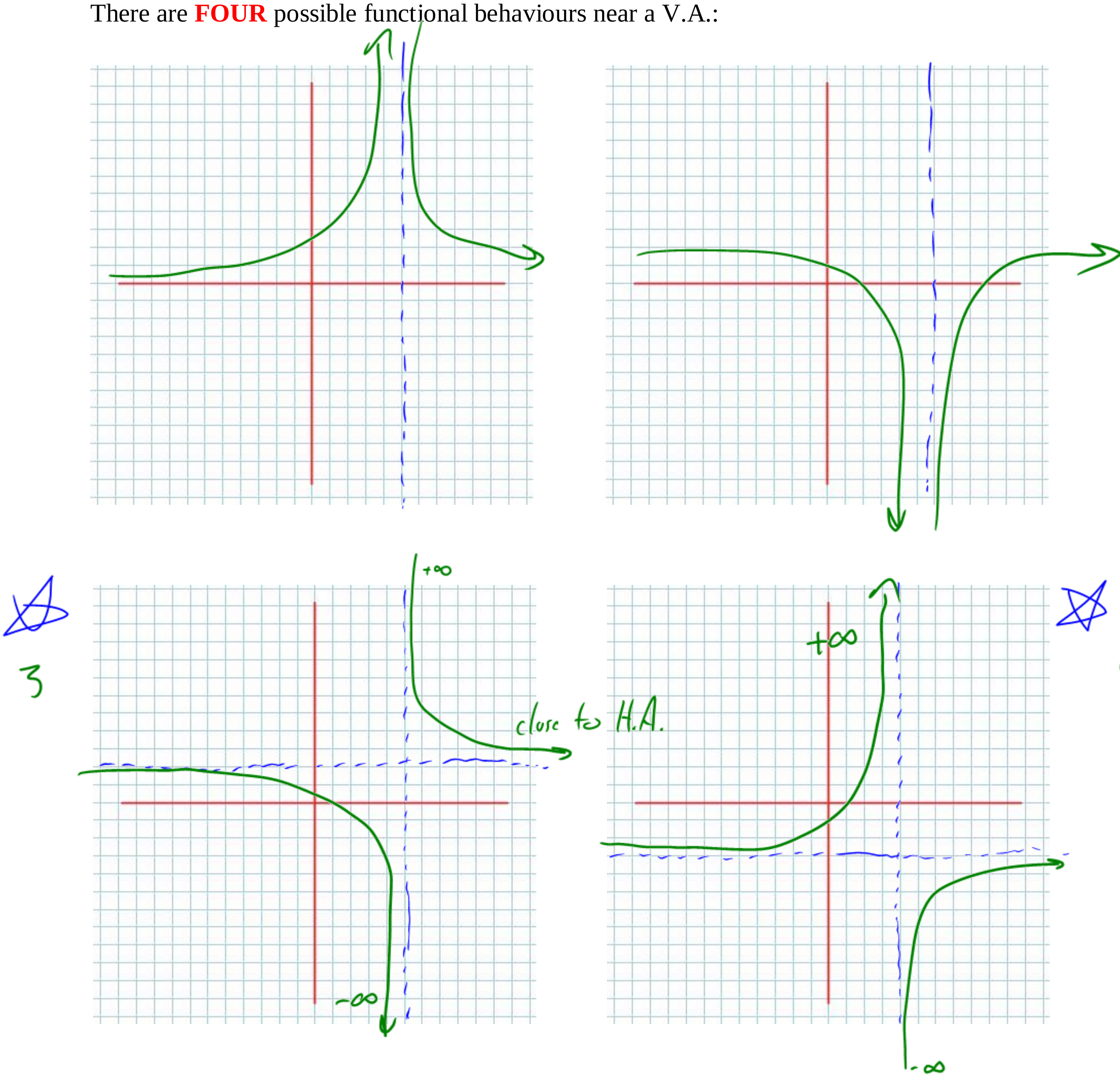
always increasing or always decreasing.

meaning no turning points.



