Advanced Functions

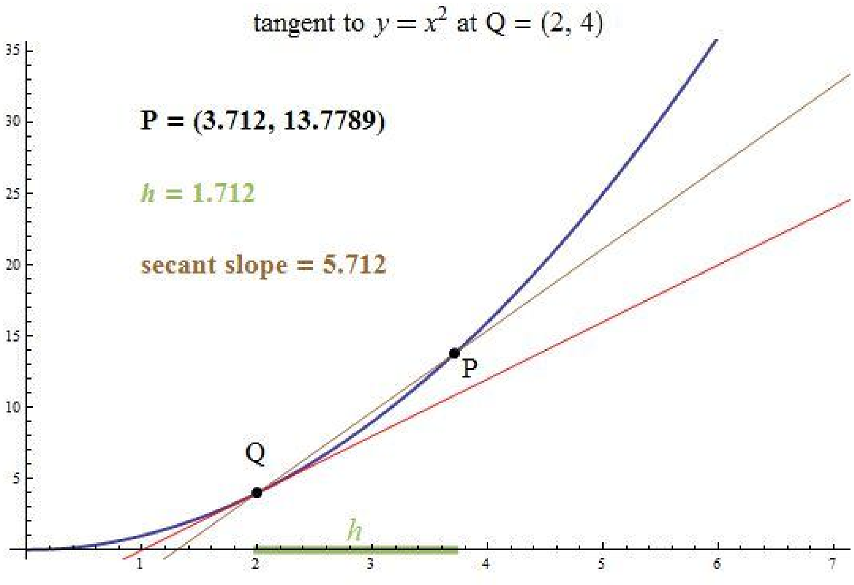
Fall 2017

Course Notes

Unit 9 – Rates of Change

We will learn

- *how to calculate an average rate of change of a function, given the function as a table of values, or a sketch, or an equation*
- *how to estimate the instantaneous rate of change of a function*
- *how to interpret the meaning of the average rate of change of a function over an interval of the function's domain*
- *how to interpret the meaning of the instantaneous rate of a change of a function at a single value of the domain.*
- *how to solve problems using rates of change*



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Chapter 9 – Rates of Change and the Tangent Problem

Contents with suggested problems from the Nelson Textbook (Chapter 2)

9.1 Average Rate of Change – The AROC

From Physics we learn that we can calculate the average velocity of some moving object through the formula

$$v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

In general we can calculate the **A**verage **R**ate **O**f **C**hange [AROC], for some given function $f(x)$, over an interval of time (the domain) $t \in [t_1, t_2]$ using the formula:

$$AROC = \frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Example 9.1.1

Consider the displacement function $s(t) = 100 - 4.9t^2$, which is being used to describe the displacement (s in m) of a falling body from the top of a 100m high cliff after t seconds.

Over the given time intervals determine the average rate of change (the AROC) of displacement for a stone dropped from the edge of the cliff:

a) $t = 0$ to $t = 1$ seconds. b) $t \in [1, 2]$ (seconds). c) $t \in [0, 3]$.

