# Advanced Functions

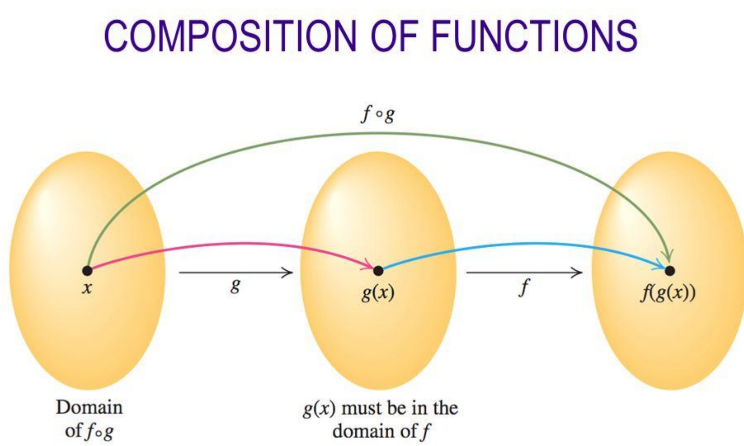
*Fall 2017*

*Course Notes*

# Unit 8 – Combinations of Functions

*We will learn*

- *how to use basic arithmetic to construct new functions from given functions*
- *how to describe the composition of two functions numerically, graphically and algebraically*
- *key characteristics of the newly created functions*

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# Chapter 8 – Combinations of Functions

*Contents with suggested problems from the Nelson Textbook (Chapter 9)*

**8.1 Sums and Differences of Functions**

Pg. 451 #1c, 4, Finish 5 – 7, 9, 11

**8.2 Product and Quotient Combinations**

Pg. 537 - 539 #1bd,3,8bd,10,15

Pg. 542 # 1aef, 2 (for #1aef)

**8.3 Composition of Functions**

READ Ex. 4 Pg550 making sure you understand the concept of a function composed with its inverse.

Pg. 552 – 553 #1abf, 2bdf, 5bcdf (use tech to graph), 6bcd, 7bcdef, 13

# 8.1 Sums and Differences of Functions

**Definition 8.1.1**

Given two functions $f(x)$, and $g(x)$ with domains $D_f$ and $D_g$ respectively, then we can **construct** new functions:

$$F(x) = (f+g)(x) \qquad\qquad G(x) = (f-g)(x)$$

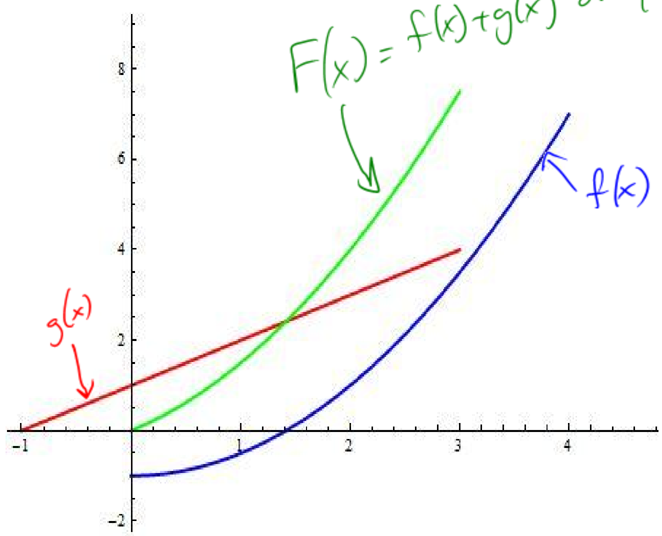
where the **meaning** of the notation for the "sum" and "difference" functions is as follows:

$F(x) = f(x) + g(x)$ $\qquad$ $G(x) = f(x) - g(x)$

**Example 8.1.1**

Consider the sketch (so that we can get at the domain of sum and difference functions):

$F(x) = f(x) + g(x)$ or $(f+g)(x)$




$D_f = [0, 4]$

$D_g = [-1, 3]$

The domain of $F(x)$ can only contain the elements common to $D_f$ and $D_g$

$D_F = [0, 3]$

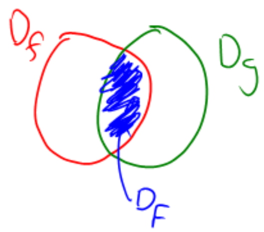
$F(-1) = f(-1) + g(-1)$

Doesn't exist + 0

In general, given functions $f(x)$ and $g(x)$ with domains $D_f$ and $D_g$, then the combined function

$$F(x) = (f \pm g)(x)$$

has the domain $D_F = D_f \cap D_g$ → intersection. ↳ what they have in common




**Example 8.1.2**

Determine the domain of $F(x) = (f - g)(x)$ for $f(x) = \sqrt{x}$, and $g(x) = \log\left(\underbrace{-(x-2)}_{>0}\right)$.