Functional Behaviour Near A Vertical Asymptote

There are **FOUR** possible functional behaviours near a V.A.:

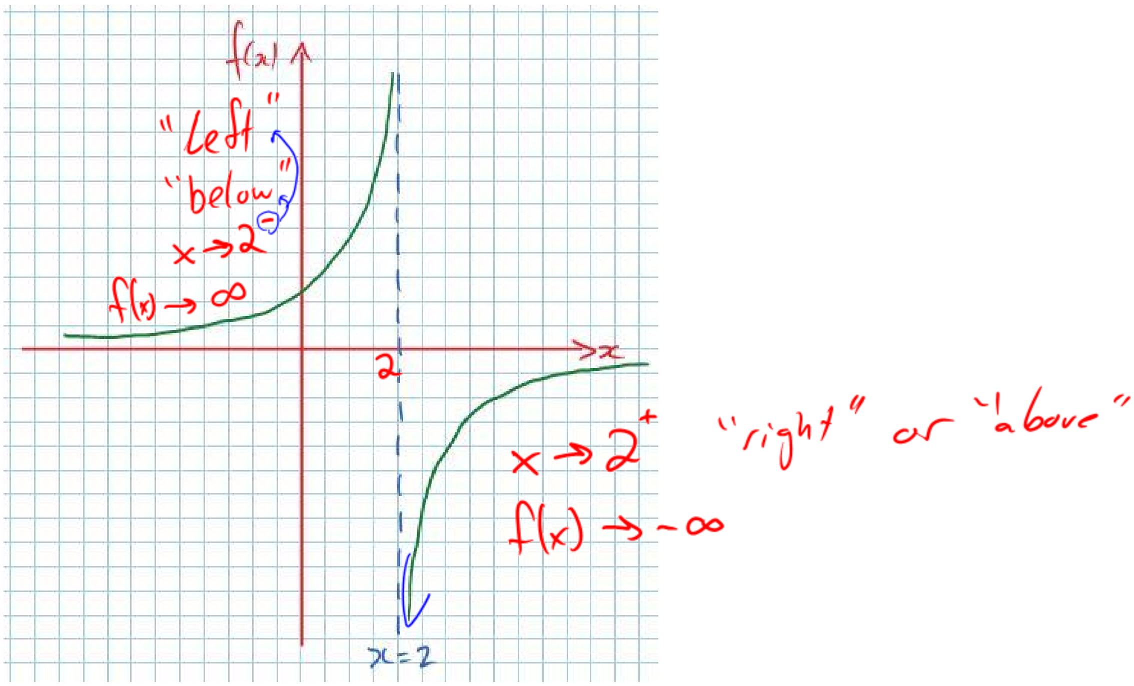


For functions of the form $f(x)=\dfrac{ax+b}{cx+d}$ we will see behaviours like 3 and 4

The questions is, **how do we know which**?

We need to **analyze** the function **near the V.A**.
We need to become familiar with some **Notation.**

Consider some rational function with a sketch of its graph which looks like:



Example 4.2.1

Determine the functional behaviour of $f(x) = \dfrac{2x+1}{x-3}$ near its V.A.

V.A. is $x = 3$

$x \to 3^{-}$	$x \to 3^{+}$
$f(x) \to -\infty$	$f(x) \to +\infty$
Pick a number really close to 3 AND less than 3.	Pick a number above 3 and close to 3
$x = 2.99$, $f(2.99) = -698$	$x = 3.001$, $f(x) = 7002$
$f(2.999) = -6998$	$\therefore$ getting bigger
$\therefore$ getting smaller	

We now have the tools to sketch some graphs!

Example 4.2.2

Sketch the graph of the given function. State the domain, range, intervals of increase/decrease and where the function is positive and negative.

