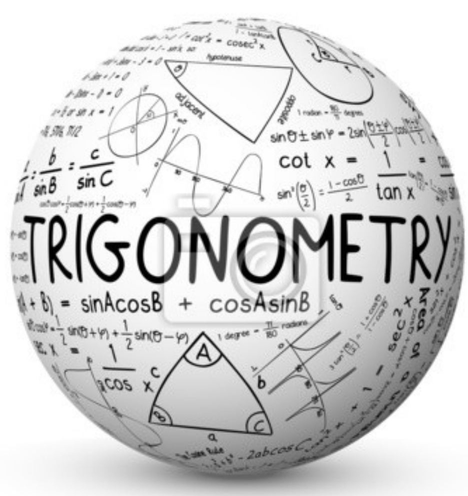
Advanced Functions

Fall 2017
Course Notes

Unit 6 – Trigonometric Identities and Equations

We will learn

- *about Equivalent Trigonometric Relationships*
- *how to use compound angle formulas to determine exact values for trig ratios which DON'T involve the two special triangles*
- *techniques for proving trigonometric identities*
- *how to solve linear and quadratic trigonometric equations using a variety of strategies*



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Chapter 6 – Trigonometric Identities and Equations

Contents with suggested problems from the Nelson Textbook (Chapter 7)

6.1 Basic Trigonometric Equivalencies

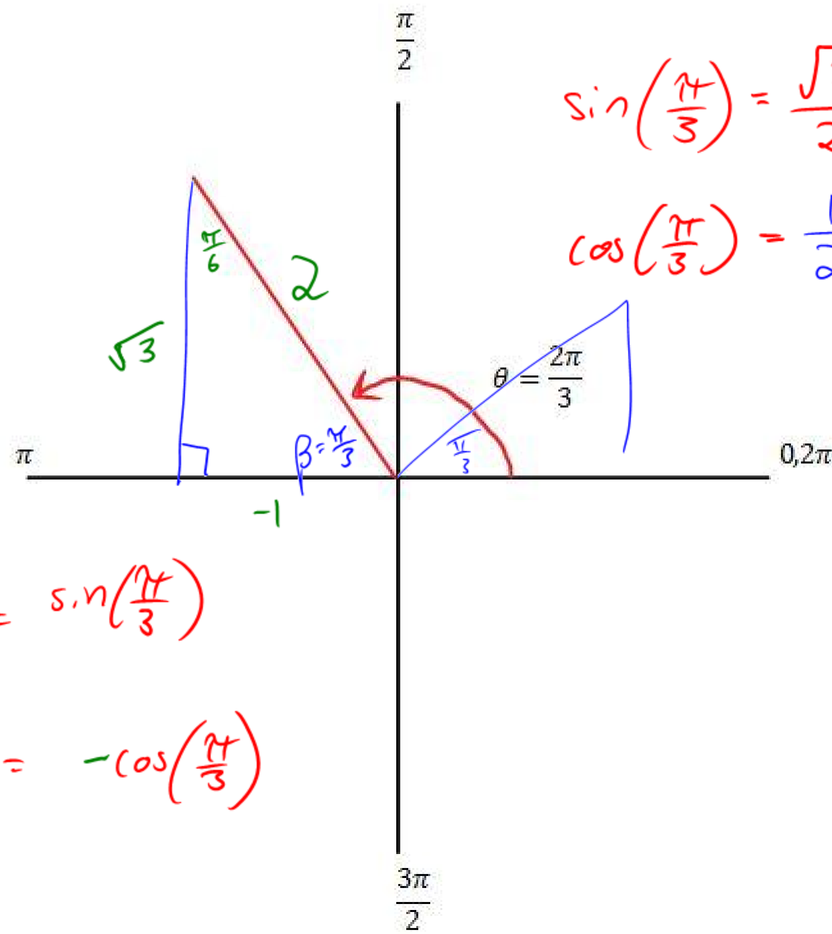
We have already seen a very basic trigonometric equivalency when we considered angles of rotation. For example, consider the angle of rotation for $\theta = \frac{2\pi}{3}$:

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$



$$\therefore \quad \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

$$\therefore \quad \cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$$

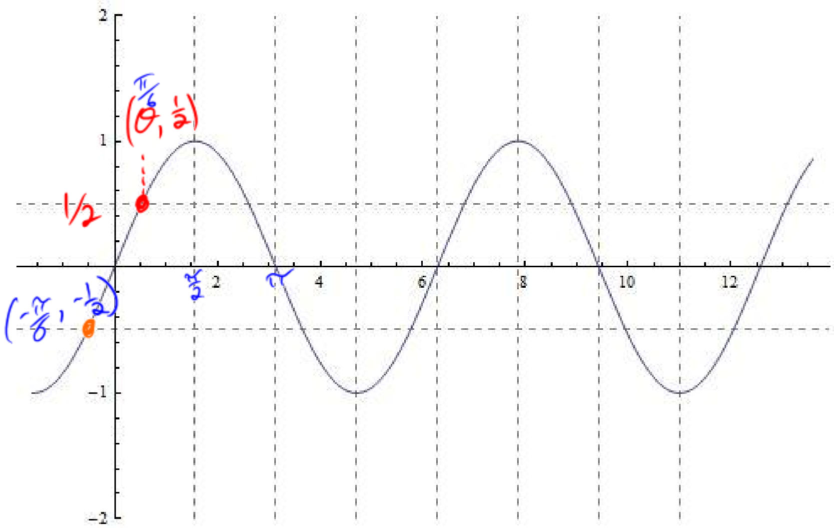
Periodic Equivalencies

Example 6.1.1

Consider the sketch of the function $f(\theta) = \sin(\theta)$

period is 2π

$\therefore \sin\theta$ is 2π periodic



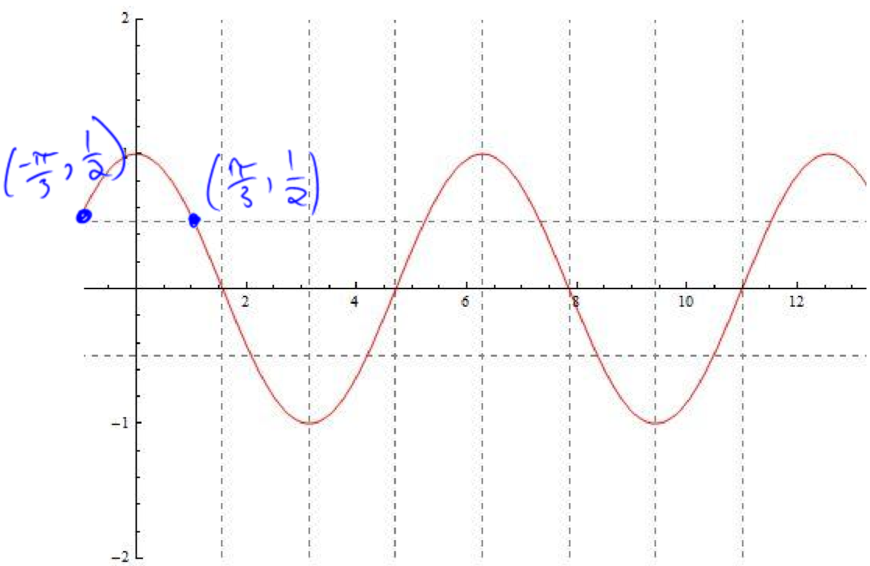
$\sin\theta = \sin(\theta + 2\pi)$

$\sin\theta = -\sin(-\theta)$

$\therefore$ sine is an odd function

Example 6.1.2

Consider $g(x) = \cos(x)$



- cosine is 2π periodic

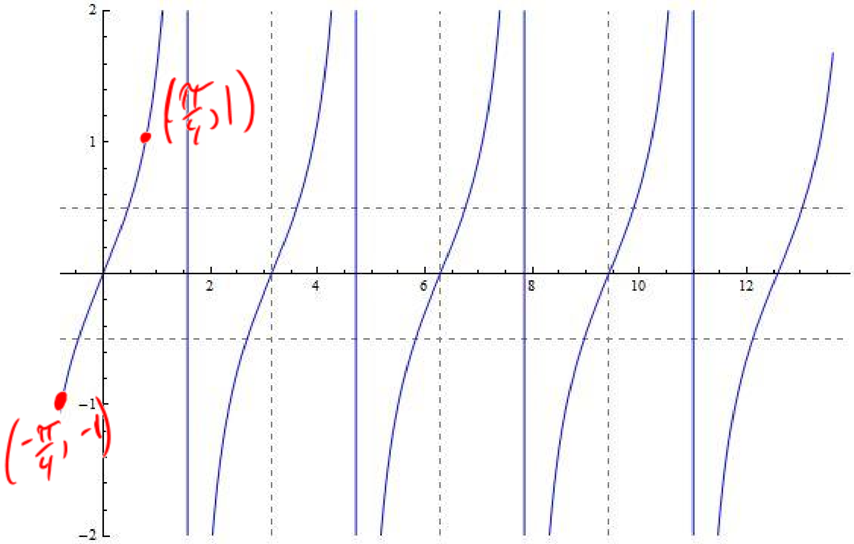
$\rightarrow \cos\theta = \cos(\theta + 2\pi)$

$\rightarrow \cos\theta = \cos(-\theta)$

$\therefore$ cosine is an even fn.

Example 6.1.3

Consider $h(\theta) = \tan(\theta)$



→ $\tan\theta$ is π periodic

→ $\tan\theta = \tan(\theta + \pi)$

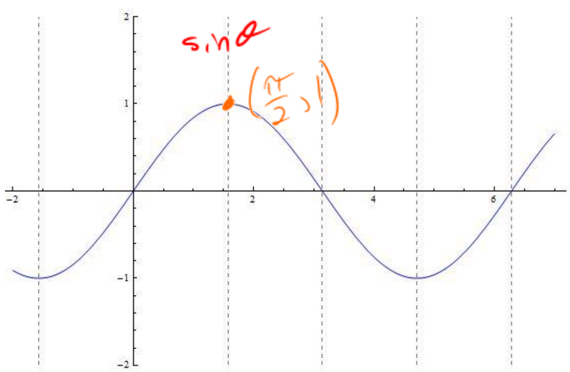
→ $\tan\theta = -\tan(-\theta)$

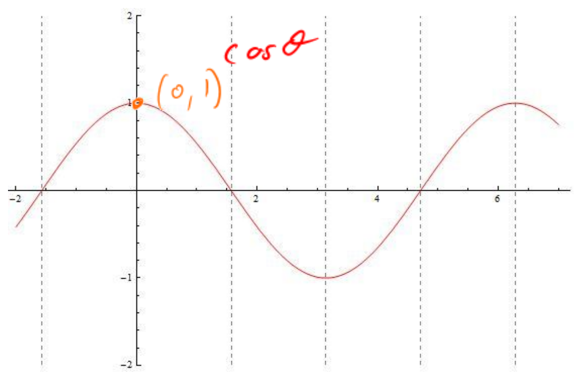
$\therefore$ tangent is an odd fn.

Shift Equivalencies

Example 6.1.4

Consider the sketches of the graphs for $f(\theta) = \sin(\theta)$ and $g(\theta) = \cos(\theta)$





$\sin(\theta) = \cos(\theta - \frac{\pi}{2})$ || $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$

