Chapter

5

Rational Functions, Equations, and Inequalities

GOALS

You will be able to

- Graph the reciprocal functions of linear and quadratic functions
- Identify the key characteristics of rational functions from their equations and use these characteristics to sketch their graphs
- Solve rational equations and inequalities with and without graphing technology
- Determine average and instantaneous rates of change in situations that are modelled by rational functions

? When polluted water flows into a clean pond, how does the concentration of pollutant in the pond change over time? What type of function would model this change?

5 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
1	R-3
2, 3, 4, 8	R-4
7	R-8

SKILLS AND CONCEPTS You Need

1. Factor each expression.

a) $x^2 - 3x - 10$
b) $3x^2 + 12x - 15$
c) $16x^2 - 49$
d) $9x^2 - 12x + 4$
e) $3a^2 + a - 30$
f) $6x^2 - 5xy - 21y^2$

2. Simplify each expression. State any restrictions on the variables, if necessary.

a) $\frac{12 - 8s}{4}$
b) $\frac{6m^2n^4}{18m^3n}$
c) $\frac{9x^3 - 12x^2 - 3x}{3x}$
d) $\frac{25x - 10}{5(5x - 2)^2}$
e) $\frac{x^2 + 3x - 18}{9 - x^2}$
f) $\frac{a^2 + 4ab - 5b^2}{2a^2 + 7ab - 15b^2}$

3. Simplify each expression, and state any restrictions on the variable.

a) $\frac{3}{5} \times \frac{7}{9}$
b) $\frac{2x}{5} \div \frac{x^2}{15}$
c) $\frac{x^2 - 4}{x - 3} \div \frac{x + 2}{12 - 4x}$
d) $\frac{x^3 + 4x^2}{x^2 - 1} \times \frac{x^2 - 5x + 6}{x^2 - 3x}$

4. Simplify each expression, and state any restrictions on the variable.

a) $\frac{2}{3} + \frac{6}{7}$
b) $\frac{3x}{4} + \frac{5x}{6}$
c) $\frac{1}{x} + \frac{4}{x^2}$
d) $\frac{5}{x - 3} - \frac{2}{x}$
e) $\frac{2}{x - 5} + \frac{y}{x^2 - 25}$
f) $\frac{6}{a^2 - 9a + 20} - \frac{8}{a^2 - 2a - 15}$

5. Solve and check.

a) $\frac{5x}{8} = \frac{15}{4}$
b) $\frac{x}{4} + \frac{1}{3} = \frac{5}{6}$
c) $\frac{4x}{5} - \frac{3x}{10} = \frac{3}{2}$
d) $\frac{x + 1}{2} - \frac{2x - 1}{3} = -1$

6. Sketch the graph of the reciprocal function $f(x) = \frac{1}{x}$ and describe its characteristics. Include the domain and range, as well as the equations of the asymptotes.

7. List the transformations that need to be applied to $y = \frac{1}{x}$ to graph each of the following reciprocal functions. Then sketch the graph.

a) $f(x) = \dfrac{1}{x + 3}$

b) $f(x) = \dfrac{2}{x - 1}$

c) $f(x) = -\dfrac{1}{2x} - 3$

d) $f(x) = \dfrac{2}{-3(x - 2)} + 1$

8. Describe the steps that are required to divide two rational expressions. Use your description to simplify $\dfrac{9y^2 - 4}{4y - 12} \div \dfrac{9y^2 + 12 + 4}{18 - 6y}$.

APPLYING What You Know

Painting Houses

Tony can paint the exterior of a house in six working days. Rebecca takes nine days to complete the same painting job.

? How long will Rebecca and Tony take to paint a similar house, if they work together?

A. What fraction of the job can Tony complete in one day? What fraction of the job can Rebecca complete?

B. Write a numerical expression to represent the fraction of the job that Rebecca and Tony can complete in one day, if they work together.

C. Let x represent the number of days that Rebecca and Tony, working together, will take to complete the job. Explain why $\frac{1}{x}$ represents the fraction of the job Rebecca and Tony will complete in one day when they work together.

D. Use your answers for parts B and C to write an equation. Determine the **lowest common denominator** for the **rational expressions** in your equation. Rewrite the equation using the lowest common denominator.

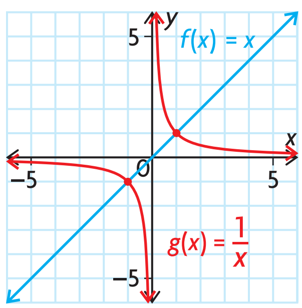
E. Solve the equation you wrote in part D by collecting like terms and comparing the numerators on the two sides of the equation.

F. What is the amount of time Rebecca and Tony will take to paint a similar house, when they work together?

5.1 Graphs of Reciprocal Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software



GOAL

Sketch the graphs of reciprocals of linear and quadratic functions.

INVESTIGATE the Math

Owen has noted some connections between the graphs of $f(x) = x$ and its reciprocal function $g(x) = \frac{1}{x}$.

- Both graphs are in the same quadrants for the same x-values.
- When $f(x) = 0$, there is a vertical asymptote for $g(x)$.
- $f(x)$ is always increasing, and $g(x)$ is always decreasing.

? How are the graphs of a function and its reciprocal function related?

A. Explain why the graphs of $f(x) = x$ and $g(x) = \frac{1}{x}$ are in the same quadrants over the same intervals. Does this relationship hold for $m(x) = -x$ and $n(x) = -\frac{1}{x}$? Does this relationship hold for any function and its reciprocal function? Explain.

B. What graphical characteristic in the reciprocal function do the zeros of the original function correspond to? Explain.

C. Explain why the reciprocal function $g(x) = \frac{1}{x}$ is decreasing when $f(x) = x$ is increasing. Does this relationship hold for $n(x) = -\frac{1}{x}$ and $m(x) = -x$? Explain how the increasing and decreasing intervals of a function and its reciprocal are related.

D. What are the y-coordinates of the points where $f(x)$ and $g(x)$ intersect? Will the points of intersection for any function and its reciprocal always have the same y-coordinates? Explain.

E. Explain why the graph of $g(x)$ has a horizontal asymptote. What is the equation of this asymptote? Will all reciprocal functions have the same horizontal asymptote? Explain.

F. On graph paper, draw the graph of $p(x) = x^2 - 4$. In a table like the one below, note the characteristics of the graph of $p(x)$ and use this information to help you determine the characteristics of the reciprocal function $q(x) = \frac{1}{x^2 - 4}$.

Characteristics	$p(x) = x^2 - 4$	$q(x) = \frac{1}{x^2 - 4}$
zeros and/or vertical asymptotes		
interval(s) on which the graph is above the *x*-axis (all values of the function are positive)		
interval(s) on which the graph is below the *x*-axis (all values of the function are negative)		
interval(s) on which the function is increasing		
interval(s) on which the function is decreasing		
point(s) where the *y*-value is 1		
point(s) where the *y*-value is -1		

G. On the same graph, draw the vertical asymptotes for the reciprocal function. Then use the rest of the information determined in part F to draw the graph for $q(x) = \frac{1}{x^2 - 4}$.

H. Verify your graphs by entering $p(x)$ and $q(x)$ in a graphing calculator using the "friendly" window setting shown.

I. Repeat parts F to H for the following pairs of functions.

a) $p(x) = x + 2$ and $q(x) = \frac{1}{x + 2}$

b) $p(x) = 2x - 3$ and $q(x) = \frac{1}{2x - 3}$

c) $p(x) = (x - 2)(x + 3)$ and $q(x) = \frac{1}{(x - 2)(x + 3)}$

d) $p(x) = (x - 1)^2$ and $q(x) = \frac{1}{(x - 1)^2}$

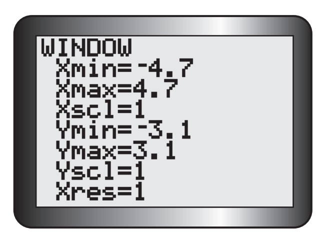
J. Write a summary of the relationships between the characteristics of the graphs of

a) a linear function and its reciprocal function

b) a quadratic function and its reciprocal function

Tech *Support*

On a graphing calculator, the length of the display screen contains 94 pixels, and the width contains 62 pixels. When the domain, *X*max-*X*min, is cleanly divisible by 94, and the range, *Y*max-*Y*min, is cleanly divisible by 62, the window is friendly. This means that you can trace without using "ugly" decimals. A friendly window is useful when working with rational functions.



Use brackets when entering reciprocal functions in the *Y* = editor of a graphing calculator. For example, to graph the function $f(x) = \frac{1}{x^2 - 4}$, enter $Y1 = \frac{1}{(x^2 - 4)}$.

Reflecting

K. How did knowing the positive/negative intervals and the increasing/decreasing intervals for $p(x) = x^2 - 4$ help you draw the graph for $p(x) = \frac{1}{x^2 - 4}$?

L. Why are some numbers in the domain of a function excluded from the domain of its reciprocal function? What graphical characteristic of the reciprocal function occurs at these values?

M. What common characteristics are shared by all reciprocals of linear and quadratic functions?

APPLY the Math

EXAMPLE 1 Connecting the characteristics of a linear function to its corresponding reciprocal function

Given the function $f(x) = 2 - x$,

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 2 - x$ is a linear function.

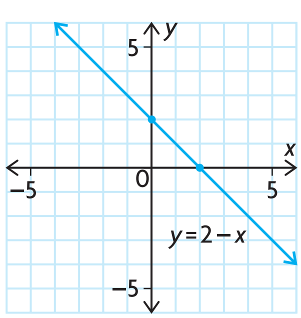
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R}\}$

The domain and range of most linear functions are the set of real numbers.

From the equation, the y-intercept is 2.
$f(x) = 0$ when $0 = 2 - x$
$x = 2$

The x-intercept is 2.

A linear function $f(x) = mx + b$ has y-intercept b.
The x-intercept occurs where $f(x) = 0$.



Sketch the graph of $f(x)$ to determine the positive and negative intervals. The line $y = 2 - x$ is above the x-axis for all x-values less than 2 and below the x-axis for all x-values greater than 2.

$f(x)$ is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.
$f(x)$ is decreasing when $x \in (-\infty, \infty)$.

This is a linear function with a negative slope, so it is decreasing over its entire domain.

b) The reciprocal function is $g(x) = \frac{1}{2 - x}$.

All the y-values of points on the reciprocal function are reciprocals of the y-values on the original function.

$D = \{x \in \mathbf{R} | x \neq 2\}$

$R = \{y \in \mathbf{R} | y \neq 0\}$

The y-intercept is 0.5.