a) $AROC = \frac{s(1) - s(0)}{1 - 0}$

$= \frac{(100 - 4.9(1)^2) - (100 - 4.9(0)^2)}{1}$

$= 95.1 - 100$

$= -4.9 \, m/sec$

b) $AROC = \frac{s(2) - s(1)}{2 - 1}$

$= \frac{80.4 - 95.1}{1}$

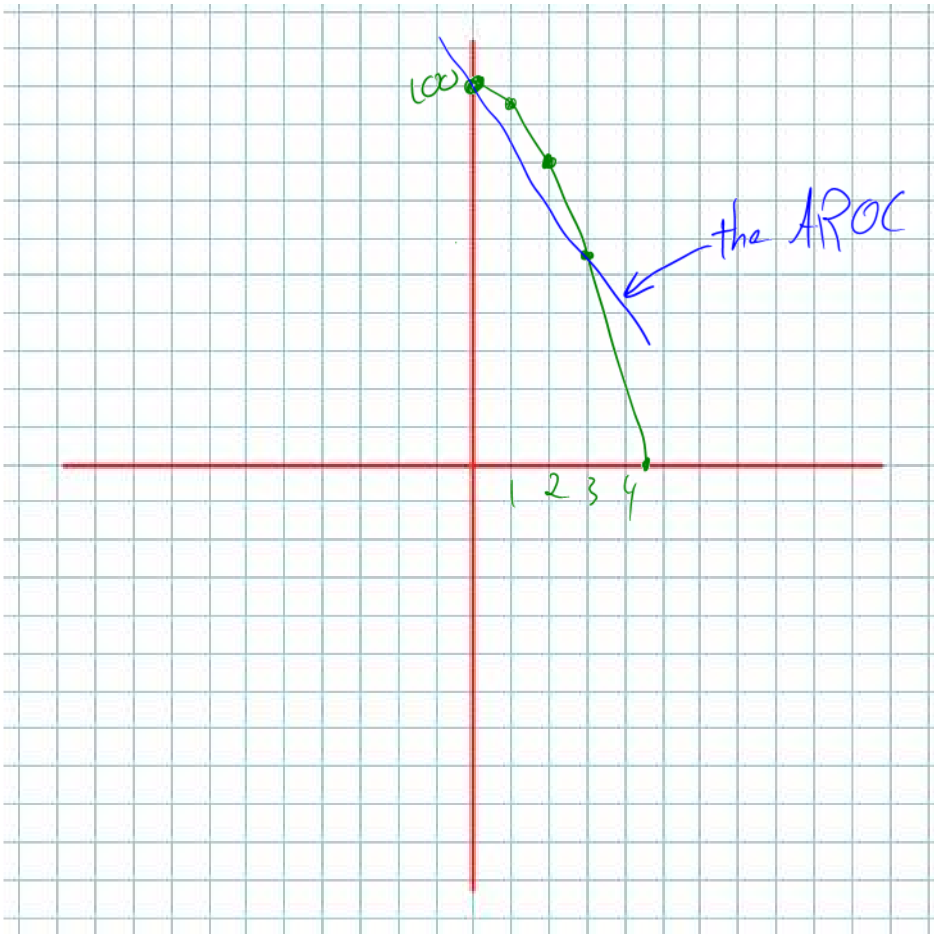
$= -14.7 \, m/s$

c) $AROC = \frac{s(3) - s(0)}{3 - 0}$

$= \frac{55.9 - 100}{3}$

$= -14.7 \, m/s$

A picture of the situation in example 9.1.1:



the AROC from 0 to 3.

This line is a secant of a curve.
A secant touches a curve at 2 points.

The Slope of a Secant is an AROC for some given fn over a closed interval.

Class/Homework for Section 9.1

Pg. 76 – 77 #1 (important question), 2, 4, 9, 10

9.2 Instantaneous Rate of Change – The IROC (*part 1*)

~~Last day~~ One minute ago, we learned that for some given function $f(x)$ we can **calculate** the AROC of that function (over some interval of the domain) as the **slope of a secant**.