Test $\theta = 406$ $\quad \sin(406) = -0.67025...$

$\cos(406 - \frac{\pi}{2}) = \cos(404.429) = -0.67025$

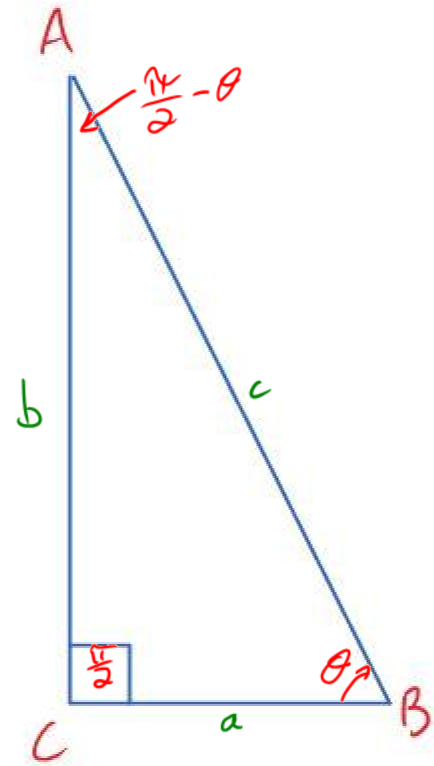
Cofunction Identities

Consider the right angle triangle

$$\sin\theta = \frac{b}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$$

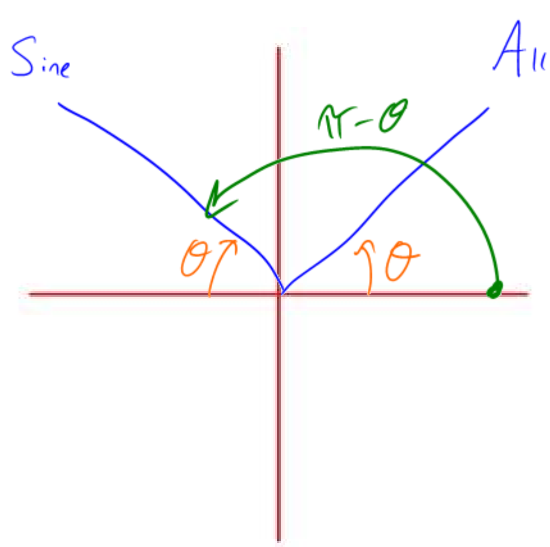
$$\cos\theta = \frac{a}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \frac{b}{a} = \cot\left(\frac{\pi}{2} - \theta\right)$$



Using CAST, relating angles of rotation to π and 2π

Compare Q1 and Q2

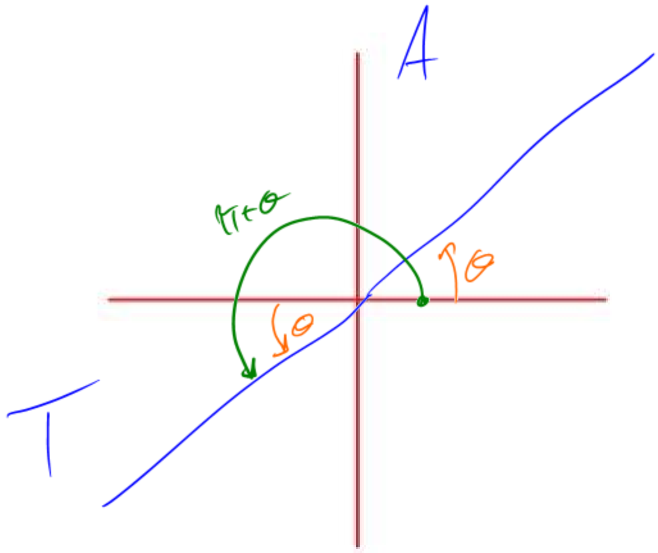


$$\sin\theta = \sin(\pi - \theta)$$

$$\cos\theta = -\cos(\pi - \theta)$$

$$\tan\theta = -\tan(\pi - \theta)$$

Compare Q1 and Q3

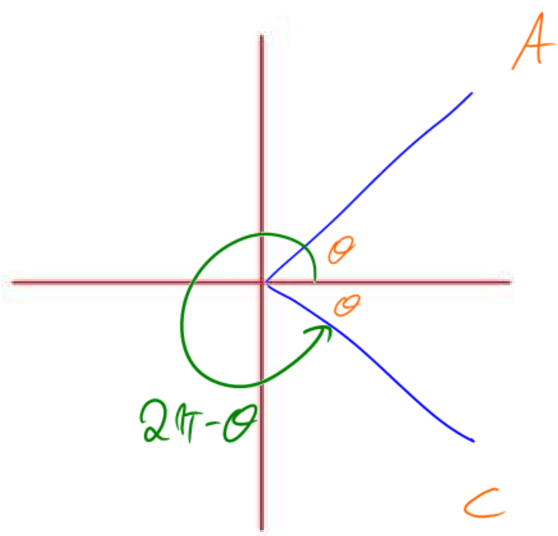


$$\sin\theta = -\sin(\pi+\theta)$$

$$\cos\theta = -\cos(\pi+\theta)$$

$$\tan\theta = \tan(\pi+\theta)$$

Compare Q1 and Q4



$$\sin\theta = -\sin(2\pi-\theta)$$

$$\cos\theta = \cos(2\pi-\theta)$$

$$\tan\theta = -\tan(2\pi-\theta)$$

Example 6.1.5

From your text: Pg. 392 #3

Use a cofunction identity to find an equivalency:

a) $\sin\left(\frac{\pi}{6}\right)$ $= \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$= \cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

d) $\cos\left(\frac{5\pi}{16}\right)$ $= \sin\left(\frac{\pi}{2} - \frac{5\pi}{16}\right)$

$= \sin\left(\frac{8\pi}{16} - \frac{5\pi}{16}\right)$

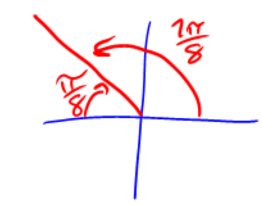
$= \sin\left(\frac{3\pi}{16}\right)$

Example 6.1.6

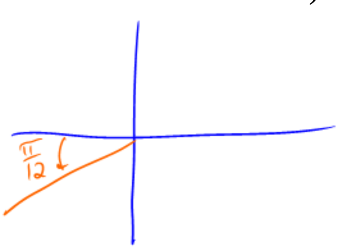
From your text: Pg. 393 #5

Using the related acute angle, find an equivalent expression:

a) $\sin\left(\frac{7\pi}{8}\right)$ $= \sin\left(\frac{\pi}{8}\right)$



b) $\cos\left(\frac{13\pi}{12}\right)$ $= -\cos\left(\frac{\pi}{12}\right)$



Class/Homework for Section 6.1

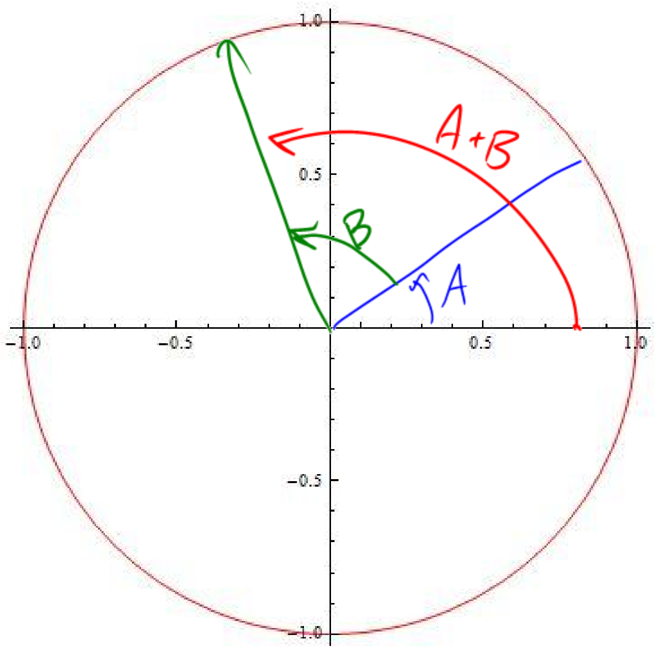
Pg. 392 – 393 #3cdef, 5cdef

6.2 Compound Angle Formulae

Here we learn to find **exact trig ratios** for **non-special angles**!