The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 0$.

There is a vertical asymptote at the zero of the original function. The reciprocals of a linear function always have the x-axis as a horizontal asymptote.

The reciprocal function is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.

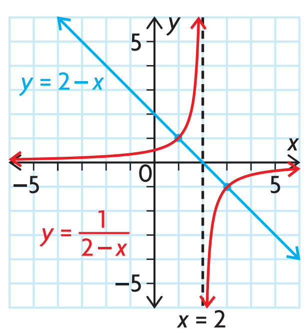
The positive/negative intervals are always the same for both functions.

It is increasing when $x \in (-\infty, 2)$ and when $x \in (2, \infty)$.

Because the original function is always decreasing, the reciprocal function is always increasing.

The graph of $g(x) = \frac{1}{2 - x}$ intersects the graph of $g(x) = 2 - x$ at $(1, 1)$ and $(3, -1)$.

The reciprocal of 1 is 1, and the reciprocal of -1 is -1. Thus, the two graphs intersect at any points with these y-values.



Use all this information to sketch the graph of the reciprocal function.

EXAMPLE 2 Connecting the characteristics of a quadratic function to its corresponding reciprocal function

Given the function $f(x) = 9 - x^2$

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 9 - x^2$ is a quadratic function.

$D = \{x \in \mathbf{R}\}$

The domain of a quadratic function is the set of real numbers.

$f(0) = 9$, so the y-intercept is 9.

$R = \{y \in \mathbf{R} | y \leq 9\}$

The graph of $f(x)$ is a parabola that opens down. The vertex is at (0, 9), so $y \leq 9$.

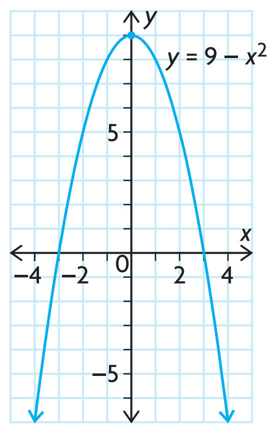
$f(x) = 0 \Rightarrow 9 - x^2 = 0$

$(3 - x)(3 + x) = 0$

$x = \pm 3$

Factor and determine the x-intercepts.

The x-intercepts are -3 and 3.



The parabola $y = 9 - x^2$ is above the x-axis for x-values between -3 and 3. The graph is below the x-axis for x-values less than -3 and for x-values greater than 3.

$f(x)$ is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$.

$f(x)$ is increasing when $x \in (-\infty, 0)$ and decreasing when $x \in (0, \infty)$.

The y-values increase as x increases from $-\infty$ to 0. The y-values decrease as x increases from 0 to ∞.

b) The reciprocal function is $g(x) = \frac{1}{9 - x^2}$.

The vertical asymptotes are $x = -3$ and $x = 3$.

$D = \{x \in \mathbf{R} \mid x \neq \pm 3\}$

Vertical asymptotes occur at each zero of the original function, so these numbers must be excluded from the domain.

The horizontal asymptote is $y = 0$.

The y-intercept is $\frac{1}{9}$.

The reciprocals of all quadratic functions have the x-axis as a horizontal asymptote. The y-intercept of the original function is 9, so the y-intercept of the reciprocal function is $\frac{1}{9}$.

There is a local minimum value at $\left(0, \frac{1}{9}\right)$.

$R = \left\{y \in \mathbf{R} \mid y < 0 \text{ or } y \geq \frac{1}{9}\right\}$

When the original function has a local maximum point, the reciprocal function has a corresponding local minimum point.

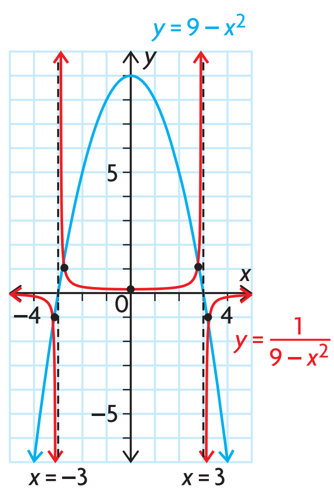
The reciprocal function is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$. It is decreasing when $x \in (-\infty, -3)$ and when $x \in (-3, 0)$, and increasing when $x \in (0, 3)$ and when $x \in (3, \infty)$.

The positive/negative intervals are always the same for both functions. Where the original function is decreasing, excluding the zeros, the reciprocal function is increasing (and vice versa).

$$\begin{aligned} f(x) = 1 \text{ when } 9 - x^2 &= 1 \\ -x^2 &= 1 - 9 \\ -x^2 &= -8 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} f(x) = -1 \text{ when } 9 - x^2 &= -1 \\ -x^2 &= -1 - 9 \\ -x^2 &= -10 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \end{aligned}$$

A function and its reciprocal intersect at points where $y = \pm 1$. Solve the corresponding equations to determine the x-coordinates of the points of intersection.

The graph of $g(x) = \frac{1}{9 - x^2}$ intersects the graph of $f(x) = 9 - x^2$ at $(-2\sqrt{2}, 1)$, $(2\sqrt{2}, 1)$ and at $(-\sqrt{10}, -1)$, $(\sqrt{10}, -1)$.



Use all this information to sketch the graph of the reciprocal function.

In Summary

Key Idea

- You can use key characteristics of the graph of a linear or quadratic function to graph the related reciprocal function.

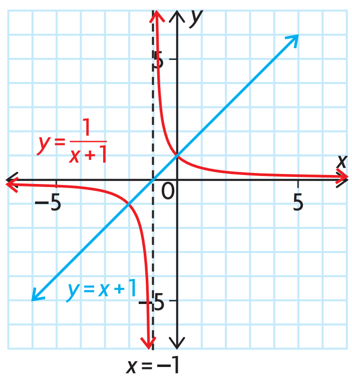
Need to Know

- All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
- The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- A reciprocal function will always have $y = 0$ as a horizontal asymptote if the original function is linear or quadratic.
- A reciprocal function has the same positive/negative intervals as the original function.

(*continued*)

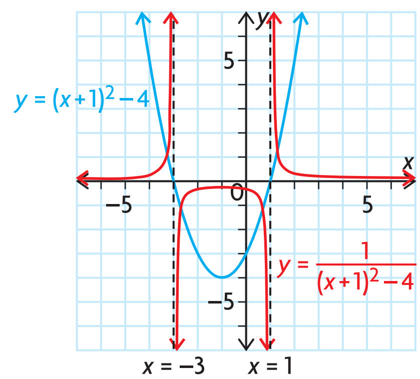
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1.
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

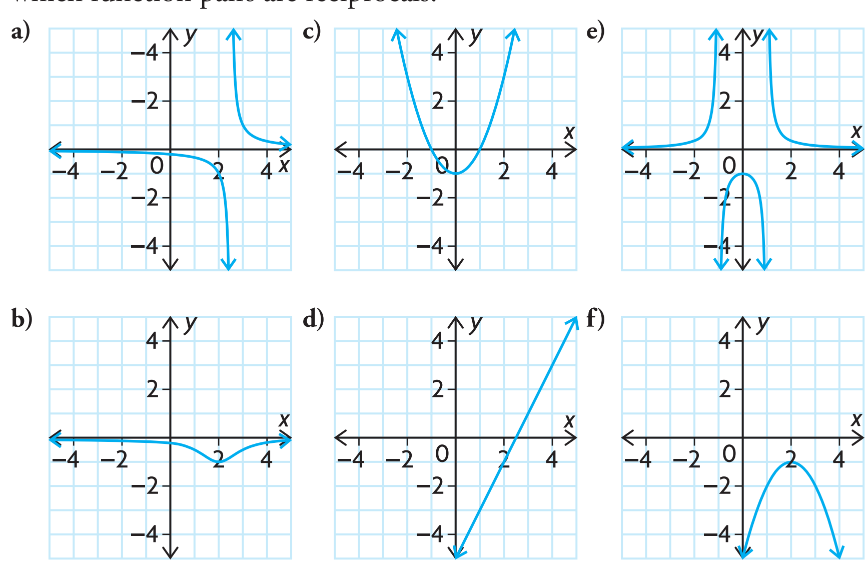
A quadratic function and its reciprocal



Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

CHECK Your Understanding

1. Match each function with its equation on the next page. Then identify which function pairs are reciprocals.



A $y = \dfrac{1}{-(x - 2)^2 - 1}$

B $y = \dfrac{1}{x^2 - 1}$

C $y = \dfrac{1}{2x - 5}$

D $y = x^2 - 1$

E $y = -(x - 2)^2 - 1$

F $y = 2x - 5$

2. For each pair of functions, determine where the zeros of the original function occur and state the equations of the vertical asymptotes of the reciprocal function, if possible.

a) $f(x) = x - 6, g(x) = \dfrac{1}{x - 6}$

b) $f(x) = 3x + 4, g(x) = \dfrac{1}{3x + 4}$

c) $f(x) = x^2 - 2x - 15, g(x) = \dfrac{1}{x^2 - 2x - 15}$

d) $f(x) = 4x^2 - 25, g(x) = \dfrac{1}{4x^2 - 25}$

e) $f(x) = x^2 + 4, g(x) = \dfrac{1}{x^2 + 4}$

f) $f(x) = 2x^2 + 5x + 3, g(x) = \dfrac{1}{2x^2 + 5x + 3}$

3. Sketch the graph of each function. Use your graph to help you sketch the graph of the reciprocal function.

a) $f(x) = 5 - x$ **b)** $f(x) = x^2 - 6x$

PRACTISING

4. a) Copy and complete the following table.

x	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$f(x)$	16	14	12	10	8	6	4	2	0	−2	−4	−6
$\frac{1}{f(x)}$												

b) Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$.

c) Find equations for $y = f(x)$ and $y = \frac{1}{f(x)}$.