$D_f = [0, \infty)$

$D_g = (-\infty, 2)$

$\therefore D_F = [0, 2)$

$\frac{-(x-2)}{-1} > \frac{0}{-1}$

$x - 2 < 0$

$x < 2$

**Example 8.1.3**

Given $f(x) = x^3 - 4x + 1$, $D_f$ =[-4,5] and $g(x) = 2x^2 - 1$, $D_g = (0,6]$,

Determine: a) the order of $F(x) = (g - f)(x)$

b) $D_F$

c) an algebraic representation for $F(x)$

a) The order of $F(x)$ is the largest exponent, $\therefore$ 3.

b) $D_F = (0, 5]$

c) $F(x) = \underbrace{2x^2 - 1}_{g(x)} - \underbrace{(x^3 - 4x + 1)}_{f(x)}$

$F(x) = 2x^2 - 1 - x^3 + 4x - 1$

$F(x) = -x^3 + 2x^2 + 4x - 2$

**Example 8.1.4**

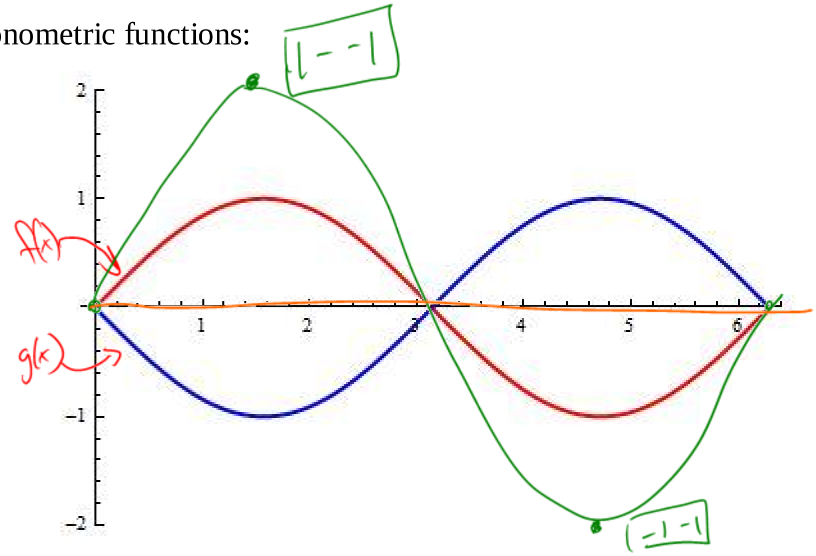
Consider the sketches of the trigonometric functions:

$f(x) = \sin(x),\ D_f = [0, 2\pi]$

$g(x) = \cos\left(x + \frac{\pi}{2}\right),\ \mathrm{D}_g = [0, 2\pi]$

Sketch a) $F(x) = (f - g)(x)$

b) $G(x) = (f + g)(x)$




**Example 8.1.5**

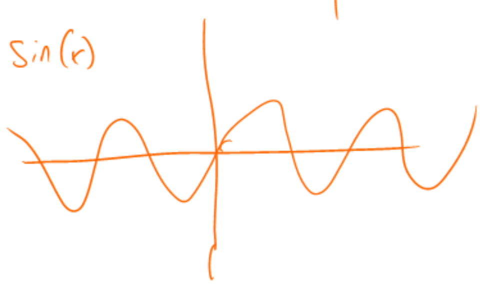
Consider the functions $f(x) = x,\ \mathrm{g}(x) = \sin(x),\ x \in \mathbb{R}$.

What does $F(x) = (f + g)(x)$ look like? (see Ex. 3 Pg. 526)

$f(x) = x$

$F(x) = X + \sin(x)$

$g(x) = \sin(x)$



**Example 8.1.6**

From your text: Pg. 528 #1ae

Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$.