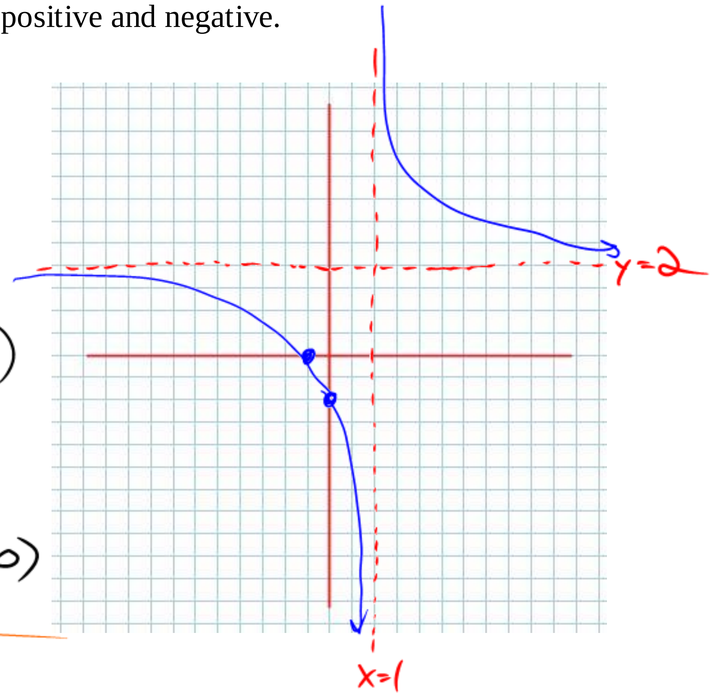
a) $f(x) = \dfrac{2x+1}{x-1}$

V.A. = $x = 1$
H.A. = $y = 2$
x-int: $x = -\frac{1}{2}$
y-int: $y = -1$

$D_f: \{x \in \mathbb{R} / x \neq 1\}$

$R_f: f(x) \in (-\infty, 2) \cup (2, \infty)$

$f(x)$ is decreasing on $(-\infty, 1) \cup (1, \infty)$



$f(x) > 0$ on $(-\infty, -\frac{1}{2}) \cup (1, \infty)$

$f(x) < 0$ on $(-\frac{1}{2}, 1)$

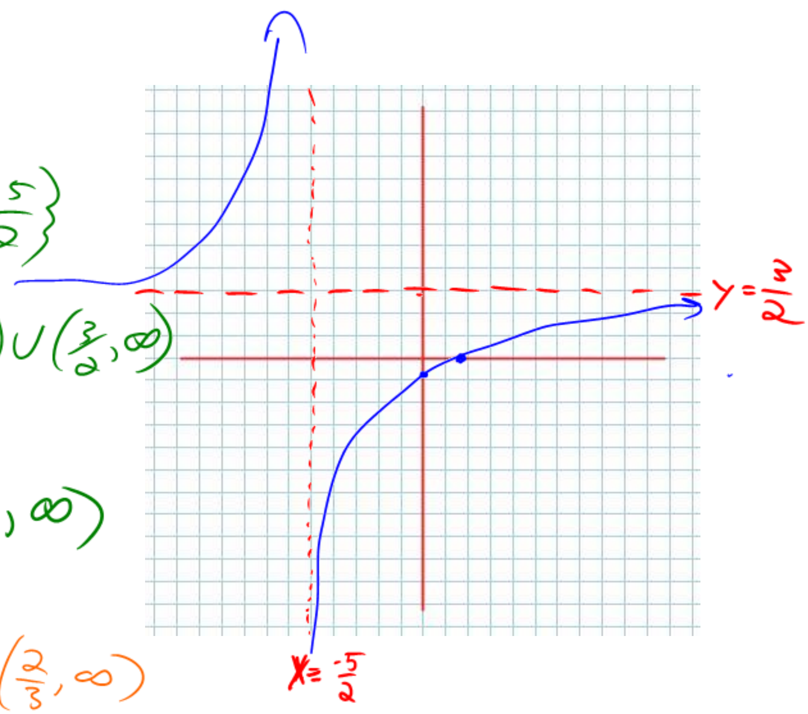
b) $g(x) = \dfrac{3x-2}{2x+5}$

V.A: $x = \frac{-5}{2}$
H.A.: $y = \frac{3}{2}$
x-int: $x = \frac{2}{3}$
y-int: $y = -\frac{2}{5}$