5. State the equation of the reciprocal of each function, and determine the equations of the vertical asymptotes of the reciprocal. Verify your results using graphing technology.

a) $f(x) = 2x$

b) $f(x) = x + 5$

c) $f(x) = x - 4$

d) $f(x) = 2x + 5$

e) $f(x) = -3x + 6$

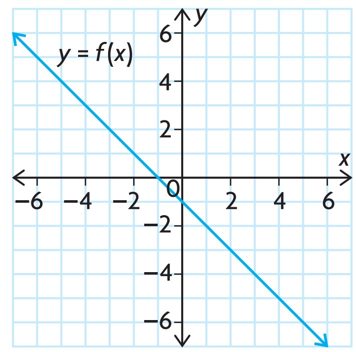
f) $f(x) = (x - 3)^2$

g) $f(x) = x^2 - 3x - 10$

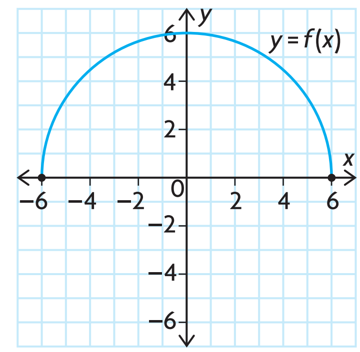
h) $f(x) = 3x^2 - 4x - 4$

6. Sketch the graph of the reciprocal of each function.

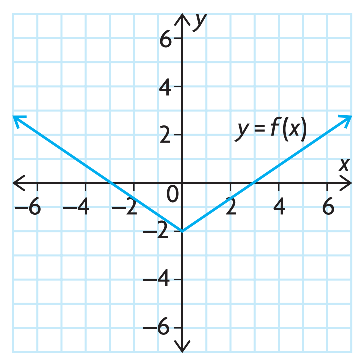
a)



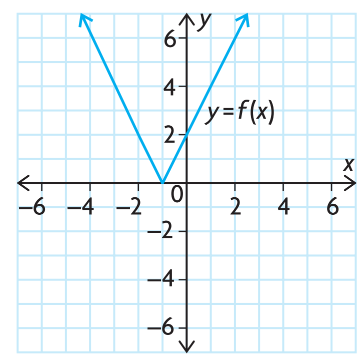
c)



b)



d)



7. Sketch each pair of graphs on the same axes. State the domain and range of each reciprocal function.

a) $y = 2x - 5, y = \dfrac{1}{2x - 5}$ **b)** $y = 3x + 4, y = \dfrac{1}{3x + 4}$

8. Draw the graph of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes.

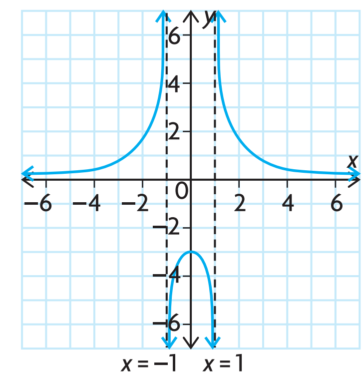
a) $f(x) = x^2 - 4$
b) $f(x) = (x - 2)^2 - 3$
c) $f(x) = x^2 - 3x + 2$
d) $f(x) = (x + 3)^2$
e) $f(x) = x^2 + 2$
f) $f(x) = -(x + 4)^2 + 1$

9. K For each function, determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals. State the equation of the reciprocal function. Then sketch the graphs of the original and reciprocal functions on the same axes.

a) $f(x) = 2x + 8$
b) $f(x) = -4x - 3$
c) $f(x) = x^2 - x - 12$
d) $f(x) = -2x^2 + 10x - 12$

10. Why do the graphs of reciprocals of linear functions always have vertical asymptotes, but the graphs of reciprocals of quadratic functions sometimes do not? Provide sketches of three different reciprocal functions to illustrate your answer.

11. An equation of the form $y = \frac{k}{x^2 + bx + c}$ has a graph that closely matches the graph shown. Find the equation. Check your answer using graphing technology.



12. **A** A chemical company is testing the effectiveness of a new cleaning solution for killing bacteria. The test involves introducing the solution into a sample that contains approximately 10 000 bacteria. The number of bacteria remaining, $b(t)$, over time, t, in seconds is given by the equation $b(t) = 10\ 000\frac{1}{t}$.

a) How many bacteria will be left after 20 s?
b) After how many seconds will only 5000 bacteria be left?
c) After how many seconds will only one bacterium be left?
d) This model is not always accurate. Determine what sort of inaccuracies this model might have. Assume that the solution was introduced at $t = 0$.
e) Based on these inaccuracies, what should the domain and range of the equation be?

13. **T** Use your graphing calculator to explore and then describe the key characteristics of the family of reciprocal functions of the form $g(x) = \frac{1}{x + n}$. Make sure that you include graphs to support your descriptions.

a) State the domain and range of $g(x)$.
b) For the family of functions $f(x) = x + n$, the y-intercept changes as the value of n changes. Describe how the y-intercept changes and how this affects $g(x)$.
c) If graphed, at what point would the two graphs $f(x)$ and $g(x)$ intersect?

14. **C** Due to a basketball tournament, your friend has missed this class. Write a concise explanation of the steps needed to graph a reciprocal function using the graph of the original function (without using graphing technology). Use an example, and explain the reason for each step.

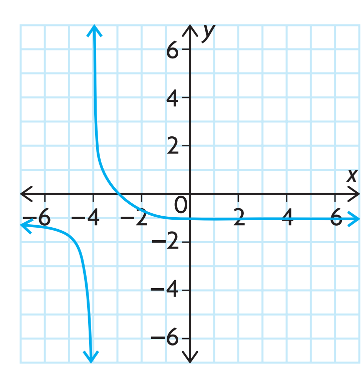
Extending

15. Sketch the graphs of the following reciprocal functions.

a) $y = \frac{1}{\sqrt{x}}$ **c)** $y = \frac{1}{2^x}$

b) $y = \frac{1}{x^3}$ **d)** $y = \frac{1}{\sin x}$

16. Determine the equation of the function in the graph shown.



5.2 Exploring Quotients of Polynomial Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software

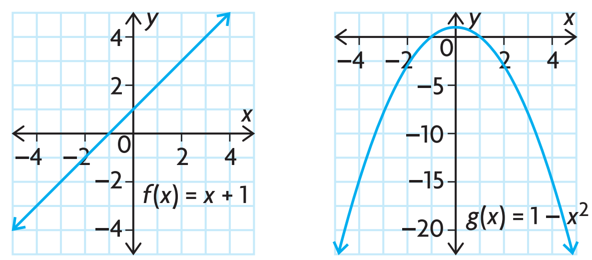
GOAL

Explore graphs that are created by dividing polynomial functions.

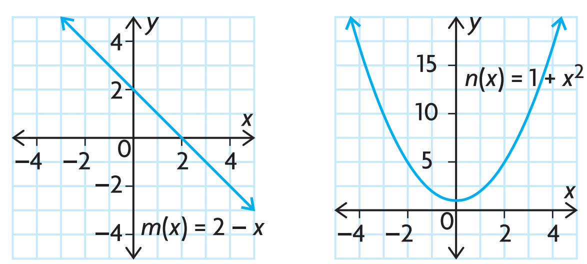
EXPLORE the Math

Each row shows the graphs of two polynomial functions.

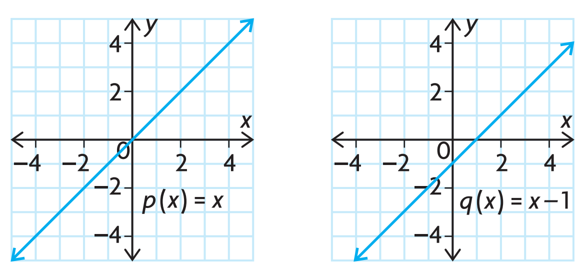
A.



B.



C.

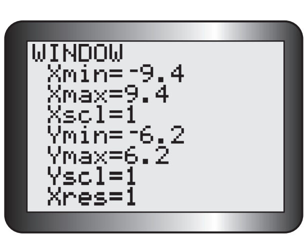


rational function

a function that can be expressed as $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, $q(x) \neq 0$ (e.g., $f(x) = \frac{3x^2 - 1}{x + 1}$, $x \neq -1$, and $f(x) = \frac{1 - x}{x^2}$, $x \neq 0$, are rational functions, but $f(x) = \frac{1 + x}{\sqrt{2 - x}}$, $x \neq 2$, is not because its denominator is not a polynomial)

? What are the characteristics of the graphs that are created by dividing two polynomial functions?

A. Using the given functions, write the equation of the rational function $y = \frac{f(x)}{g(x)}$. Enter this equation into Y1 of the equation editor of a graphing calculator. Graph this equation using the window settings shown, and draw a sketch.



B. Describe the characteristics of the graph you created in part A by answering the following questions:

i) Where are the zeros?

ii) Are there any asymptotes? If so, where are they?

iii) What are the domain and range of this function?

iv) Is it a **continuous function**? Explain.

v) Are there any values of $y = \frac{f(x)}{g(x)}$ that are undefined? What feature(s) of the graph is (are) related to these values?

vi) Describe the end behaviours of this function.

vii) Is the resulting graph a function? Explain.

C. Write the equation defined by $y = \frac{g(x)}{f(x)}$. Predict how the graph of this function will differ from the graph of $y = \frac{f(x)}{g(x)}$. Graph this function using your graphing calculator, and draw a sketch.

Tech ***Support***

When entering a rational function into a graphing calculator, use brackets around the expression in the numerator and the expression in the denominator.

D. Describe the characteristics of the graph you created in part C by answering the questions in part B.

E. Repeat parts A through D for the functions in the other two rows.

F. Using graphing technology, and the same window settings you used in part A, explore the graphs of the following rational functions. Sketch each graph on separate axes, and note any holes or asymptotes.

i) $f(x) = \frac{x^2 - 1}{x - 1}$

ii) $f(x) = \frac{3}{x + 1}$

iii) $f(x) = \frac{x + 1}{x^2 - 2x - 3}$

iv) $f(x) = \frac{x + 1}{x + 2}$

v) $f(x) = \frac{0.5x^2 + 1}{x - 1}$

vi) $f(x) = \frac{x^2 + 2x}{x + 1}$

vii) $f(x) = \frac{9x}{1 + x^2}$

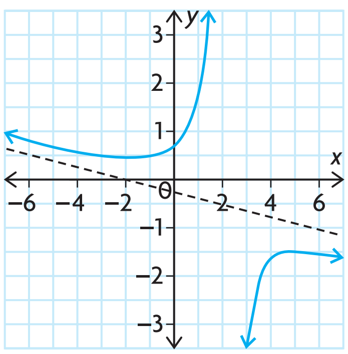
viii) $f(x) = \frac{2x^2 - 3}{x^2 + 1}$

G. Examine the graphs of the functions in parts i) and v) of part F at the point where $x = 1$. Explain why $f(x) = \frac{x^2 - 1}{x - 1}$ has a hole where $x = 1$, but $f(x) = \frac{0.5x^2 + 1}{x - 1}$ has a vertical asymptote. Identify the other functions in part F that have holes and the other functions that have vertical asymptotes.