↳ no calculators.

Consider the picture:



Can we find $\sin(A+B)$ if we know $\sin(A)$ and $\sin(B)$?

or $\cos(A+B)$?

or $\tan(A+B)$?

$\neq \tan A + \tan B$

Yes, of course!

Your text has a nice proof of one of the six compound angle formulas (there are six of them!...see Pg. 394)

Namely your text proves that $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

Using some trig equivalencies (from 6.1) we will find the other 5 compound angle formulae.

Example 6.2.1

Find a formula for $\cos(A+B)$

Consider $\cos(\underset{A}{A} - (\underbrace{-B}_{B})) = \cos A \boxed{\cos(-B)} + \sin A \boxed{\sin(-B)}$

cosine is even, $\therefore \cos(-B) = \cos B$

$= \cos A \cos B + \sin A(-\sin B)$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Sine is odd

$\therefore \sin\theta = -\underline{\sin(-\theta)}$

$-\sin\theta = \sin(-\theta)$

$$\therefore \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Example 6.2.2

Determine a compound angle formula for $\sin(A+B)$ using **a cofunction identity** and a **cosine compound angle formula**.

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$$\sin(\underbrace{A+B}_{\theta}) = \cos\left(\frac{\pi}{2} - (A+B)\right)$$

$$= \cos\left(\underbrace{\left(\frac{\pi}{2} - A\right)}_{A} - B\right) \qquad \text{use } \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

In the same way, $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$\therefore \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

Example 6.2.3

Using the fact that $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$ determine a compound angle formula for $\tan(A+B))$.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

Through some mathemagic

$$\therefore \tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$

30° 45° 60°

Example 6.2.4

From your text: Pg. 400 #3acd

Express each given angle as a compound angle using a pair of special triangle angles

a) $75^\circ = 30^\circ + 45^\circ$

$\frac{\pi}{6}, \frac{2\pi}{6}$

c) $-\frac{\pi}{6} = \frac{\pi}{6} - \frac{2\pi}{6} = \frac{\pi}{6} - \frac{\pi}{3}$

$\frac{2\pi}{12}, \frac{3\pi}{12}, \frac{4\pi}{12}$

d) $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$

Example 6.2.5

From your text: Pg. 400 #4ac, 8bd

Determine the **EXACT** value of the trig ratio

a) $\sin(75^\circ)$

$= \sin(\overset{A}{30} + \overset{B}{45})$

$= \sin 30 \cos 45 + \sin 45 \cos 30$

$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{(1+\sqrt{3})\sqrt{2}}{(2\sqrt{2})\sqrt{2}}$ This is a valid test answer!

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

b) $\tan\left(\frac{5\pi}{12}\right)$

$= \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)}$

$= \frac{\frac{1}{\sqrt{3}} + \frac{1\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}}1 - \frac{1}{\sqrt{3}}(1)}$

$= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \div$

$= \frac{1+\sqrt{3}}{\cancel{\sqrt{3}}} \times \frac{\cancel{\sqrt{3}}}{\sqrt{3}-1}$

$= \frac{1+\sqrt{3}}{\sqrt{3}-1}$ Stop here.

{if you rationalized

$= 2+\sqrt{3}$

$30 - 45 \quad 45 - 60$

#8b) $\tan\left(-15^\circ\right)$

$= \tan(30 - 45)$

$= \dfrac{\tan 30 - \tan 45}{1 + \tan 30 \tan 45}$

$= \dfrac{\frac{1}{\sqrt{3}} - \frac{1\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}\cancel{1}}{\sqrt{3}} + \frac{1}{\sqrt{3}}(1)} = \dfrac{\frac{1-\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$

$= \dfrac{1-\sqrt{3}}{\cancel{\sqrt{3}}} \times \dfrac{\cancel{\sqrt{3}}}{\sqrt{3}+1}$

FOIL

$= \dfrac{(1-\sqrt{3})}{\sqrt{3}+1} \times \dfrac{(\sqrt{3}-1)}{\sqrt{3}-1}$

$= \dfrac{\sqrt{3} - 1 - 3 + \sqrt{3}}{\sqrt{9} - 1} = \dfrac{2\sqrt{3} - 4}{2} = \sqrt{3} - 2$

d) $\sin\left(\dfrac{13\pi}{12}\right) = -\sin\left(\dfrac{\pi}{12}\right)$

$= -\left[\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right]$

$= -\left[\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right]$

$= -\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \dfrac{-\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}}$

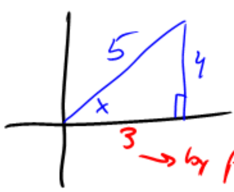
$= \dfrac{1-\sqrt{3}}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

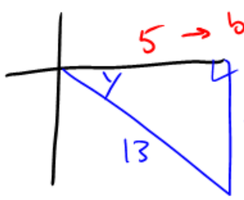
Example 6.2.6

From your text: Pg. 401 #9a

If $\sin x = \dfrac{4}{5}$ and $\sin y = -\dfrac{12}{13}$, where $0 \le x \le \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} \le y \le 2\pi$ evaluate $\cos(x+y)$.

$Q_1 \quad Q_4$





$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$

$= \dfrac{15}{65} + \dfrac{48}{65}$

$= \dfrac{63}{65}$

Class/Homework for Section 6.2

Pg. 400 – 401 #3 – 6, 8 – 10, 13

Note: Test is on Nov. 21.

6.3 Double Angle Formulae

This is a nice extension of the compound angle formulae from section 6.2.