$D_g: \{x \in \mathbb{R} / x \neq \frac{-5}{2}\}$

$R_g: g(x) \in (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

$g(x)$ is increasing on $(-\infty, \frac{-5}{2}) \cup (\frac{-5}{2}, \infty)$

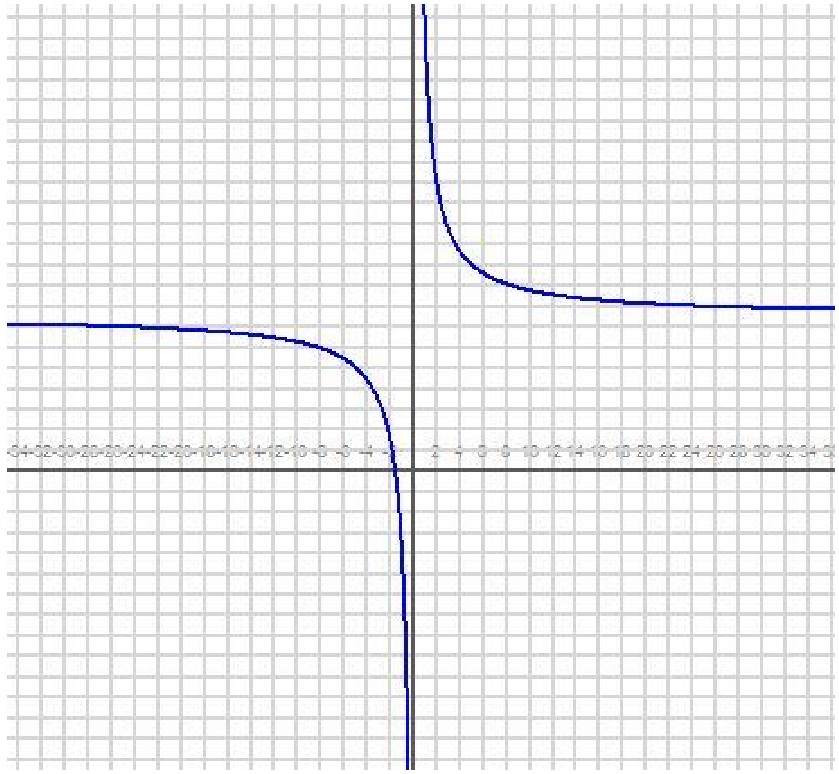


$g(x) > 0$ on $(-\infty, \frac{-5}{2}) \cup (\frac{2}{3}, \infty)$

$g(x) < 0$ on $(\frac{-5}{2}, \frac{2}{3})$

Example 4.2.3

Consider question #9 on page 274: $I(t) = \dfrac{15t + 25}{t}$



Class/Homework for Section 4.2

Pg. 272 #1, 2 (Don't use any tables of values!), 4 – 6, 9, 10

4.4 Solving Rational Equations

Solving a Rational Equation is **VERY MUCH** like solving a Polynomial Equation. Thus, this stuff is so much fun it should be illegal. But it isn't illegal unless you break a rule of algebra. Math Safe!