H. Redraw the graph of the rational function $f(x) = \frac{0.5x^2 + 1}{x - 1}$. Then enter the equation $y = 0.5x + 0.5$ into Y2 of the equation editor. What do you notice? Examine all your other sketches in this exploration to see if any of the other functions have an **oblique asymptote**.

oblique asymptote

an asymptote that is neither vertical nor horizontal, but slanted



I. Examine the equations with graphs that have horizontal asymptotes in part F. Compare the degree of the expression in the numerator with the degree of the expression in the denominator. Is there a connection between the degrees in the numerator and denominator and the existence of horizontal asymptotes? Explain. Repeat for functions with oblique asymptotes.

J. Investigate several functions of the form $f(x) = \frac{ax + b}{cx + d}$. Note similarities and differences. Without graphing, how can you predict where a horizontal asymptote will occur?

K. Investigate graphs of quotients of quadratic functions. How are they different from graphs of quotients of linear functions?

L. Summarize the different characteristics of the graphs of rational functions.

Reflecting

M. How do the zeros of the function in the numerator help you graph the rational function? How do the zeros of the function in the denominator help you graph the rational function?

N. Explain how you can use the expressions in the numerator and the denominator of a rational function to decide if the graph has

- **i)** a hole
- **ii)** a vertical asymptote
- **iii)** a horizontal asymptote
- **iv)** an oblique asymptote

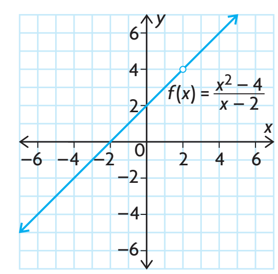
In Summary

Key Ideas

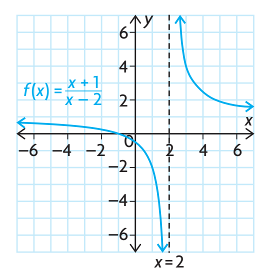
- The quotient of two polynomial functions results in a rational function which often has one or more discontinuities.
- The breaks or discontinuities in a rational function occur where the function is undefined. The function is undefined at values where the denominator is equal to zero. As a result, these values must be restricted from the domain of the function.
- The values that must be restricted from the domain of a rational function result in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).
- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

Need to Know

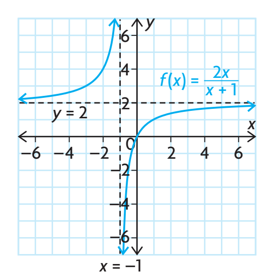
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a hole at $x = a$ if $\frac{p(a)}{q(a)} = \frac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x - a)$. For example, $f(x) = \frac{x^2 - 4}{x - 2}$ has the common factor of $(x - 2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x = 2$.



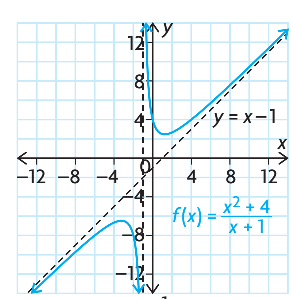
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a vertical asymptote at $x = a$ if $\frac{p(a)}{q(a)} = \frac{p(a)}{0}$. For example, $f(x) = \frac{x + 1}{x - 2}$ has a vertical asymptote at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x) = \frac{2x}{x + 1}$ has a horizontal asymptote at $y = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x) = \frac{x^2 + 4}{x + 1}$ has an oblique asymptote.



FURTHER Your Understanding

1. Without using graphing technology, match each equation with its corresponding graph. Explain your reasoning.

a) $y = \dfrac{-1}{x - 3}$

b) $y = \dfrac{x^2 - 9}{x - 3}$

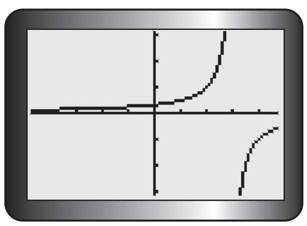
c) $y = \dfrac{1}{(x + 3)^2}$

d) $y = \dfrac{x}{(x - 1)(x + 3)}$

e) $y = \dfrac{1}{x^2 + 5}$

f) $y = \dfrac{x^2}{x - 3}$

A



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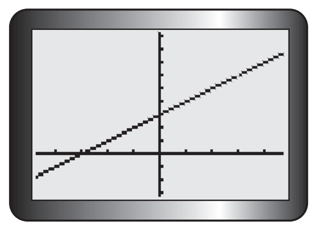

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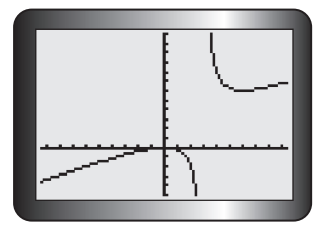

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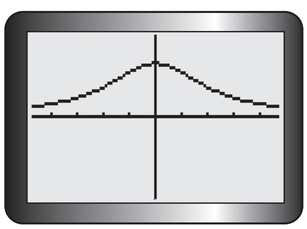

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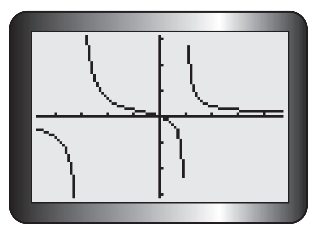
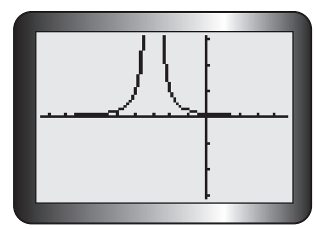


F



2. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal or oblique asymptotes.

a) $y = \dfrac{x}{x + 4}$

b) $y = \dfrac{1}{2x + 3}$

c) $y = \dfrac{2x + 5}{x - 6}$

d) $y = \dfrac{x^2 - 9}{x + 3}$

e) $y = \dfrac{1}{(x + 3)(x - 5)}$

f) $y = \dfrac{-x}{x + 1}$

g) $y = \dfrac{3x - 6}{x - 2}$

h) $y = \dfrac{-4x + 1}{2x - 5}$

i) $y = \dfrac{8x}{4x + 1}$

j) $y = \dfrac{x + 4}{x^2 - 16}$

k) $y = \dfrac{x}{5x - 3}$

l) $y = \dfrac{-3x + 1}{2x - 8}$

3. Write an equation for a rational function with the properties as given.

a) a hole at $x = 1$

b) a vertical asymptote anywhere and a horizontal asymptote along the x-axis

c) a hole at $x = -2$ and a vertical asymptote at $x = 1$

d) a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 2$

e) an oblique asymptote, but no vertical asymptote

5.3 Graphs of Rational Functions of the Form $f(x) = \frac{ax + b}{cx + d}$

GOAL

Sketch the graphs of rational functions, given equations of the form $f(x) = \frac{ax + b}{cx + d}$.

YOU WILL NEED

- graph paper
- graphing calculator or graphing software

INVESTIGATE the Math

The radius, in centimetres, of a circular juice blot on a piece of paper towel is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where t is measured in seconds. According to this model, the maximum size of the blot is determined by the location of the horizontal asymptote.

? How can you find the equation of the horizontal asymptote of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$?

A. Without graphing, determine the domain, intercepts, vertical asymptote, and positive/negative intervals of the simple rational function $f(x) = \frac{x}{x + 1}$.

B. Copy the following tables, and complete them by evaluating $f(x)$ for each value of x. Examine the **end behaviour** of $f(x)$ by observing the trend in $f(x)$ as x grows positively large and negatively large. What value does $f(x)$ seem to approach?

$x \to \infty$	
x	$f(x) = \frac{x}{x + 1}$
10	
100	
1 000	
10 000	
100 000	
1 000 000	

$x \to -\infty$	
x	$f(x) = \frac{x}{x + 1}$
−10	
−100	
−1 000	
−10 000	
−100 000	
−1 000 000	

C. Write an equation for the horizontal asymptote of the function in part B.

D. Repeat parts A, B, and C for the functions $g(x) = \frac{4x}{x + 1}$, $h(x) = \frac{2x}{3x + 1}$, and $m(x) = \frac{3x - 2}{2x - 5}$.

E. Verify your results by graphing all the functions in part D on a graphing calculator. Note similarities and differences among the graphs.

F. Make a list of the equations of the functions and the equations of their horizontal asymptotes. Discuss how the degree of the numerator compares with the degree of the denominator. Explain how the leading coefficients of x in the numerator and the denominator determine the equation of the horizontal asymptote.

G. Determine the equation of the horizontal asymptote of the juice blot function $r(t) = \frac{1 + 2t}{1 + t}$. What does this equation tell you about the eventual size of the juice blot?

Reflecting

H. How do the graphs of rational functions with linear expressions in the numerator and denominator compare with the graphs of reciprocal functions?

I. Explain how you determined the equation of a horizontal asymptote from

i) end behaviour tables

ii) the equation of the function

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine how a graph approaches a vertical asymptote

Determine how the graph of $f(x) = \frac{3x - 5}{x + 2}$ approaches its vertical asymptote.

Solution

$f(x) = \frac{3x - 5}{x + 2}$ has a vertical asymptote with the equation $x = -2$. Near this asymptote, the values of the function will grow very large in a positive direction or very large in a negative direction.

$f(x)$ is undefined when $x = -2$.

There is no common factor in the numerator and denominator.

Choose a value of x to the left and very close to -2. This value is less than -2.

$$f(-2.1) = \frac{3(-2.1) - 5}{(-2.1) + 2}$$
$$= 113$$

The graph of a rational function never crosses a vertical asymptote, so choose x-values that are very close to the vertical asymptote, on both sides, to determine the behaviour of the function.

On the left side of the vertical asymptote, the values of the function are positive. As $x \to -2, f(x) \to \infty$.

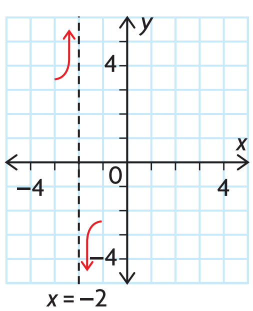
The function increases to large positive values as x approaches -2 from the left.

Choose a value of x to the right and very close to -2. This value is greater than -2.

$$f(-1.9) = \frac{3(-1.9) - 5}{(-1.9) + 2}$$
$$= -107$$

On the right side of the vertical asymptote, the values of the function are negative. As $x \to -2, f(x) \to -\infty$.

The function decreases to small negative values as x approaches -2 from the right.



Make a sketch to show how the graph approaches the vertical asymptote.