Recall the compound Angle Formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

1) $\sin(2A) = \sin(A + \overset{\text{"B"}}{A})$

$= \sin A \cos A + \sin A \cos A$

$= 2\sin A \cos A$

Recall:

$\sin^2\theta + \cos^2\theta = 1$

$\rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\rightarrow \cos^2\theta = 1 - \sin^2\theta$

2) $\cos(2A) = \cos(A + \overset{\text{"B"}}{A})$

$= \cos A \cos A - \sin A \sin A$

① $\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$

② $\cos(2A) = 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

③ $\cos(2A) = \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

3) $\tan(2A) = \tan(A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

Example 6.3.1

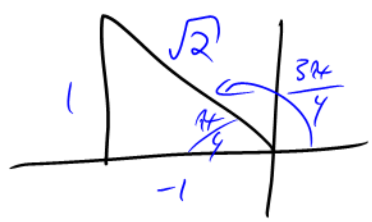
From your text: Pg. 407 #2ae

Express as a single trig ratio and evaluate:

a) $2\sin\left(\underset{A}{45^\circ}\right)\cos\left(\underset{A}{45^\circ}\right)$

$= \sin(2(45))$

$= \sin(90) = 1$



e) $1 - 2\sin^2\left(\overset{A}{\frac{3\pi}{8}}\right)$

$= \cos\left(\cancel{2}\left(\frac{3\pi}{\cancel{8}_4}\right)\right)$

$= \cos\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

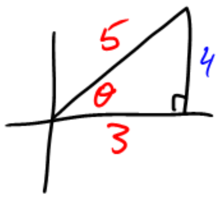
$= \frac{-\sqrt{2}}{2}$

Example 6.3.2

From your text: Pg. 407 #4

Q$_1$

Determine the values of $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$ given $\cos(\theta) = \frac{3}{5}$, $\boxed{0 \le \theta \le \frac{\pi}{2}}$



$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$= \frac{2}{1}\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$
$$= \frac{24}{25}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$= \frac{9}{25} - \frac{16}{25}$$
$$= \frac{-7}{25}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2(\theta)}$$
$$= \frac{\frac{2}{1}\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{\frac{9}{9}1 - \frac{16}{9}} = \frac{\frac{8}{3}}{\frac{-7}{9}} \div$$
$$= \frac{8}{\cancel{3}} \times \frac{-\cancel{9}^{\,3}}{7} = -\frac{24}{7}$$

Example 6.3.3

From your text: Pg. 408 #12

→ using only sine in the end.

Use the appropriate angle and double angle formulae to determine a formula for:

a) $\sin(3\theta)$ $= \sin(\overset{A}{2\theta} + \overset{B}{\theta})$

$\sin(A+B) =$
$\sin A\cos B + \sin B\cos A$

$\cos^2\theta = 1 - \sin^2\theta$

$$= \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + \sin\theta(1 - 2\sin^2\theta)$$
$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

Example 6.3.4

From your text: Pg. 407 #8

Determine the value of a in the following:

$$2\tan(x)-\tan(2x)+2a=1-\tan(2x)\cdot\tan^2(x)$$

$$2a = 1 - \tan(2x)\tan^2 x - 2\tan(x) + \tan(2x)$$

$$2a = \tan(2x) - \tan(2x)\tan^2(x) + 1 - 2\tan(x)$$

$$2a = \tan(2x)(1-\tan^2(x)) + 1 - 2\tan(x)$$

$$2a = \left(\frac{\cancel{2\tan(x)}}{\cancel{1-\tan^2(x)}}\right)\underbrace{\cancel{(1-\tan^2(x))}}_{1} + 1 \cancel{- 2\tan(x)}$$

$$2a = 1$$

$$a = \frac{1}{2}$$

Class/Homework for Section 6.3

Pg. 407 – 408 Finish #2, 4, 12 – Do # 6, 7

6.4 Trigonometric Identities

Pure Mathematical Joy.

Proving Trigonometric Identities is so much fun, it's plainly ridiculous. I should be paid extra for letting you play with these proofs! We will be using **ALGEBRA** (remember the rules?). Inside our algebra we will be using the following tools:

Reciprocal Identities

e.g. $\csc(\theta) = \dfrac{1}{\sin(\theta)}$ $\qquad \sin\theta = \dfrac{1}{\csc\theta}$

Quotient Identities

e.g. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, or $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

The Pythagorean Trig Identities

$\sin^2(\theta) + \cos^2(\theta) = 1$ $\qquad 1 + \tan^2(\theta) = \sec^2(\theta)$ $\qquad 1 + \cot^2(\theta) = \csc^2(\theta)$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$ $\qquad \Rightarrow \tan^2\theta = \sec^2\theta - 1$ $\qquad \Rightarrow \cot^2\theta = \csc^2\theta - 1$

or $\cos^2\theta = 1 - \sin^2\theta$ $\qquad$ or $1 = \sec^2\theta - \tan^2\theta$ $\qquad$ or $1 = \csc^2\theta - \cot^2\theta$

The Compound Angle Formulae

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

The Double Angle Formulae

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \begin{cases} \cos(2\theta) = 1 - 2\sin^2\theta \\ \cos(2\theta) = 2\cos^2\theta - 1 \end{cases}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

(And let's not forget our friends, "The Trig Equivalencies" such as the Cofunction Identities!)

A General Rule of Thumb

~~Covert~~ Convert everything to sine and cosine

Example 6.4.1

Prove $1+\tan^2(x) = \sec^2(x)$

L.S. $1 + \tan^2(\theta)$

$= \frac{1}{1} + \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{1}{\cos^2\theta}$

$= \sec^2\theta$ $\square$

$\therefore$ L.S = R.S.

Example 6.4.2

Prove $\sin(x+y)\cdot\sin(x-y)=\cos^2(y)-\cos^2(x)$

$(a+b)(a-b)$
$=a^2-b^2$

L.S. $\sin(x+y)\cdot\sin(x-y)$

$=(\sin x\cos y+\sin y\cos x)(\sin x\cos y-\sin y\cos x)$

$\sin^2\theta = 1-\cos^2\theta$

$=\sin^2 x\cos^2 y-\sin^2 y\cos^2 x$

$=(1-\cos^2 x)\cos^2 y-(1-\cos^2 y)\cos^2 x$ careful with the negative.

$=\cos^2 y-\cancel{\cos^2 x\cos^2 y}-\cos^2 x+\cancel{\cos^2 x\cos^2 y}$

$=\cos^2 y-\cos^2 x$

$\square$ $\therefore$ L.S. = R.S.