Determine: a) $(f+g)(x)$ e) $(f+f)(x)$

$= f(x) + g(x)$

a) $(f+g)(x) = \{(-4, 6), (-2, 5), (1, 5), (4, 10)\}$

b) $(f+f)(x) = \{(-4, 8), (-2, 8), (1, 6), (3, 10), (4, 12)\}$

**Example 8.1.7**

From your text: Pg. 529 #7ab

Given $f(x) = \dfrac{1}{3x-4}$ and $g(x) = \dfrac{1}{x-2}$, determine $(f+g)(x)$ and $D_{(f+g)}$

$(f+g)(x) = f(x) + g(x)$

$= \dfrac{1(x-2)}{3x-4} + \dfrac{1(3x-4)}{x-2}$

$= \dfrac{x-2+3x-4}{(3x-4)(x-2)} = \dfrac{4x-6}{(3x-4)(x-2)}$

$D_f: x \neq \frac{4}{3}, \ (-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$

$D_g: x \neq 2, \ (-\infty, 2) \cup (2, \infty)$

$D_{f+g}: (-\infty, \frac{4}{3}) \cup (\frac{4}{3}, 2) \cup (2, \infty)$

*Class/Homework for Section 8.1*

*Pg. 528 - 530 #1ce, 2, 3, 6ad, 10 (recall even/odd fns), 11 (see ex3 pg. 526), 12, 9ac (not symmetry)*

# 8.2 Product and Quotient Combinations

**Definition 8.2.1**

Given two functions $f(x)$, and $g(x)$ with domains $D_f$ and $D_g$ respectively, then we can **construct** new functions:

$F(x) = (f \cdot g)(x)$

$= f(x) \cdot g(x)$

$D_{f \cdot g} = D_f \cap D_g$

takes the common elements

$G(x) = \left(\frac{f}{g}\right)(x)$

$= \frac{f(x)}{g(x)}$

$D_{\frac{f}{g}} = D_f \cap D_g$ , $g(x) \neq 0$

**Example 8.2.1**

Determine $(f \cdot g)(x)$ given $f(x) = \{(-2,3),(-1,5),(0,3),(1,-3),(2,-5)\}$ and $g(x) = \{(-1,4),(0,-2),(1,7),(2,-2),(3,2)\}$

$(f \cdot g)(x) = \{(-1,20),(0,-6),(1,-21),(2,10)\}$

**Example 8.2.2**

From your text: Pg. 537 #2 for #1e

Sketch the given pair of functions on the same set of axes. State their domains. Use your sketch to draw $(f \cdot g)(x)$. State $(f \cdot g)(x)$ and $D_{f \cdot g}$.