KEY (*this is a major key for you music buffs*)

Multiplying by the Multiplicative Inverse of the Common Denominator is wonderful to use **WHEN YOU HAVE** something like:

$$\text{RATIONAL}_1 + \text{RATIONAL}_2 = \text{RATIONAL}_3$$

e.g. $\dfrac{3}{x-2} = \dfrac{4(x+5)}{x} + \dfrac{3}{2}$ C.D. is $2x(x-2)$

$$2x\cancel{(x-2)}\left(\frac{3}{\cancel{x-2}}\right) = 2\cancel{x}(x-2)\left(\frac{4(x+5)}{\cancel{x}}\right) + \cancel{2}x(x-2)\left(\frac{3}{\cancel{2}}\right)$$

$$6x = 8(x-2)(x+5) + 3x(x-2)$$

$x \neq 2,\ x \neq 0$

Make Sure To Keep **RESTRICTIONS ON X** In Mind

Expand, "stuff" = 0, solve.

means that restrictions cannot be solutions

Example 4.4.1

a) Solve $\frac{x}{5} = \frac{9}{18}$

No restrictions.

$18x = 45$

$\frac{18x}{18} = \frac{45}{18}$

$x = \frac{5}{2}$

b) Solve $\left(\frac{1}{x} - \frac{5x}{3} = \frac{2}{5}\right)(5)(3)(x)$

RESTRICTIONS

$x \neq 0$

C.D.

$(x)(3)(5) = 15x$

$15 - 25x^2 = 6x$

$0 = 25x^2 + 6x - 15$

d.n.f.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-6 \pm \sqrt{6^2 - 4(25)(-15)}}{2(25)}$

{ work → show this work

$x = -0.9$ and $x = 0.66$

c) Solve $\left(\frac{3}{x}+\frac{4}{x+1}=\frac{2}{1}\right) x(x+1)$

RESTRICTIONS

$x \neq -1, 0$ | $x(x+1)$

$3(x+1) + 4x = 2x(x+1)$

$3x + 3 + 4x = 2x^2 + 2x$

$0 = 2x^2 - 5x - 3$ ⊗ -6 ⊕ -5 $-6, +1$

$0 = 2x^2 - 6x + 1x - 3$

$0 = (2x+1)(x-3)$

$x = -\frac{1}{2}$

$x = 3$

d) Solve $\left(\frac{10}{\cancel{x^2-2x}}+\frac{4}{x}=\frac{5}{x-2}\right) x(x-2)$

$x(x-2)$

RESTRICTIONS

$x \neq 0, 2$ | C.D. $x(x-2)$

$10 + 4(x-2) = 5x$

$10 + 4x - 8 = 5x$

$2 = x$

Whoah!! $x=2$ is a restriction, ∴ no solutions

e) Solve $\left(16x - \frac{5}{x+2} = \frac{15}{x-2} - \frac{60}{(x-2)(x+2)}\right)^{(x+2)(x-2)}$ Restrictions: $x \neq -2, 2$

C.D. $= (x-2)(x+2)$

$$16x\underbrace{(x+2)(x-2)}_{(x^2-4)} - 5(x-2) = 15(x+2) - 60$$

$$16x^3 - 64x - 5x + 10 = \boxed{15x + 30 - 60}$$

$$\frac{16x^3 - 84x + 40 = 0}{4}$$

$$4x^3 - 21x + 10 = 0$$

Let $f(x) = 4x^3 - 21x + 10$ T.V ± 1, ± 2, ± 5, ± 10

$f(2) = 4(2)^3 - 21(2) + 10$

$= 32 - 42 + 10$

$= 0$

$\therefore$ $(x-2)$ is a factor!

$$\begin{array}{r|rrrr} 2 & 4 & 0 & -21 & 10 \\ & & 8 & 16 & -10 \\ \hline & 4 & 8 & -5 & 0 \end{array}$$