EXAMPLE 2 Using key characteristics to sketch the graph of a rational function

For each function,

a) $f(x) = \frac{2}{x - 3}$ **b)** $f(x) = \frac{x - 2}{3x + 4}$ **c)** $f(x) = \frac{x - 3}{2x - 6}$

i) determine the domain, intercepts, asymptotes, and positive/negative intervals
ii) use these characteristics to sketch the graph of the function
iii) describe where the function is increasing or decreasing

Solution

a) $f(x) = \frac{2}{x - 3}$

i) $D = \{x \in \mathbf{R} | x \neq 3\}$

The function $f(x) = \frac{2}{x - 3}$ is undefined when $x = 3$.

$f(0) = -\frac{2}{3}$, so the y-intercept is $-\frac{2}{3}$.

$f(x) \neq 0$, so there is no x-intercept.

Any rational function equals zero when its numerator equals zero. The numerator is always 2, so $f(x)$ can never equal zero.

The line $x = 3$ is a vertical asymptote.

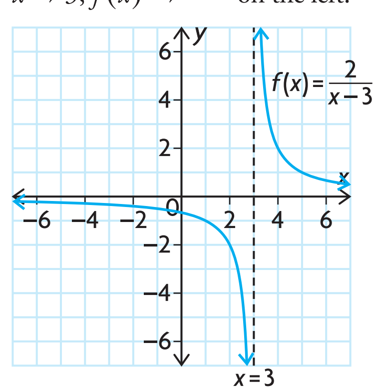
The line $y = 0$ is a horizontal asymptote.

Since the numerator and denominator do not contain the common factor $(x - 3)$, $f(x)$ has a vertical asymptote at $x = 3$. Any rational function that is formed by a constant numerator and a linear function denominator has a horizontal asymptote at $y = 0$.

$f(x)$ is negative when $x \in (-\infty, 3)$ and positive when $x \in (3, \infty)$.

The numerator is always positive, so the denominator determines the sign of $f(x)$.
$x - 3 < 0$ when $x < 3$
$x - 3 > 0$ when $x > 3$

ii) Confirm the behaviour of $f(x)$ near the vertical asymptote.
$f(3.1) = 20$, so as
$x \to 3, f(x) \to \infty$
on the right.
$f(2.9) = -20$, so as
$x \to 3, f(x) \to -\infty$ on the left.



Use all the information in part i) to sketch the graph.

iii) From the graph, the function is decreasing on its entire domain: when $x \in (-\infty, 3)$ and when $x \in (3, \infty)$.

b) $f(x) = \dfrac{x - 2}{3x + 4}$

i) $3x + 4 \neq 0$

$3x \neq -4$

$x \neq -\dfrac{4}{3}$

$D = \left\{x \in \mathbf{R} \mid x \neq -\dfrac{4}{3}\right\}$

$f(x)$ is undefined when the denominator is zero.

$f(0) = \dfrac{0 - 2}{3(0) + 4} = \dfrac{-2}{4} \text{ or } -\dfrac{1}{2}$

To determine the y-intercept, let $x = 0$.

The y-intercept is $-\dfrac{1}{2}$.

$f(x) = 0$ when $\dfrac{x - 2}{3x + 4} = 0$.

$x - 2 = 0$

$x = 2$

The x-intercept is 2.

To determine the x-intercept, let $y = 0$. Any rational function equals zero when its numerator equals zero.

The line $x = -\frac{4}{3}$ is a vertical asymptote.

This is the value that makes $f(x)$ undefined.

The line $y = \frac{1}{3}$ is a horizontal asymptote.

The ratio of the leading coefficients of the numerator and denominator is $\frac{1}{3}$.

Examine the signs of the numerator and denominator, and their quotient, to determine the positive/negative intervals.

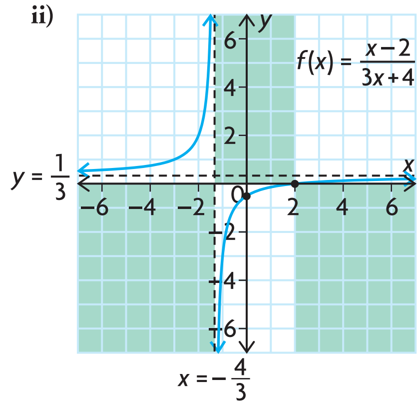
	$x < -\frac{4}{3}$	$-\frac{4}{3} < x < 2$	$x > 2$
$x - 2$	$-$	$-$	$+$
$3x + 4$	$-$	$+$	$+$
$\frac{x - 2}{3x - 4}$	$\frac{-}{-} = +$	$\frac{-}{+} = -$	$\frac{+}{+} = +$

The vertical asymptote and the x-intercept divide the set of real numbers into three intervals: $\left(-\infty, -\frac{4}{3}\right)$, $\left(-\frac{4}{3}, 2\right)$, and $(2, \infty)$. Choose numbers in each interval to evaluate the sign of each expression.

$f(x)$ is positive when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in (2, \infty)$.

$f(x)$ is negative when $x \in \left(-\frac{4}{3}, 2\right)$.

ii)



When sketching the graph, it helps to shade the regions where there is no graph. Use the positive and negative intervals as indicators for these regions. For example, since $f(x)$ is positive on $\left(-\infty, -\frac{4}{3}\right)$, there is no graph under the x-axis on this interval. Draw the asymptotes, and mark the intercepts. Then draw the graph to approach the asymptotes.

iii) From the graph, $f(x)$ is increasing on its entire domain; that is, when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in \left(-\frac{4}{3}, \infty\right)$.

c) $f(x) = \frac{x-3}{2x-6}$

i) $2x - 6 \neq 0$

$2x \neq 6$

$x \neq 3$

$D = \{x \in \mathbf{R} \mid x \neq 3\}$

$f(x)$ is undefined when the denominator is zero.

$f(0) = \frac{0-3}{2(0)-6} = \frac{-3}{-6} \text{ or } \frac{1}{2}$

To determine the y-intercept, let $x = 0$.

The y-intercept is $\frac{1}{2}$.

$f(x) \neq 0$, so there is no x-intercept.

To determine the x-intercept, let $y = 0$. Only consider when the numerator is zero; that is, when $x - 3 = 0$. Therefore, the numerator is zero at $x = 3$, but this has already been excluded from the domain.

$f(x) = \frac{x-3}{2(x-3)}$

$f(x)$ has a hole, not a vertical asymptote, where $x = 3$.

Factoring reveals a common factor $(x - 3)$ in the numerator and denominator. The graph has a hole at the point where $x = 3$.

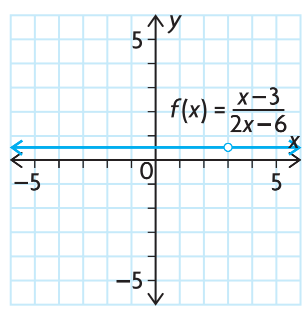
$f(x) = \frac{\cancel{x-3}^{\,1}}{2(\cancel{x-3})_{1}} = \frac{1}{2}$

The value of the function is always $\frac{1}{2}$ for all values of x, except when $x = 3$.

$f(x)$ is positive at all points in its domain.

$f(x)$ has no vertical asymptote or x-intercept. There is only one interval to consider: $(-\infty, \infty)$. For any value of x, $f(x) = \frac{1}{2}$.

ii) The graph is a horizontal line with the equation $y = \frac{1}{2}$. There is a hole at $x = 3$.



Use the information in part i) to sketch the graph.

iii) The function is neither increasing nor decreasing. It is constant on its entire domain.

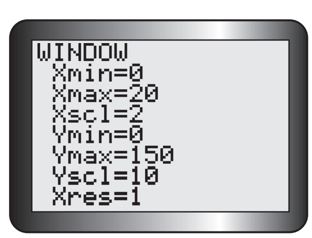
EXAMPLE 3 Solving a problem by graphing a rational function

The function $P(t) = \frac{30(7t + 9)}{3t + 2}$ models the population, in thousands, of a town t years since 1990. Describe the population of the town over the next 20 years.

Solution

$$P(t) = \frac{30(7t + 9)}{3t + 2}$$

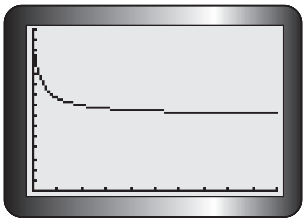
Use the equation to help you decide on the window settings. For the given context, $t \geq 0$ and $P(t) > 0$.



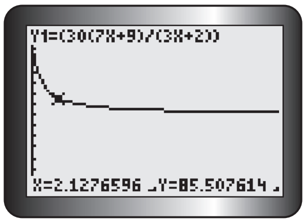
Determine the initial population in 1990, when $t = 0$.

$$P(0) = \frac{30(7(0) + 9)}{3(0) + 2} = 135$$

Graph $P(t)$ to show the population for the 20 years after 1990.



The value that makes $3t + 2 = 0$ lies outside the domain of $P(t)$. There is no vertical asymptote in the domain.

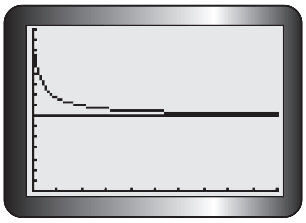


TRACE along the curve to get an idea of how the population changed.

In the first two years, the population dropped by about 50 000 people. Then it began to level off and approach a steady value.

There is a horizontal asymptote at $P = \frac{30(7)}{3} = 70$.

Use the function equation to determine the equation of the horizontal asymptote. For large values of t, $P(t) \doteq \frac{30(7t)}{3t}$. Therefore, the leading coefficients in the numerator and denominator define the equation of the horizontal asymptote.



The population of the town has been decreasing since 1990. It was 135 000 in 1990, but dropped by about 50 000 in the next two years. Since then, the population has begun to level off and, according to the model, will approach a steady value of 70 000 people by 2010.

Multiply the values of the function by 1000, since the population is given in thousands.