$f(x) = x + 2$, $g(x) = x^2 - 2x + 1$

$= (x-1)(x-1)$

$= (x-1)^2$

$a(x-h)^2 + k$

$(f \cdot g)(x) = (x+2)(x^2 - 2x + 1)$

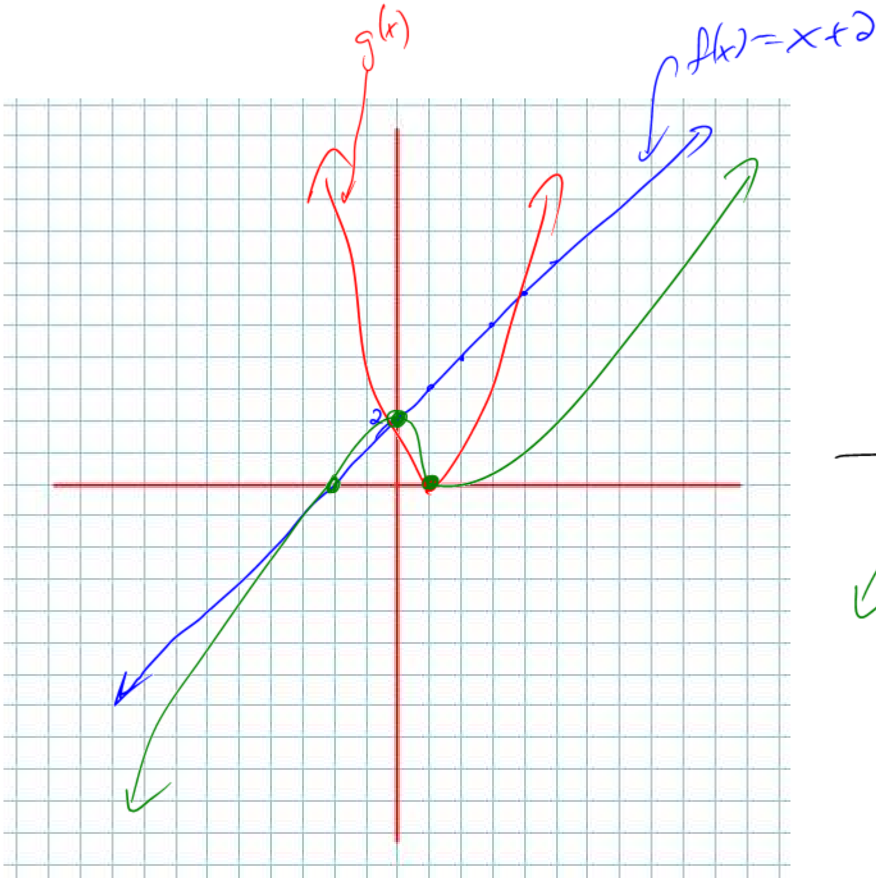
$= (x+2)(x-1)^2$

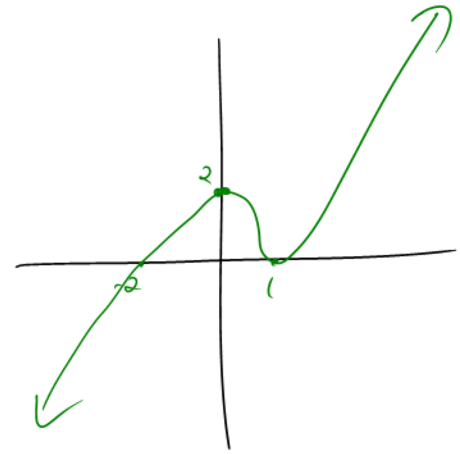
→ degree/order is 3, odd degree.

→ Leading coefficient: 1 ∴ positive

→ zeros: $x = -2$ and $\boxed{x = 1}$ order 2

→ y-int: $(0, 2)$



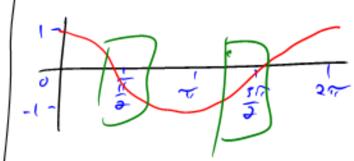

**Example 8.2.3**

Determine the domain of $D_{f\cdot g}$ and $D_{f\div g}$ given $f(x)=\sqrt{\underbrace{2x+3}_{=0}}$, and $g(x)=\sec(x)=\dfrac{1}{\cos(x)}$

$x = -\frac{3}{2}$

$D_f: \left[-\frac{3}{2}, \infty\right)$

$D_g: \{x \in \mathbb{R} \mid x \neq \ldots \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots\}$




$D_{f\cdot g}: \{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots\}$

$D_{\frac{f}{g}}: \{x \in \mathbb{R} \mid x \geq -\frac{3}{2}, x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots\}$

Note: sec(x) never equals zero! $\therefore D_{f\cdot g} = D_{\frac{f}{g}}$

**Example 8.2.4**

From your text: Pg. 542 #2 for 1d.

Sketch the given pair of function on the same set of axes. State their domains. Use your sketch to draw $\left(\frac{f}{g}\right)(x)$. State $\left(\frac{f}{g}\right)(x)$ and $D_{\frac{f}{g}}$.

$$f(x) = x + 2 \text{ and } g(x) = \sqrt{x-2}$$

$D_f = (-\infty, \infty)$

$D_g = [2, \infty)$

$D_{\frac{f}{g}} = (2, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x+2}{\sqrt{x-2}}$

$x = 2$ is a V.A.

*Class/Homework for Section 8.2 – READ ex 4 Pg. 536*

*Pg. 537 - 539 #1bd,3,8bd,10,15*
*Pg. 542 # 1aef, 2 (for #1aef)*

# 8.3 Composition of Functions

In sections 8.1 and 8.2 we examined how to combine functions (constructing new functions) through the standard algebraic operations of addition, subtraction, multiplication and division. Here we will learn another method for combining functions, but **we won't be using a standard algebraic operation.**

The concept we define as **Composition of Functions** is **very useful for Calculus** (among other things) as some of you will see next semester.

The basic idea is that given two functions $f(x)$ and $g(x)$, we can define the composition of the two by

inserting one function into the other.

The "algebraic" notation may seem a little weird, but don't make fun. Math has feelings too.