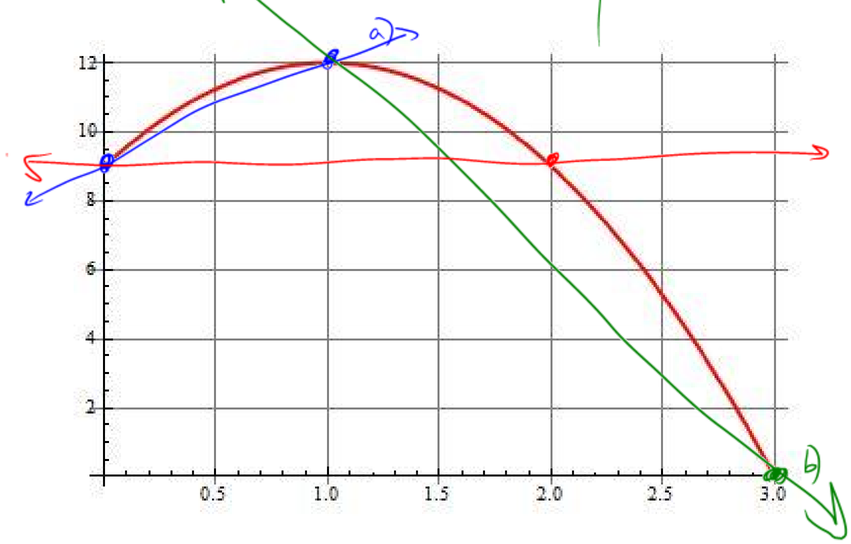
Example 9.2.1

Given the displacement function $s(t) = -3(t-1)^2 + 12$, determine the AROC of a water-balloon tossed from a 3rd floor balcony, over the (time) intervals:

a) $t \in [0,1]$ b) $t \in [1,3]$ c) $t \in [0,2]$

$\frac{12-9}{1-0} = 3$ $\frac{0-12}{3-1} = -6$ $\frac{9-9}{2-0} = \frac{0}{2} = 0!!$

A Picture

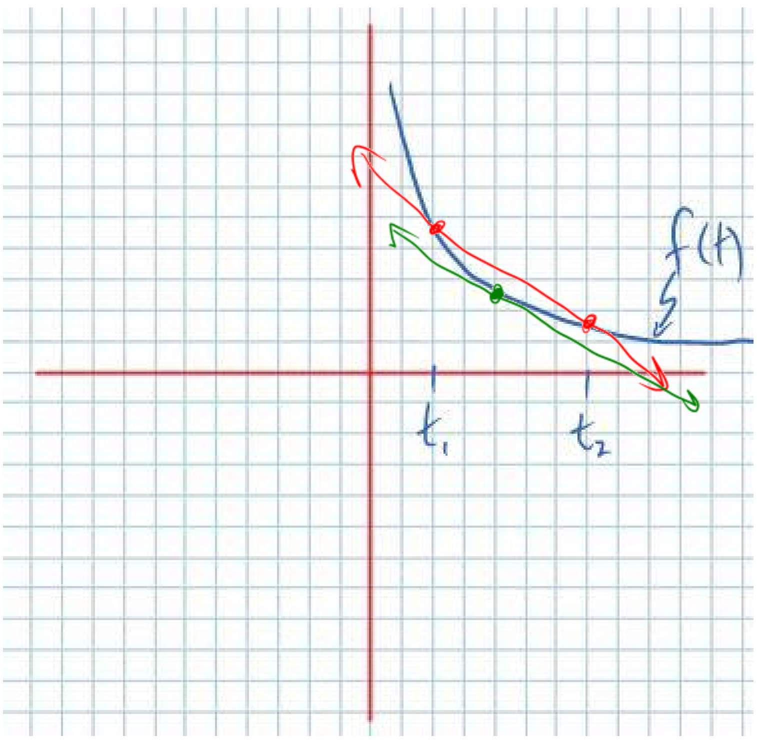


Note: Each AROC is a number which represents

Consider the question:

How can we calculate the velocity of the balloon at the **instant** $t = 2$ seconds?
Note: We CANNOT use the slope of a secant, since the secant requires two domain values, but an "instant" is at a single domain value $(t = 2 \text{ in this case})$.

Consider the following:



The **AROC** is the **slope** of a secant.

$$AROC = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Geometrically speaking, the **IROC** is also a **slope**, but it's the slope of a **TANGENT**

↳ a line which touches a curve at only one point

The Problem is this:

You cannot calculate the slope using only one point.