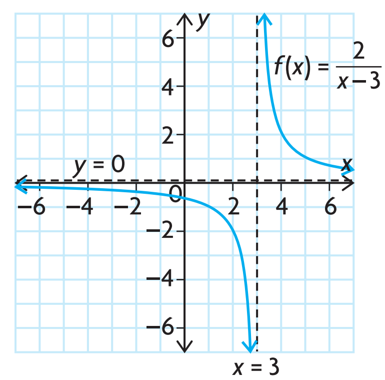
In Summary

Key Ideas

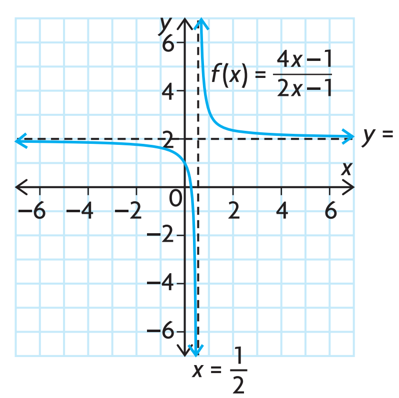
- The graphs of most rational functions of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$ have both a vertical asymptote and a horizontal asymptote.
- You can determine the equation of the vertical asymptote directly from the equation of the function by finding the zero of the denominator.
- You can determine the equation of the horizontal asymptote directly from the equation of the function by examining the ratio of the leading coefficients in the numerator and the denominator. This gives you the end behaviours of the function.
- To sketch the graph of a rational function, you can use the domain, intercepts, equations of asymptotes, and positive/negative intervals.

Need to Know

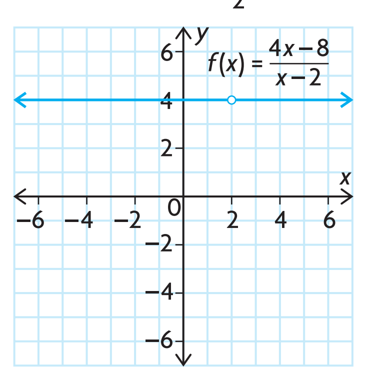
- Rational functions of the form $f(x) = \frac{b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = 0$. For example, see the graph of $f(x) = \frac{2}{x - 3}$.



- Most rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = \frac{a}{c}$. For example, see the graph of $f(x) = \frac{4x - 1}{2x - 1}$.



The exception occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factor occurs. As a result, the graph has no asymptotes. For example, see the graph of $f(x) = \frac{4x - 8}{x - 2} = \frac{4(x - 2)}{(x - 2)}$.



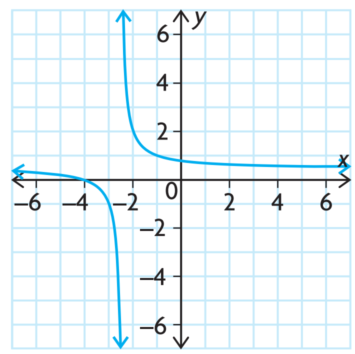
CHECK Your Understanding

1. Match each function with its graph.

a) $h(x) = \dfrac{x + 4}{2x + 5}$ **c)** $f(x) = \dfrac{3}{x - 1}$

b) $m(x) = \dfrac{2x - 4}{x - 2}$ **d)** $g(x) = \dfrac{2x - 3}{x + 2}$

A



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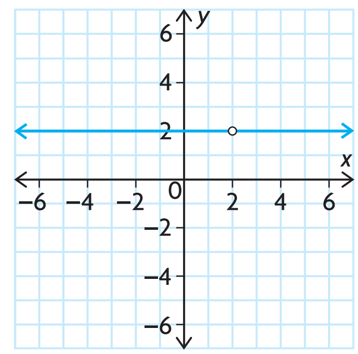

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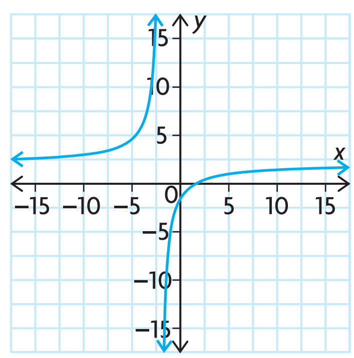

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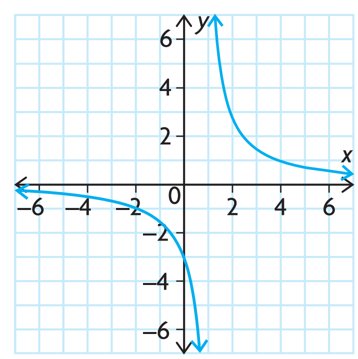


2. Consider the function $f(x) = \frac{3}{x - 2}$.

a) State the equation of the vertical asymptote.
b) Use a table of values to determine the behaviour(s) of the function near its vertical asymptote.
c) State the equation of the horizontal asymptote.
d) Use a table of values to determine the end behaviours of the function near its horizontal asymptote.
e) Determine the domain and range.
f) Determine the positive and negative intervals.
g) Sketch the graph.

3. Repeat question 2 for the rational function $f(x) = \frac{4x - 3}{x + 1}$.

PRACTISING

4. State the equation of the vertical asymptote of each function. Then choose a strategy to determine how the graph of the function approaches its vertical asymptote.

a) $y = \dfrac{2}{x+3}$ **c)** $y = \dfrac{2x+1}{2x-1}$

b) $y = \dfrac{x-1}{x-5}$ **d)** $y = \dfrac{3x+9}{4x+1}$

5. K For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \dfrac{3}{x+5}$ **c)** $f(x) = \dfrac{x+5}{4x-1}$

b) $f(x) = \dfrac{10}{2x-5}$ **d)** $f(x) = \dfrac{x+2}{5(x+2)}$

6. Read each set of conditions. State the equation of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$ that meets these conditions, and sketch the graph.

a) vertical asymptote at $x = -2$, horizontal asymptote at $y = 0$; negative when $x \in (-\infty, -2)$, positive when $x \in (-2, \infty)$; always decreasing

b) vertical asymptote at $x = -2$, horizontal asymptote at $y = 1$; x-intercept $= 0$, y-intercept $= 0$; positive when $x \in (-\infty, -2)$ or $(0, \infty)$, negative when $x \in (-2, 0)$

c) hole at $x = 3$; no vertical asymptotes; y-intercept $= (0, 0.5)$

d) vertical asymptotes at $x = -2$ and $x = 6$, horizontal asymptote at $y = 0$; positive when $x \in (-\infty, -2)$ or $(6, \infty)$, negative when $x \in (-2, 6)$; increasing when $x \in (-\infty, 2)$, decreasing when $x \in (2, \infty)$

7. T **a)** Use a graphing calculator to investigate the similarities and differences in the graphs of rational functions of the form $f(x) = \frac{8x}{nx+1}$, for $n = 1, 2, 4$, and 8.

b) Use your answer for part a) to make a conjecture about how the function changes as the values of n approach infinity.

c) If n is negative, how does the function change as the value of n approaches negative infinity? Choose your own values, and use them as examples to support your conclusions.

8. Without using a graphing calculator, compare the graphs of the rational functions $f(x) = \frac{3x + 4}{x - 1}$ and $g(x) = \frac{x - 1}{2x + 3}$.

9. The function $I(t) = \frac{15t + 25}{t}$ gives the value of an investment, in thousands of dollars, over t years.
   a) What is the value of the investment after 2 years?
   b) What is the value of the investment after 1 year?
   c) What is the value of the investment after 6 months?
   d) There is an asymptote on the graph of the function at $t = 0$. Does this make sense? Explain why or why not.
   e) Choose a very small value of t (a value near zero). Calculate the value of the investment at this time. Do you think that the function is accurate at this time? Why or why not?
   f) As time passes, what will the value of the investment approach?

10. **A** An amount of chlorine is added to a swimming pool that contains pure water. The concentration of chlorine, c, in the pool at t hours is given by $c(t) = \frac{2t}{2 + t}$, where c is measured in milligrams per litre. What happens to the concentration of chlorine in the pool during the 24 h period after the chlorine is added?

11. **C** Describe the key characteristics of the graphs of rational functions of the form $f(x) = \frac{ax + b}{cx + d}$. Explain how you can determine these characteristics using the equations of the functions. In what ways are the graphs of all the functions in this family alike? In what ways are they different? Use examples in your comparison.

Extending

12. Not all asymptotes are horizontal or vertical. Find a rational function that has an asymptote that is neither horizontal nor vertical, but slanted or oblique.

13. Use long division to rewrite $f(x) = \frac{2x^3 - 7x^2 + 8x - 5}{x - 1}$ in the form $f(x) = ax^2 + bx + c + \frac{k}{x - 1}$. What does this tell you about the end behaviour of the function? Graph the function. Include all asymptotes in your graph. Write the equations of the asymptotes.

14. Let $f(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $g(x) = \frac{x^3 + 8}{x^2 + 9}$, $h(x) = \frac{x^3 - 3x}{x + 1}$, and $m(x) = \frac{x^2 + x - 12}{x^2 - 4}$.
    a) Which of these rational functions has a horizontal asymptote?
    b) Which has an oblique asymptote?
    c) Which has no vertical asymptote?
    d) Graph $y = m(x)$, showing the asymptotes and intercepts.

5 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you use the graph of a linear or quadratic function to graph its reciprocal function?

A: If you have the graph of a linear or quadratic function, you can draw the graph of its reciprocal function as follows:

1. Draw a vertical asymptote for the reciprocal function at each zero of the original function. The x-axis is a horizontal asymptote for the reciprocal function, unless the original function is a constant function.
2. The reciprocal function has the same positive/negative intervals that the original function has, so you can shade the regions where there will be no graph.
3. Mark any points where the y-value of the original function is 1 or -1. The reciprocal function also goes through these points.
4. The y-intercept of the reciprocal function is the reciprocal of the y-intercept of the original function. Determine and mark the y-intercept of the reciprocal function.
5. If the original function is quadratic, the reciprocal function has a local maximum/minimum at the same x-value as the vertex of the quadratic function. The y-value of this local maximum/minimum is the reciprocal of the y-value of the vertex. Determine and mark the local maximum/minimum point.
6. Draw the pieces of the graph of the reciprocal function through the marked points, approaching the asymptotes.

Study *Aid*

- See Lesson 5.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1 and 2.

Q: What are rational functions, and what are the characteristics of their graphs?

A: Rational functions are quotients of polynomial functions. Their equations have the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Graphs of rational functions may have vertical, horizontal, or oblique asymptotes. Some rational functions have holes in their graphs.

Study *Aid*

- See Lesson 5.2.
- Try Mid-Chapter Review Question 3.

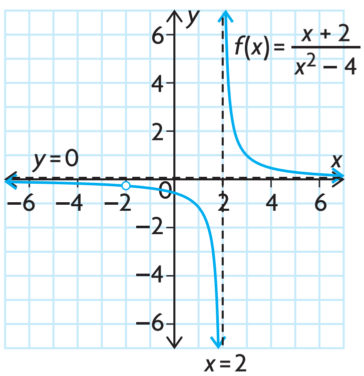
Study Aid

- See Lesson 5.2.
- Try Mid-Chapter Review Question 4.