(x) -20, (+) 8 $\quad 10, -2$

$$\therefore (x-2)(4x^2 + 8x - 5) = 0$$

$$(x-2)(4x^2 - 2x + 10x - 5) = 0$$

$$(x-2)(2x+5)(2x-1) = 0$$

$\therefore$ ~~$x = 2$~~ inadmissable

$x = \frac{1}{2}$

$x = \frac{-5}{2}$

Example 4.4.2

From your Text: Pg. 285 #10

The Turtledove Chocolate factory has two chocolate machines. Machine A takes s minutes to fill a case with chocolates, and machine B takes $s + 10$ minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

~~Work~~ Rate problem: $\dfrac{\#\text{ cases filled}}{\#\text{ of minutes}}$

A: $\dfrac{1 \text{ case}}{x}$ B: $\dfrac{1}{x+10}$

Together: $(15)(x)(x+10)\left(\dfrac{1}{x} + \dfrac{1}{x+10} = \dfrac{1}{15}\right)$

Restriction: $x \neq 0, -10$

C.D. $= (15)(x)(x+10)$

$$15(x+10) + 15x = x(x+10)$$

$$15x + 150 + 15x = x^2 + 10x$$

$$0 = x^2 - 20x - 150$$

does not factor

$\therefore$ by the Q.F.

$$x = 25.8$$

$\therefore$ Machine A is 25.8 min and Machine B is 35.8 min

Class/Homework for Section 4.4

Pg. 285 - 287 #2, 5 – 7def, 9, 12, 13

4.5 Solving Rational Inequalities

The joy, wonder and peace these bring is really quite amazing

e.g. Solve $\left(\frac{x-2}{7} \geq 0\right)7$

$$x - 2 \geq 0$$

$$x \geq 2$$

Note: For Rational Inequalities, with a variable in the denominator, you **CANNOT** multiply by the multiplicative inverse of the common denominator!!!!

Why? If the factor "x+3" is negative, cross multiplying would change the direction of the inequality.

what do you plug into the top to equal zero?

Example 4.5.1

Solve $\frac{x-2}{x+3} \geq 0$

Zero: $x=2$ Restriction: $x \neq -3$

We solve by using an Interval Chart

For the intervals, we split $(-\infty, \infty)$ at all zeros (**where the numerator is zero**), and all restrictions (**where the denominator is zero**) of the (SINGLE) rational expression. Keep in mind that it may take a good deal of algebraic manipulation to get a SINGLE rational expression…

Intervals	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
T.V.	-4	0	3
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
Ratio	$+$	$-$	$+$

restriction

$$\therefore \frac{x-2}{x+3} \geq 0 \text{ when } x \in (-\infty, -3) \cup [2, \infty)$$

Example 4.5.2

Solve $\frac{1}{x+5} < 5$ $-5 \quad -5$

DO NOT CROSS MULTIPLY or else

- Get everything on one side
- Simplify into a single Rational Expression using a common denominator
- Interval Chart it up

$$\frac{1}{x+5} - \frac{5}{1}\frac{(x+5)}{(x+5)} < 0$$

$$\frac{1-5x-25}{x+5} < 0$$

$$\frac{-5x-24}{x+5} < 0$$

Zero: $x = \frac{-24}{5}$ or -4.8

restriction: $x \neq -5$

Interval	$(-\infty, -5)$	$(-5, -4.8)$	$(-4.8, \infty)$
T.V.	-6	-4.9	0
$-5x-24$	+	+	−
$x+5$	−	+	+
ratio	−	+	−

$\therefore \frac{1}{x+5} < 5$ for $x \in (-\infty, -5) \cup (-4.8, \infty)$

Example 4.5.3

Solve $\dfrac{x^2+3x+2}{x^2-16} \geq 0$

FACTORED FORM IS YOUR FRIEND

$$\frac{(x+2)(x+1)}{(x-4)(x+4)} \geq 0$$

Zeros: $x=-2$, $x=-1$

Restrictions: $x \neq -4$, $x \neq 4$

Intervals	$(-\infty,-4)$	$(-4,-2)$	$(-2,-1)$	$(-1,4)$	$(4,\infty)$
T.V.	-5	-3	-1.5	0	5
$x+1$	$-$	$-$	$-$	$+$	$+$
$x+2$	$-$	$-$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$	$+$	$+$
Ratio	$+$	$-$	$+$	$-$	$+$

$$\therefore \frac{x^2+3x+2}{x^2-16} \geq 0 \text{ for } x \in (-\infty,-4) \cup [-2,-1] \cup (4,\infty)$$