Example 6.4.3

Prove $\sin(\theta)\cdot\tan(\theta)=\sec(\theta)-\cos(\theta)$

R.S. $\sec\theta-\cos\theta$

$=\dfrac{1}{\cos\theta}-\dfrac{\cos\theta}{1}$

$=\dfrac{1}{\cos\theta}-\dfrac{\cos^2\theta}{\cos\theta}$

$=\dfrac{\sin^2\theta}{\cos\theta}$

$=\dfrac{\sin\theta\,\sin\theta}{\cos\theta}$ $=\tan\theta$ $=\sin\theta\tan\theta$

$\square$

$\therefore$ L.S. = R.S.

Example 6.4.4

Prove $\tan(x)\cdot\tan(y)=\dfrac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}$

$$\frac{(3)\,1}{(3)\,2}+\frac{1(2)}{3(2)}$$

$$=\frac{3}{(3)(2)}+\frac{2}{(3)(2)}=\frac{5}{6}$$

$$\frac{1}{x}+\frac{1}{y}$$

$$=\frac{y+x}{xy}$$

R.S. $\dfrac{\tan x+\tan y}{\cot x+\cot y}$

$$=\frac{\tan x+\tan y}{\frac{1}{\tan x}+\frac{1}{\tan y}}$$

$$=\frac{\tan x+\tan y}{\underbrace{\frac{\tan y+\tan x}{(\tan x)(\tan y)}}_{\text{flipped}}}\div$$

$$=\frac{\cancel{\tan x+\tan y}}{1}\times\frac{(\tan x)(\tan y)}{\cancel{\tan y+\tan x}}$$

$$=(\tan x)(\tan y)\ \square$$

$\therefore$ L.S. = R.S

Example 6.4.5

From your text: Pg. 417 #9a

Prove $\dfrac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin(\theta)\cos(\theta)}=1-\tan(\theta)$

$(a-b)(a+b)$

$= a^2-b^2$

L.S. $\dfrac{\boxed{\cos^2\theta-\sin^2\theta}}{\cos^2\theta+\sin\theta\cos\theta}$ difference of squares

FACTOR

x^2+xy

$=x(x+y)$

$$=\frac{\cancel{(\cos\theta+\sin\theta)}(\cos\theta-\sin\theta)}{\cos\theta\cancel{(\cos\theta+\sin\theta)}}$$

$$=\frac{\cos\theta-\sin\theta}{\cos\theta}$$

$$\frac{2x^2-8x}{2x}$$

$$=\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=x-4$$

$$=1-\tan\theta$$

Class/Homework for Section 6.4

Pg. 417 – 418 #8 – 11

6.5 Linear Trigonometric Equations

By this time, asking you to solve a "linear equation" is almost an insult to your intelligence. BUT it is never an insult to ask you to solve problems with math. Instead it is a special treat to be able to spend time thinking mathematically. And so, **you're very welcome**.

e.g. Solve the linear equation

$$3x - 4 = 9$$

$+4 \quad +4$

$$3x = 13$$

$$\frac{3x}{3} = \frac{13}{3}$$

$$x = \frac{13}{3}$$

→ you undo the operation acting on the "x"

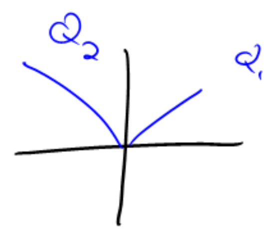
inverse operation

Example 6.5.1

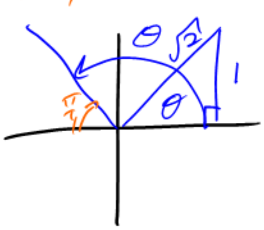
From your text: Pg. 427 #6

For $\theta \in [0, 2\pi]$, solve the linear trigonometric equation

a) $\sin(\theta) = \dfrac{1}{\sqrt{2}}$ exactly, and using a calculator.



Exactly:



$$\theta_1 = \frac{\pi}{4}$$

$$\theta_2 = \frac{3\pi}{4}$$

$$\sin(\theta) = \frac{1}{\sqrt{2}}$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

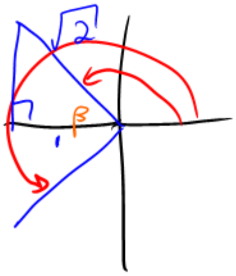
$$\theta_1 = 0.7854 = \beta$$

$$\theta_2 = \pi - \beta$$

$$= 3.14 - 0.79$$

$$= 2.35$$

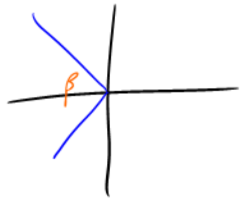
e) $\cos(\theta) = -\frac{1}{\sqrt{2}}$ exactly and using a calculator.



$\beta = \frac{\pi}{4}$

$\therefore \theta_1 = \frac{3\pi}{4}$

$\theta_2 = \frac{5\pi}{4}$



$\cos\beta = \frac{1}{\sqrt{2}}$

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\beta = 0.79$

$\theta_1 = 3.14 - 0.79 = 2.35$

$\theta_2 = 3.14 + 0.79 = 3.93$

Example 6.5.2

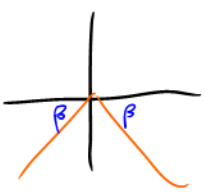
From your text: Pg. 427 #7

Using a calculator, determine solutions for $0^\circ \le \theta \le 360^\circ$

a) $2\sin(\theta) = -1$

Note: Our Domain is in Degrees!!

$\sin(\theta) = -\frac{1}{2}$



$\sin\beta = \frac{1}{2}$

$\beta = \sin^{-1}\left(\frac{1}{2}\right)$

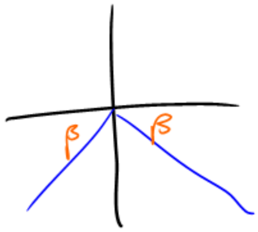
$\beta = 30^\circ$

$\therefore \theta_1 = 180 + 30 = 210^\circ$

$\theta_2 = 360 - 30 = 330^\circ$

$\therefore \theta = 210^\circ$ and 330°

d) $-3\sin(\theta)\overset{+1}{-1}=\overset{+1}{1}$ (correct to one decimal place)