**Definition 8.3.1**

Given two functions $f(x)$ and $g(x)$ we write the composition of $f(x)$ and $g(x)$ as

$$(f \circ g)(x)$$

We can also write the composition of $g(x)$ and $f(x)$ as

$$(g \circ f)(x)$$

## The "Algebraic Meaning" of Composition

$$(f \circ g)(x) = f(g(x))$$

composed with

outer fn

$$(g \circ f)(x) = g(f(x))$$

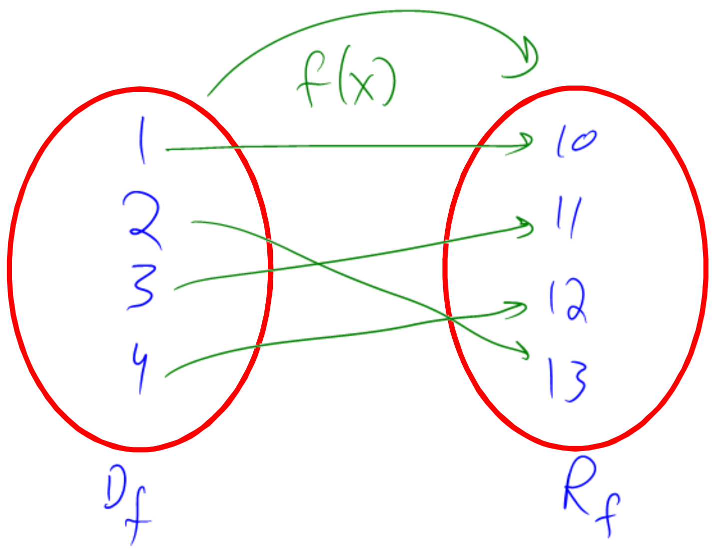
inner fn

Note: It is **very helpful** to keep in mind the distinction between the **inner** and **outer** functions

# The Domain of a Composition of Functions

Recall the basic "**machinery**" of any function: **"Plug a (domain) number into the function, and get a (range) number out."**

**Picture**

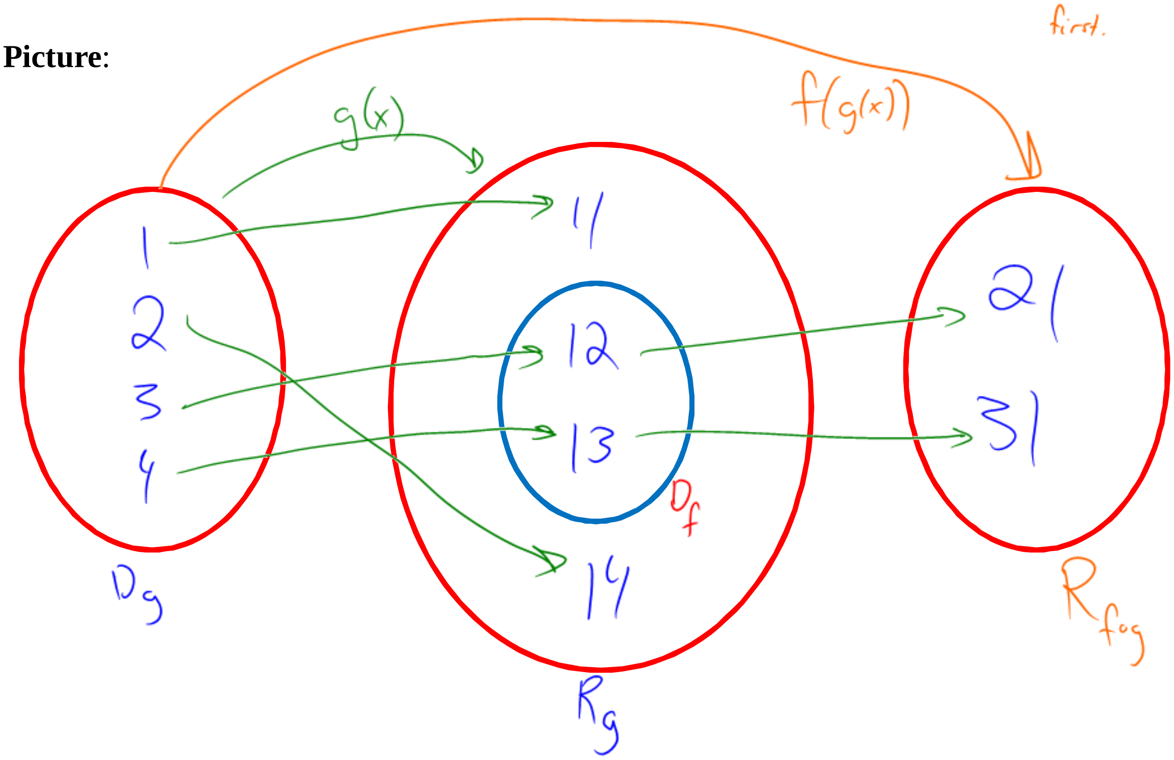



Recall further that many functions cannot claim "all real numbers" as their (natural) domain.