Example 4.5.4

Solve $\frac{3}{x+2} \le x$ $-x$ $-x$

$$\frac{3}{x+2} - \frac{x^{(x+2)}}{1_{(x+2)}} \le 0$$

$$\frac{3 - x^2 - 2x}{x+2} \le 0$$

$$\div -1 \left(\frac{-x^2 - 2x + 3}{x+2}\right) \le 0 \left(\div -1\right)$$

this flipped.

$$\frac{x^2 + 2x - 3}{x+2} \ge 0$$

$$\frac{(x+3)(x-1)}{x+2} \ge 0$$

Zeros: $x = -3$ and $x = 1$
Restriction: $x \ne -2$

Intervals	$(-\infty, -3)$	$(-3, -2)$	$(-2, 1)$	$(1, \infty)$
T.V.	-4	-2.5	0	2
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+2$	$-$	$-$	$+$	$+$
Ratios	$-$	$+$	$-$	$+$

$\therefore \frac{3}{x+2} \le x$ for $x \in [-3, -2) \cup [1, \infty)$

restriction

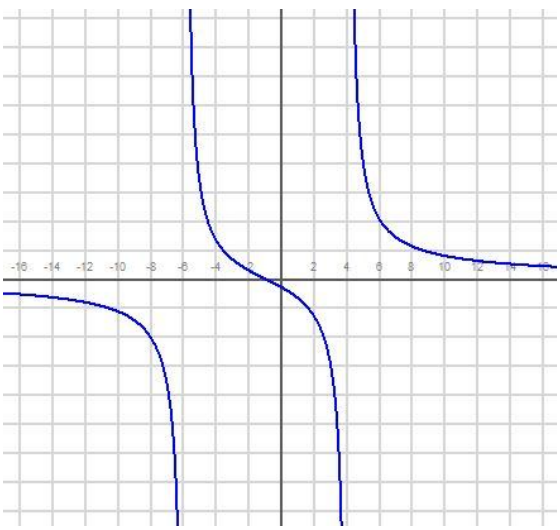
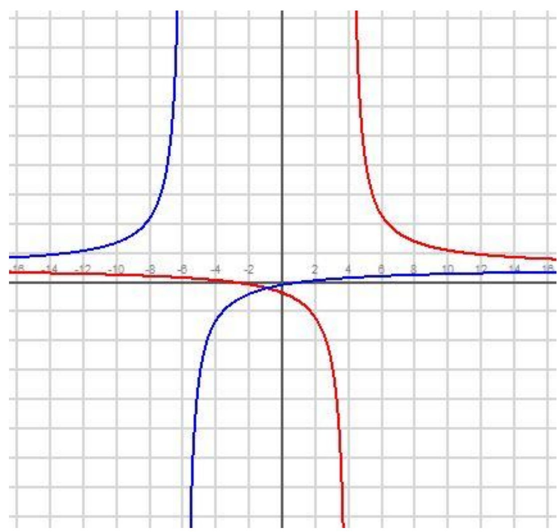
Example 4.5.5

From your Text: Pg. 296 #6a

Using **Graphing Tech**

Solve $\frac{x+3}{x-4} \geq \frac{x-1}{x+6}$

Note: There are **TWO** methods, both of which require a **FUNCTION** (let $f(x) = ...$ returns)

1) Get a Single Function (on one side of the inequality)	2) Use Two Functions (one for each side)
	

Class/Homework for Section 4.5

Pg. 295 - 297 #1, 3, 4 – 6 (def), 9, 11