Q: **Most rational functions have one or more discontinuities. Where and why do these discontinuities occur? When is a rational function continuous?**

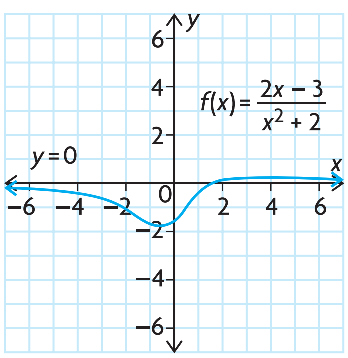
A: If the polynomial function in the denominator of a rational function has one or more zeros, the rational function will be discontinuous at these points. If a value of x can be zero in both the numerator and the denominator of a rational function (that is, if the numerator and denominator have a common linear factor), the result is a hole. This type of discontinuity is called a point discontinuity.

If a zero in the denominator does not correspond to a zero in the numerator, there will be a vertical asymptote at the x-value. This is called an infinite discontinuity.



For example, $f(x) = \frac{x + 2}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)}$ has a point discontinuity where $x = -2$ because -2 is a zero of both the denominator, $q(x) = x^2 - 4$, and the numerator, $p(x) = x + 2$. The graph of $f(x)$ has an infinite discontinuity where $x = 2$ because 2 is a zero of $q(x)$ but not of $p(x)$. The graph also has a hole at $x = -2$ and a vertical asymptote at $x = 2$. Note that $\frac{p(-2)}{q(-2)} = \frac{0}{0}$, but $\frac{p(2)}{q(2)} = \frac{4}{0}$.

If the polynomial function in the denominator of a rational function does not have any zeros, the rational function is continuous. Its graph is a smooth curve, with no breaks.



For example, $f(x) = \frac{2x - 3}{x^2 + 2}$ is a continuous rational function with a horizontal asymptote at $y = 0$.

Study Aid

- See Lesson 5.3, Example 2.
- Try Mid-Chapter Review Questions 5 to 8.

Q: **How do you determine the equations of the vertical and horizontal asymptotes of a rational function of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$?**

A: You can determine the equations of the vertical and horizontal asymptotes directly from the equation of a rational function of the form $f(x) = \frac{b}{cx + d}$ or $f(x) = \frac{ax + b}{cx + d}$. The vertical asymptote occurs at the zero of the function in the denominator. The equation of the vertical asymptote is $x = -\frac{d}{c}$. The horizontal asymptote describes the end behaviour of the function when $x \to \pm\infty$.

All rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have $y = \frac{a}{c}$ as a horizontal asymptote.

All rational functions of the form $f(x) = \frac{b}{cx + d}$ have $y = 0$ (the x-axis) as a horizontal asymptote.

PRACTICE Questions

Lesson 5.1

1. State the reciprocal of each function, and determine the locations of any vertical asymptotes.

a) $f(x) = x - 3$
b) $f(q) = -4q + 6$
c) $f(z) = z^2 + 4z - 5$
d) $f(d) = 6d^2 + 7d - 3$

2. For each function, determine the domain and range, intercepts, positive/negative intervals, and intervals of increase/decrease. Use this information to sketch the graphs of the function and its reciprocal.

a) $f(x) = 4x + 6$
b) $f(x) = x^2 - 4$
c) $f(x) = x^2 + 6$
d) $f(x) = -2x - 4$

Lesson 5.2

3. Different characteristics of the graph of a rational function are created by different characteristics of the function. List at least four characteristics of a graph and the characteristic of the function that causes each one. Make sure that you include a characteristic of a continuous rational function in your list.

4. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal asymptotes (other than the x-axis) or oblique asymptotes.

a) $y = \dfrac{x}{x - 2}$

b) $y = \dfrac{x - 1}{3x - 3}$

c) $y = \dfrac{-7x}{4x + 2}$

d) $y = \dfrac{x^2 + 2}{x - 6}$

e) $y = \dfrac{1}{x^2 + 2x - 15}$

Lesson 5.3

5. List the functions that had a horizontal asymptote in question 4, and give the equation of the horizontal asymptote.

6. For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \dfrac{5}{x - 6}$

b) $f(x) = \dfrac{3x}{x + 4}$

c) $f(x) = \dfrac{5x + 10}{x + 2}$

d) $f(x) = \dfrac{x - 2}{2x - 1}$

7. Kevin is trying to develop a reciprocal function to model some data that he has. He wants the horizontal asymptote to be $y = 7$. He also wants the graph to decrease and approach $y = 7$ as x approaches infinity, so he chooses the equation $y = \frac{7x + 6}{x}$. Then he decides that he needs the vertical asymptote to be $x = -1$, so he changes the equation to $y = \frac{7x + 6}{x + 1}$. What happened to the graph of Kevin's function? Did it give him the result he wanted? Explain why or why not.

8. For the function $f(x) = \frac{7x - m}{2 - nx}$, find the values of m and n such that the vertical asymptote is at $x = 6$ and the x-intercept is 5.

9. Create a rational function that has a domain of $\{x \in \mathbf{R} | x \neq -2\}$ and no vertical asymptote. Describe the graph of this function.

5.4 Solving Rational Equations

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Connect the solution to a rational equation with the graph of a rational function.

LEARN ABOUT the Math

When they work together, Stuart and Lucy can deliver flyers to all the homes in their neighbourhood in 42 min. When Lucy works alone, she can finish the deliveries in 13 min less time than Stuart can when he works alone.

? When Stuart works alone, how long does he take to deliver the flyers?

EXAMPLE 1 Selecting a strategy to solve a rational equation

Determine the time that Stuart takes to deliver the flyers when he works alone.

Solution A: Creating an equation and solving it using algebra

Let s minutes be the time that Stuart takes to deliver the flyers when working alone.

Lucy takes $(s - 13)$ minutes when working alone.

Choose a variable to represent Stuart's time and use it to write an expression for Lucy's time.

Lucy delivers the flyers in 13 min less time than Stuart.

The fraction of deliveries made in one minute

- by Stuart working alone is $\frac{1}{s}$
- by Lucy working alone is $\frac{1}{s - 13}$
- by Stuart and Lucy working together is $\frac{1}{42}$

$$\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$$

Compare the rates at which they work.
For example, if Stuart took 80 min to deliver all the flyers, he would deliver $\frac{1}{80}$ of the flyers per minute.
$s > 13$ because Stuart takes longer than Lucy to deliver the flyers, and it is not possible for the denominators to be zero.

Multiply by the LCD.

$$42s(s-13)\left(\frac{1}{s}+\frac{1}{s-13}\right)=42s(s-13)\left(\frac{1}{42}\right)$$

$$\frac{42s(s-13)}{s}+\frac{42s(s-13)}{s-13}=\frac{42s(s-13)}{42}$$

$$\frac{42\overset{1}{\cancel{s}}(s-13)}{\underset{1}{\cancel{s}}}+\frac{42s\overset{1}{\cancel{(s-13)}}}{\underset{1}{\cancel{s-13}}}=\frac{\overset{1}{\cancel{42}}s(s-13)}{\underset{1}{\cancel{42}}}$$

$$42(s-13)+42s=s(s-13)$$

There are no common factors in the denominators, so the LCD (lowest common denominator) is the product of the three denominators $42s(s-13)$. Multiply each term by the LCD, and then simplify the resulting rational expressions to remove all the denominators.

$$\begin{aligned}42s-546+42s&=s^2-13s\\0&=s^2-97s+546\\0&=(s-91)(s-6)\\s&=6\text{ or }91\end{aligned}$$

Solve the quadratic equation by factoring or by using the quadratic formula.

$s > 13$ so 6 is not an admissible solution.

Remember to look for inadmissible solutions by carefully considering both the context and the information given in the problem.

$$\begin{aligned}\text{LS}&=\frac{1}{s}+\frac{1}{s-13}\\&=\frac{1}{91}+\frac{1}{91-13}\\&=\frac{1}{91}+\frac{1}{78}\\&=\frac{78+91}{7098}\\&=\frac{169}{7098}\\&=\frac{1}{42}\end{aligned}$$

Check the solution, $s = 91$, by substituting it into the original equation.

$$\text{RS}=\frac{1}{42}$$

Since LS = RS, $s = 91$ is the solution.

It will take Stuart 91 min to deliver the flyers when working alone.

Solution B: Using the graph of a rational function to solve a rational equation

The equation that models the problem is $\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$, where s represents the time, in minutes, that Stuart takes to deliver the flyers when working alone.

$$\frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42} = 0$$

Subtract $\frac{1}{42}$ from each side.

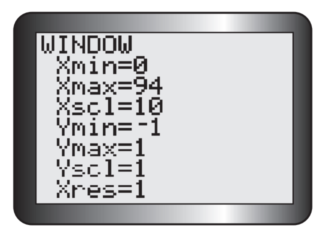
To solve the equation, find the zeros of the function $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

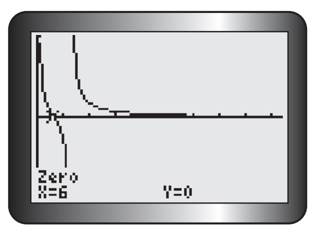
Graph $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

Use the zero operation to determine the zeros.

From the equation, you can expect the graph to have vertical asymptotes at $s = 0$ and $s = 13$.

Since you are only interested in finding the zeros, you can limit the y-values to those close to zero.





Tech Support

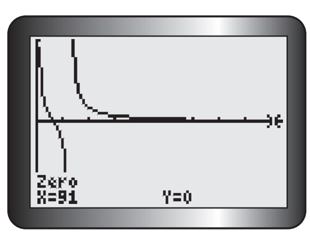
For help determining the zeros on a graphing calculator, see Technical Appendix, T-8.

The first zero for $f(s)$ is $s = 6$.
Reject this solution since $s > 13$.

You know that Lucy takes 13 min less time than Stuart takes, so Stuart must take longer than 13 min.

The solution $s = 6$ is inadmissible.

Determine the other zero.



The solution is $s = 91$.

Stuart takes 91 min to deliver the flyers when working alone.

Reflecting

A. In Solution A, explain how a rational equation was created using the times given in the problem.

B. In Solution B, explain how finding the zeros of a rational function provided the solution to the problem.

C. Where did the inadmissible root obtained in Solution A show up in the graphical solution in Solution B? How was this root dealt with?