We will consider two techniques for **ESTIMATING** the so-called

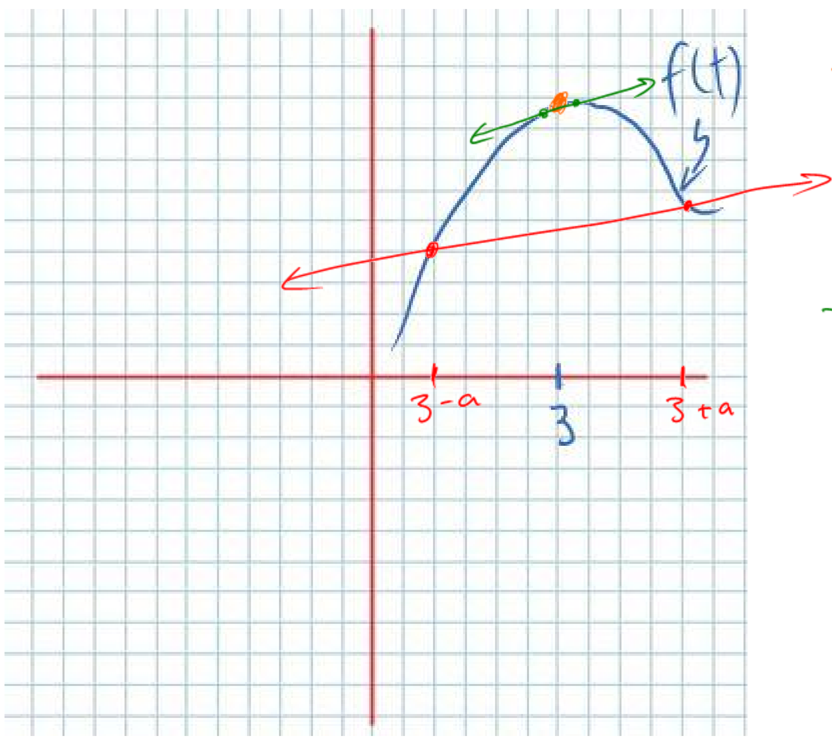
INSTANTANEOUS RATE OF CHANGE (IROC)

Note: To estimate the IROC (a number we CAN'T calculate) we will be calculating the slopes of secants because we can do that calculation!

1) Using a "**centered interval**" and **squeezing the interval** to get better and better **estimates**

A Geometric View

Consider the picture:



→ pick a value, call it "a"

→ squeeze the interval by picking smaller "a".

An Algebraic View

Example 9.2.2

Estimate the IROC for $s(t) = -2(t-1)^3 + 3$ at $t = 2$.

$[2-a, 2+a]$

Consider $a = 1$

$\therefore$ the AROC of $[1, 3]$

$$= \frac{s(3) - s(1)}{3 - 1} = \frac{-13 - 3}{2} = -8 \text{ units}$$

This "a" is too big!

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consider $a = 0.1$ The AROC of $[1.9, 2.1]$

$$= \frac{s(2.1) - s(1.9)}{2.1 - 1.9}$$

$$= -6.02$$

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consider $a = 0.01$ The AROC of $[1.99, 2.01]$

$$= \frac{s(2.01) - s(1.99)}{2.01 - 1.99}$$

$$= -6.002$$

$\therefore$ The IROC ~ -6 units a $t = 2$

approximately

Example 9.2.3

From your text: Pg. 87 #5

Using a centered interval approach, determine an estimate for the IROC at $x = 3\text{sec}$ of the height (in m) of an object, which is moving according to $h(x) = -5x^2 + 3x + 65$.

use $a = 0.01$, $[2.99, 3.01]$

$$IROC \sim AROC = \frac{h(3.01) - h(2.99)}{3.01 - 2.99}$$

$$= -27 \, m/s$$

Class/Homework for Section 9.2

Pg. 86 – 87 #4ac, 6, 8, 9, 10 (centered interval only)

9.3 Instantaneous Rate of Change – The IROC

(*part 2*)

The Difference Quotient

Suppose we wish to calculate the Instantaneous Rate of Change of some function, $f(x)$, at $x = 2$. Last day we saw three things:

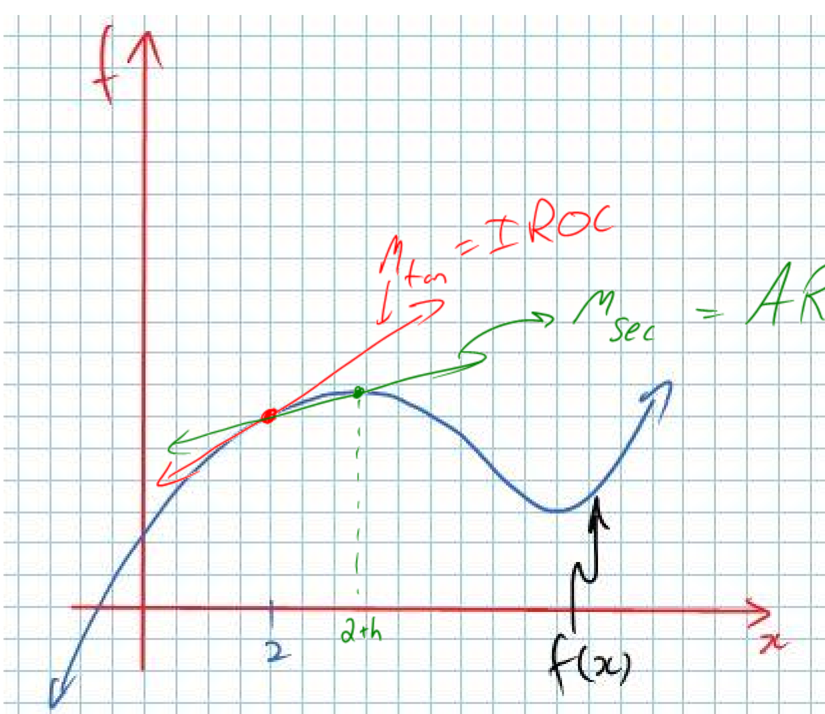
1) IROC = the slope of the tangent at $x = 2$

2) we cannot calculate the slope of a tangent using standard techniques

3) we can estimate the IROC using the AROC

Rather than using a "centered interval" approach, we now consider the so-called **Difference Quotient** (which can be much more useful than the centered interval approach).

Consider the sketch:



= AROC, by making h really small, we can approximate the IROC.

$$m_{sec} = \frac{f(2+h) - f(2)}{\cancel{2}+h - \cancel{2}} = \frac{f(2+h) - f(2)}{h} \approx m_{tan} = \text{IROC}$$