$$\frac{-3\sin\theta}{-3}=\frac{2}{-3}$$

$$\sin\theta=\frac{-2}{3}$$

$$\sin\beta=\frac{2}{3}$$

$$\beta=\sin^{-1}\left(\frac{2}{3}\right)$$

$$\beta=41.8^\circ$$

$$\theta=180+41.8$$
$$=221.8^\circ$$
$$\theta=360-41.8$$
$$=318.2^\circ$$

Example 6.5.3

From your text: Pg. 427 #8

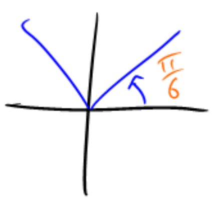
~~Using a calculator~~ determine solutions to the equations for $\boxed{0\le x\le 2\pi}$. radians

a) $3\sin(x)=\sin(x)+1$

$-\sin x$ $\quad$ $-\sin x$

$$2\sin(x)=1$$

$$\sin(x)=\frac{1}{2}$$



$$x=\frac{\pi}{6},\ \frac{5\pi}{6}$$

Example 6.5.4

From your text: Pg. 427 #9f

Solve for $x \in [0, 2\pi]$

$$8 + 4\cot(x) = 10$$

$-8 \qquad -8$

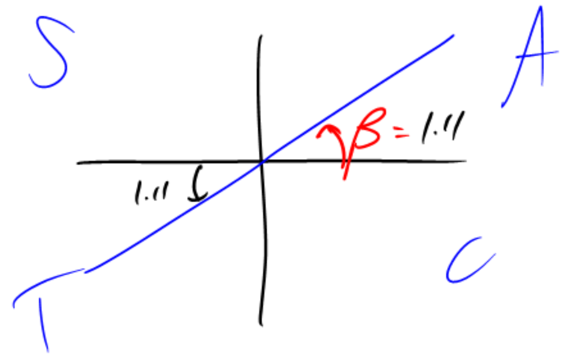
$$\frac{4\cot(x)}{4} = \frac{2}{4}$$

flip $\cot(x) = \frac{1}{2}$ flip

$$\tan(x) = 2$$

$$\beta = \tan^{-1}(2)$$

$$\beta = 1.11$$



$$\therefore x = 1.11$$

and

$$x = 3.14 + 1.11$$

$$x = 4.25$$

Class/Homework for Section 6.5

Pg. 427 – 428 #6, 7def, 8, 9abc

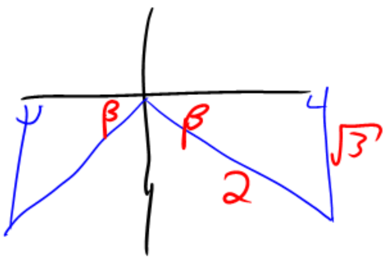
6.6 Quadratic Trigonometric Equations

Before moving on to Quadratic Trigonometric Equations, we need to consider a mind stretching problem, because it's good stretch from time to time (*in Baseball parlance, this would be the Lesson 6 Stretch*).

Example 6.6.1

Don't be afraid of the **3**! (though it does give one some concern…)

Solve $\sin(3x) = -\dfrac{\sqrt{3}}{2}$ exactly on $x \in [0, 2\pi]$



$\sin\beta = \dfrac{\sqrt{3}}{2}$

$\beta = \dfrac{\pi}{3}$

$\theta = \pi + \dfrac{\pi}{3}$ | $\theta = 2\pi - \dfrac{\pi}{3}$

$\theta = \dfrac{4\pi}{3}$ | $\theta = \dfrac{5\pi}{3}$

$f(x) = \sin(3x)$

Period: $\dfrac{2\pi}{3}$

→ 2 answers for every period.

∴ we will have 6!! solutions

Now we need the 6 answers:

$3x = \dfrac{4\pi}{3}$ | $3x = \dfrac{5\pi}{3}$

$x = \dfrac{4\pi}{9}$ (1) | $x = \dfrac{5\pi}{9}$ (2) These are in 0 to $\dfrac{2\pi}{3}$ (First period)

Add the period

$x = \dfrac{4\pi}{9} + \dfrac{2\pi}{3}$ | $x = \dfrac{5\pi}{9} + \dfrac{2\pi}{3}$

$x = \dfrac{10\pi}{9}$ (3) | $x = \dfrac{11\pi}{9}$ (4)

Add the period again!

$x = \dfrac{10\pi}{9} + \dfrac{2\pi}{3}$ | $x = \dfrac{11\pi}{9} + \dfrac{2\pi}{3}$

$x = \dfrac{16\pi}{9}$ (5) | $x = \dfrac{17\pi}{9}$ (6)

$$\therefore x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$$

In Quadratic Trigonometric Functions the highest power on the trig ‘factor’ will be 2.

Example 6.6.2

From your text: Pg. 436 #4: Solve, to the nearest degree, for $0^\circ \le \theta \le 360^\circ$

b) $\cos^2(\theta) = 1$

Axis Angles

$\cos\theta = \pm 1$

$\cos\theta = 1$

$\theta = 0, 2\pi$

$0^\circ, 360^\circ$

$\cos\theta = -1$

$\theta = \pi$

$= 180^\circ$

$\therefore \theta = 0, \pi, 2\pi$

$\theta = 0^\circ, 180^\circ, 360^\circ$

f) $2\sin^2(\theta) = 1$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\sin\theta = -\frac{1}{\sqrt{2}}$

$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Example 6.6.3

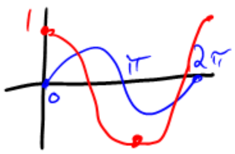
From your text: Pg. 436 #5: Solve for $0^\circ \le x \le 360^\circ$

b) $\underbrace{\boxed{\sin(x)}}_{=0}\underbrace{(\cos(x)-1)}_{=0}=0$

$\overset{=0}{x}\,\overset{=0}{(\cancel{x}-1)}=0$

$x=0,\ x=1$

$\sin x = 0$



$x = 0, \pi, 2\pi$

$0^\circ, 180^\circ, 360^\circ$

$\cos x - 1 = 0$

$\cos x = 1$

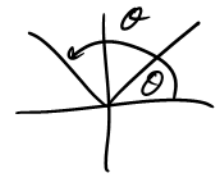
$x = 0, 2\pi$

$0^\circ, 360^\circ$

$\therefore\ x = 0, \pi, 2\pi$ or $0^\circ, 180^\circ, 360^\circ$

d) $\boxed{\cos(x)}\overset{=0}{\underbrace{\left(2\sin(x)-\sqrt{3}\right)}_{=0}}=0$