e.g. Determine the domain of $f(x) = \sqrt{x+1}$

Now we consider the "machinery" for the composition function $(f \circ g)(x) = f\left(\underbrace{g}_{\text{first.}}(x)\right)$

**Picture**:



## Algebraic Definition of the Domain of a Composition of Two Functions

Given two functions $f(x)$ and $g(x)$ with domains $D_f$ and $D_g$ respectively, then the domain of the composition of $f(x)$ and $g(x)$ is given by:

range of $g(x)$

$$D_{(f\circ g)} = \{x \in \mathbb{R} \mid x \in D_g \quad \text{such that } g(x) \in D_f \}$$

Using words we might write that the **domain of a composition of functions** $(f \circ g)(x) = f(g(x))$ is **the set of all *x* values** which belong to the domain of the

___inner___ function which have range values which are in the domain of the

___outer___ function.

**Example 8.3.1**

Given $f(x) = 3x+1$ and $g(x) = x^3 - 1$ determine:

a) $(f \circ g)(0)$

$= f(g(0))$

$= f(0^3 - 1)$

$= f(-1) = 3(-1)+1 = -2$

$\therefore (0,-2)$

b) $g(f(0))$

$g(3(0)+1)$

$g(1) = 1^3 - 1$

$g(1) = 0$

$\therefore (0,0)$

c) $(f \circ f)(1)$

$f(f(1))$

$= f(3(1)+1)$

$= f(4)$

$= 3(4)+1$

$= 13$

$\therefore (1,13)$

**Example 8.3.2**

From your text: Pg. 552 #2 ac"g"

Given $f(x) = \{(0,1),(1,2),(2,5),(3,10)\}$ and $g(x) = \{(2,0),(3,1),(4,2),(5,3),(6,4)\}$

determine: a) $(g \circ f)(2)$ c) $f(g(5))$ g) $g(f(3))$

a)

$g(f(2))$

$= g(5)$

$= 3$

$\therefore (2,3)$

c)

$= f(3)$

$= 10$

$\therefore (5,10)$

g)

$= g(10)$

Does not exist!

$x = 3 \notin D_{g\circ f}$

## Something Silly but Entirely Serious

Given $f(x) = 2x^2 - 1$ determine:

a) $f(2)$ b) $f(A)$ c) $f(\square)$ d) $f(\square + \triangle)$

$= 2(2)^2 - 1$
$= 7$

$= 2A^2 - 1$

$= 2\square^2 - 1$

$= 2(\square + \triangle)^2 - 1$

**Example 8.3.3**

From your text: Pg 552 #6ae

Given the functions $f(x)$ and $g(x)$ determine functional equations for $f(g(x))$ and $g(f(x))$ and determine their domains.

a) $f(x) = 3x$ and $g(x) = \sqrt{x-4}$

$f(g(x)) = f(\sqrt{x-4}) = 3\sqrt{x-4}$

$D_{f\circ g} = [4, \infty)$

$g(f(x)) = g(3x) = \sqrt{3x-4}$

$D_{g\circ f} = [\frac{4}{3}, \infty)$

~~Note: In General~~

e) $f(x) = 10^x$ and $g(x) = \log(x)$ $x > 0$

$f(g(x)) = f(\boxed{\log(x)}) = 10^{\log(x)} = x$

$x > 0$

$D_{f\circ g}: (0, \infty)$

$g(f(x)) = g(10^x) = \log(10^x) = x$

$D_g = (-\infty, \infty)$ $R_g = (0, \infty)$

Since $D_f = R_g$, $D_{g\circ f} = (-\infty, \infty)$

**Example 8.3.4**

From your text: Pg. 553 #7a

Given $h(x)$ find two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$.

a) $h(x) = \sqrt{x^2 + 6}$

$g(x) = x^2$
$f(x) = \sqrt{x+6}$

$g(x) = x^2 + 6$
$f(x) = \sqrt{x}$

$g(x) = x^2 + 3$
$f(x) = \sqrt{x+3}$

*Class/Homework for Section 8.3*

*READ Ex. 4 Pg550 making sure you understand the concept of a function composed with its inverse.*
*Pg. 552 – 553 #1abf, 2bdf, 5bcdf (use tech to graph), 6bcd, 7bcdef, 13*