Now, we understand that

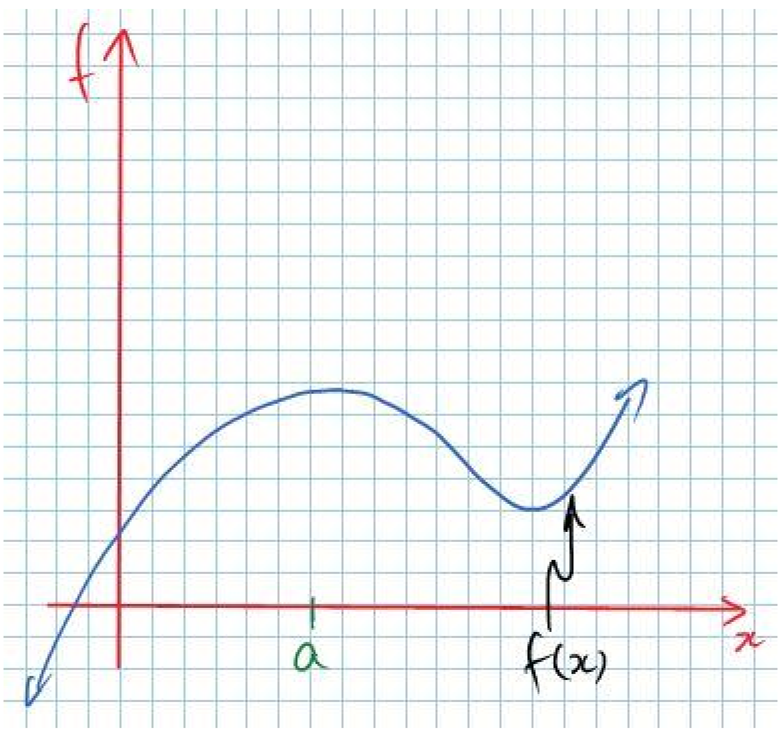
$$IROC \sim AROC = m_{\text{sec}} =$$

Definition of the Difference Quotient

In general, if we wish to approximate the IROC of $f(x)$ at some (general) domain value $x = a$, then

$$IROC \sim AROC = \frac{f(a+h) - f(a)}{h}, \text{ for small } h.$$

Note: "h" can be either positive or negative. Consider the sketch:



Example 9.3.1

Given $s(t) = 2t^2 - 3t - 5$, determine a difference quotient which will estimate the IROC of $s(t)$ ~~at $t = a$~~. Use that difference quotient to estimate the IROC at $t = 3$ using $h = 0.0001$.

IROC ~ AROC = $\dfrac{s(a+h) - s(a)}{h}$, for small h.

For $a = 3$ and $h = 0.0001$

$$\text{IROC} \sim \frac{s(3.0001) - s(3)}{0.0001}$$

$$\sim \frac{\left(2(3.0001)^2 - 3(3.0001) - 5\right) - \left(2(3)^2 - 3(3) - 5\right)}{0.0001}$$

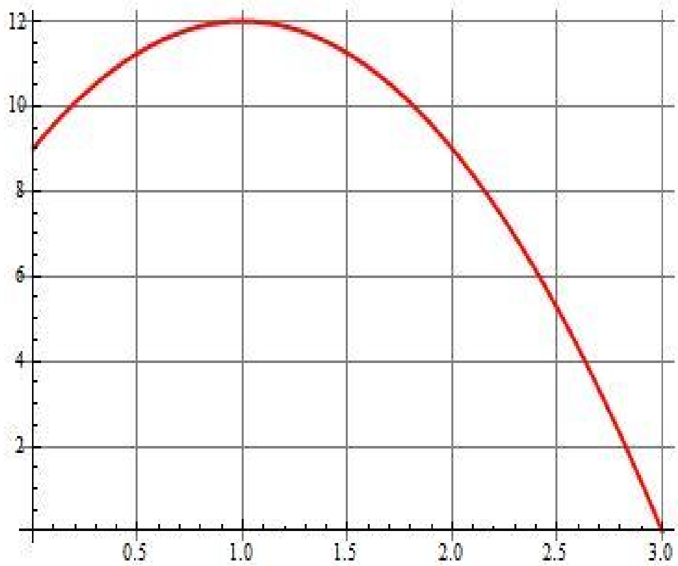
$$\sim \frac{4.00090002 - 4}{0.0001}$$

$$\sim 9.0002$$

$\therefore$ The IROC is 9 units at $t = 3$.

Example 9.3.2

Consider the water-balloon problem from Example 9.2.1. The water-balloon "flies" according to the function $s(t) = -3(t-1)^2 + 12$. Estimate the instantaneous velocity (the IROC) of the balloon when it hits the ground (at $t = 3$ sec).



Class/Homework

Determine an estimate for the IROC of the given function at the indicated domain value using a difference quotient. Use $h = 0.001$ for your estimation.

a) $f(x) = x^2 - 3x + 1$ at $x = 2$ *IROC* ~ 1

b) $h(t) = 2^t - 3$ at $t = 0$ *IROC* ~ 0.693

c) $g(x) = \sin(x)$ at $x = \pi$ *IROC* ~ -1

d) $s(t) = \dfrac{t+1}{t-2}$ at $t = 3$ *IROC* ~ -3

e) $g(x) = x^3 + 2$ at $x = 3$ *IROC* ~ 27