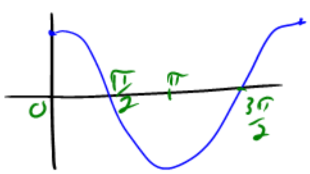
$2\sin(x) - \sqrt{3} = 0$



$\sin(x) = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

$\cos x = 0$



$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$= 90^\circ, 270^\circ$

$\therefore\ x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$

$= 60^\circ, 90^\circ, 120^\circ, 270^\circ$

Example 6.6.4

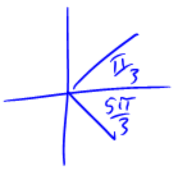
From your text: Pg. 436 #6: Solve for $0 \le x \le 2\pi$

d) $\underbrace{\left(2\cos(x)-1\right)}_{=0}\underbrace{\left(2\sin(x)+\sqrt{3}\right)}_{=0}=0$

$2\cos(x)-1=0$

$\cos(x)=\frac{1}{2}$

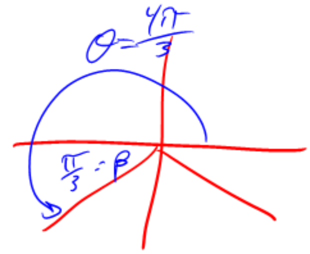
$x=\frac{\pi}{3}$ and $\frac{5\pi}{3}$



$2\sin(x)+\sqrt{3}=0$

$\sin(x)=\frac{-\sqrt{3}}{2}$

$x=\frac{4\pi}{3}$ and $\frac{5\pi}{3}$



$\therefore x=\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Example 6.6.5

From your text: Pg. 436 #7: Solve for $0 \le \theta \le 2\pi$ to the nearest hundredth (if necessary).

a) $2\cos^2(\theta)+\cos(\theta)-1=0$

Let $x=\cos\theta$

$2x^2+x-1=0$

$(2x-1)(x+1)=0$

$\underbrace{(2\cos\theta-1)}_{=0}\underbrace{(\cos\theta+1)}_{=0}=0$

$\cos\theta=\frac{1}{2}$

$\theta=\frac{\pi}{3}, \frac{5\pi}{3}$

$\cos\theta=-1$

$\theta=\pi$

$\therefore \theta=\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

e) $3\tan^2(\theta)-2\tan(\theta)=1$

Let $x=\tan\theta$

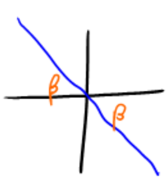
$3x^2-2x-1=0$

$(3x+1)(x-1)=0$

$3\tan^2\theta-2\tan\theta-1=0$

$\underbrace{(3\tan\theta+1)}_{=0}\underbrace{(\tan\theta-1)}_{=0}=0$

$\tan\theta=\frac{-1}{3}$



$\tan\beta=\frac{1}{3}$

$\beta=\tan^{-1}\left(\frac{1}{3}\right)$

$\beta=0.32$

$\theta=\pi-0.32=2.82$

$\theta=2\pi-0.32=5.96$

$\tan\theta=1$

$\theta=\frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore \theta=\frac{\pi}{4}, 2.82, \frac{5\pi}{4}, 5.96$

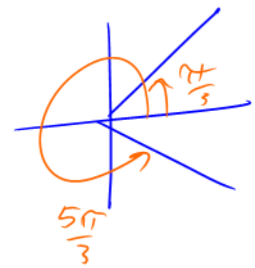
Example 6.6.6 (decimals are between the sixes!)

From your text: Pg. 436 #8: Solve for $x \in [0, 2\pi]$

a) $\sec(x) \cdot \csc(x) - 2\csc(x) = 0$

Factor $\csc(x)$!

$\underbrace{\csc(x)}_{=0}\underbrace{(\sec(x) - 2)}_{=0} = 0$



flip $\csc(x) = \frac{0}{1}$ flip

$\sin(x) = \frac{1}{0}$

$\therefore$ no solution

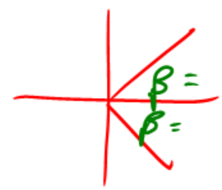
flip $\sec(x) = \frac{2}{1}$ flip

$\cos(x) = \frac{1}{2}$

$\therefore x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

c) $2\sin(x) \cdot \sec(x) - 2\sqrt{3}\sin(x) = 0$

$\underbrace{2\sin(x)}_{=0}\underbrace{(\sec(x) - \sqrt{3})}_{=0} = 0$



$2\sin(x) = 0$

$\sin(x) = 0$

$\therefore x = 0, \pi, 2\pi$

flip $\sec(x) = \sqrt{3}$ flip

$\cos(x) = \frac{1}{\sqrt{3}}$ NOT SPECIAL!

$\cos\beta = \frac{1}{\sqrt{3}}$

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.96$ ✓ | $2\pi - 0.96 = 5.32$

$\therefore x = 0.96, 5.32$

$\therefore x = 0, 0.96, \pi, 5.32, 2\pi$

$$\cos(2x) = 2\cos^2 x - 1$$

Example 6.6.7

From your text: Pg. 437 #9: Solve for $x \in [0, 2\pi]$. Round to two decimal places.

a) $5\cos(2x) - \cos(x) + 3 = 0$

$$5(2\cos^2 x - 1) - \cos x + 3 = 0$$

$$10\cos^2 x - 5 - \cos x + 3 = 0$$

$$10\cos^2 x - \cos x - 2 = 0$$

$$\underbrace{(5\cos x + 2)}_{}{}^{=0}\underbrace{(2\cos x - 1)}_{}{}^{=0} = 0$$

Let $x = \cos x$

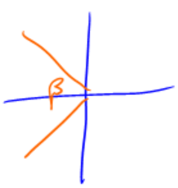
$$10x^2 - x - 2 = 0$$

$$\underbrace{10x^2 - 5x}_{5x}\ \underbrace{+4x - 2}_{+2} = 0$$

$$(5x + 2)(2x - 1) = 0$$

$$5\cos x + 2 = 0$$

$$\cos x = \frac{-2}{5}$$



$$\cos\beta = \frac{2}{5}$$

$$\beta = \cos^{-1}\left(\frac{2}{5}\right) = 1.16$$

$$x = \pi - 1.16 = 1.98$$

$$x = \pi + 1.16 = 4.3$$

$$2\cos x - 1 = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore, $x = \frac{\pi}{3}, 1.98, 4.3, \frac{5\pi}{3}$

Class/Homework for Section 6.6

Pg. 436 – 437 #4ade, 5acef, 6ac, 7 – 9