APPLY the Math

EXAMPLE 2 Using an algebraic strategy to solve simple rational equations

Solve each rational equation.

a) $\frac{x-2}{x-3} = 0$ **b)** $\frac{x+3}{x-4} = \frac{x-1}{x+2}$

Solution

a)

$$\frac{x-2}{x-3} = 0, x \neq 3$$

Determine any restrictions on the value of x.

$$\cancel{(x-3)}^{1}\frac{(x-2)}{\cancel{(x-3)}_{1}} = 0(x-3)$$

$$x - 2 = 0$$

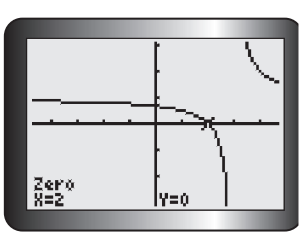
$$x = 2$$

Multiply both sides of the equation by the LCD, $(x - 3)$.

Add 2 to each side.

To verify, graph $f(x) = \frac{x-2}{x-3}$ and use the zero operation to determine the zero.

From the equation, the graph will have a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 1$.



The solution is $x = 2$.

b) $\dfrac{x+3}{x-4} = \dfrac{x-1}{x+2},\ x \neq -2, 4$

Note the restrictions.

$$(x-4)(x+2)\left(\frac{x+3}{x-4}\right) = (x-4)(x+2)\left(\frac{x-1}{x+2}\right)$$

Multiply each side of the equation by the LCD, $(x-4)(x+2)$.

$$\overset{1}{\cancel{(x-4)}}(x+2)\left(\frac{x+3}{\underset{1}{\cancel{x-4}}}\right) = (x-4)\overset{1}{\cancel{(x+2)}}\left(\frac{x-1}{\underset{1}{\cancel{x+2}}}\right)$$

Simplify.

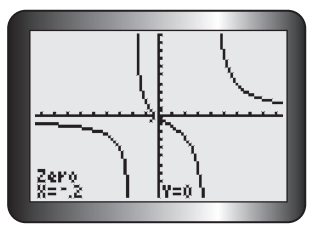
$$(x+2)(x+3) = (x-4)(x-1)$$
$$x^2 + 5x + 6 = x^2 - 5x + 4$$

Expand. Subtract x^2, and add $5x$ to both sides.

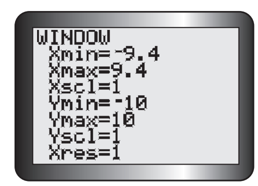
$$10x + 6 = 4$$
$$10x = -2$$
$$x = -0.2$$

Solve the resulting linear equation.

To verify, graph $f(x) = \dfrac{x+3}{x-4} - \dfrac{x-1}{x+2}$ and use the zero operation to determine the zero.



Adjust the window settings so you can view enough of the graph to see all the possible zeros.



The solution is $x = -0.2$.

EXAMPLE 3 Connecting the solution to a problem with the zeros of a rational function

Salt water is flowing into a large tank that contains pure water. The concentration of salt, c, in the tank at t minutes is given by $c(t) = \dfrac{10t}{25+t}$, where c is measured in grams per litre. When does the salt concentration in the tank reach 3.75 g/L?

Solution

If the salt concentration is 3.75, $c(t) = 3.75$.

$$\frac{10t}{25+t} = 3.75$$

$$(25+t)\left(\frac{10t}{25+t}\right) = 3.75(25+t)$$

$$\overset{1}{\cancel{(25+t)}}\frac{10t}{\underset{1}{\cancel{(25+t)}}} = 3.75(25+t)$$

$$10t = 93.75 + 3.75t$$

Set the function expression equal to 3.75. $25 + t \neq 0$, and because t measures the time since the salt water started flowing, $t \geq 0$.

Multiply both sides of the equation by the LCD, $(25 + t)$, and solve the resulting linear equation.

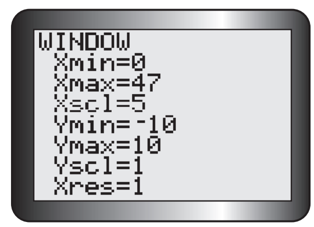
$$10t - 3.75t = 93.75$$
$$6.25t = 93.75$$
$$\frac{6.25t}{6.25} = \frac{93.75}{6.25}$$
$$t = 15$$

Use inverse operations to solve for t.

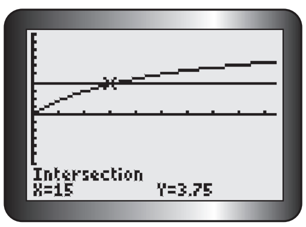
It takes 15 min for the salt concentration to reach 3.75 g/L.

To verify, graph $f(t) = \frac{10t}{25 + t}$ and $g(t) = 3.75$, and determine where the functions intersect.

Use an appropriate window setting, based on the domain, $t \geq 0$.



Use the intersect operation.



The salt concentration reaches 3.75 g/L after 15 min.

Tech Support

For help determining the point of intersection between two functions, see Technical Appendix, T-12.

EXAMPLE 4 Using a rational function to model and solve a problem

Rima bought a case of concert T-shirts for \$450. She kept two T-shirts for herself and sold the rest for \$560, making a profit of \$10 on each T-shirt. How many T-shirts were in the case?

Solution

Let the number of T-shirts in the case be x.

Buying price per T-shirt $= \frac{450}{x}$

Selling price per T-shirt $= \frac{560}{x - 2}$

Rima paid \$450 for x T-shirts, so each T-shirt cost her $\$\frac{450}{x}$.

She kept two for herself, which left $x - 2$ T-shirts for her to sell.

Rima sold $x - 2$ T-shirts for \$560, so she charged $\$\frac{560}{x - 2}$ for each one.

$$\frac{560}{x-2} - \frac{450}{x} = 10$$

She made a profit of \$10 on each T-shirt, so the difference between the selling price and the buying price was \$10.

$$x(x-2)\left(\frac{560}{x-2} - \frac{450}{x}\right) = 10x(x-2)$$

Multiply both sides of the equation by the LCD, $x(x-2)$.

$$\frac{560x\cancel{(x-2)}^{1}}{\cancel{x-2}_{1}} - \frac{450\cancel{x}^{1}(x-2)}{\cancel{x}_{1}} = 10x(x-2)$$

$$560x - 450(x-2) = 10x(x-2)$$
$$560x - 450x + 900 = 10x^2 - 20x$$
$$0 = 10x^2 - 130x - 900$$

Expand and collect all terms to one side of the equation.

$$0 = 10(x^2 - 13x - 90)$$
$$0 = 10(x-18)(x+5)$$
$$x = 18 \text{ or } -5$$

Solve the resulting quadratic equation by factoring.

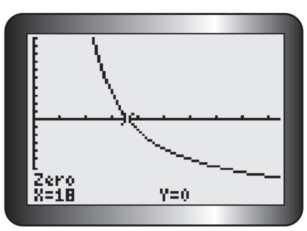
-5 is inadmissible since $x \geq 0$.
There were 18 T-shirts in the case.

You cannot have a negative number of T-shirts in the case.

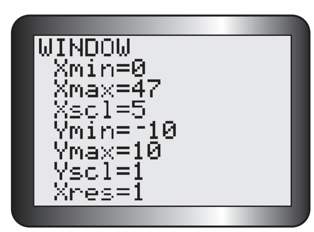
To verify, graph $f(x) = \frac{560}{x-2} - \frac{450}{x} - 10$ and determine the zeros using the zero operation.

If $\frac{560}{x-2} - \frac{450}{x} = 10$, then $\frac{560}{x-2} - \frac{450}{x} - 10 = 0$.

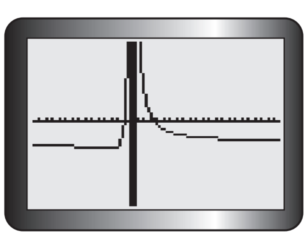
Zeros for $f(x)$ are possible solutions to the problem.



Use an appropriate window setting, based on the domain, $x \geq 0$.



The zero occurs when $x = 18$.
Zoom out to check that there are no other zeros in the domain.



The other zero is for a negative value of x, which is inadmissible in the context of this problem.

There is no other zero in the domain.
There were 18 T-shirts in the case.

In Summary

Key Ideas

- You can solve a rational equation algebraically by multiplying each term in the equation by the lowest common denominator and solving the resulting polynomial equation.
- The root of the equation $\frac{ax + b}{cx + d} = 0$ is the zero (x-intercept) of the function $f(x) = \frac{ax + b}{cx + d}$.
- You can use graphing technology to solve a rational equation or verify the solution. Determine the zeros of the corresponding rational function, or determine the intersection of two functions.

Need to Know

- The zeros of a rational function are the zeros of the function in the numerator.
- Reciprocal functions do not have zeros. All functions of the form $f(x) = \frac{1}{g(x)}$ have the x-axis as a horizontal asymptote. They do not intersect the x-axis.
- When solving contextual problems, it is important to check for inadmissible solutions that are outside the domain determined by the context.
- When using a graphing calculator to determine a zero or intersection point, you can avoid inadmissible roots by matching the window settings to the domain of the function in the context of the problem.

CHECK Your Understanding

1. Are $x = 3$ and $x = -2$ solutions to the equation $\frac{2}{x} = \frac{x - 1}{3}$? Explain how you know.

2. Solve each equation algebraically. Then verify your solution using graphing technology.

a) $\frac{x + 3}{x - 1} = 0$

b) $\frac{x + 3}{x - 1} = 2$

c) $\frac{x + 3}{x - 1} = 2x + 1$

d) $\frac{3}{3x + 2} = \frac{6}{5x}$

3. For each rational equation, write a function whose zeros are the solutions.

a) $\frac{x - 3}{x + 3} = 2$

b) $\frac{3x - 1}{x} = \frac{5}{2}$

c) $\frac{x - 1}{x} = \frac{x + 1}{x + 3}$

d) $\frac{x - 2}{x + 3} = \frac{x - 4}{x + 5}$

4. Solve each equation in question 3 algebraically, and verify your solution using a graphing calculator.

PRACTISING

5. Solve each equation algebraically.

a) $\frac{2}{x} + \frac{5}{3} = \frac{7}{x}$

b) $\frac{10}{x+3} + \frac{10}{3} = 6$

c) $\frac{2x}{x-3} = 1 - \frac{6}{x-3}$

d) $\frac{2}{x+1} + \frac{1}{x+1} = 3$

e) $\frac{2}{2x+1} = \frac{5}{4-x}$

f) $\frac{5}{x-2} = \frac{4}{x+3}$

6. Solve each equation algebraically.

a) $\frac{2x}{2x+1} = \frac{5}{4-x}$

b) $\frac{3}{x} + \frac{4}{x+1} = 2$

c) $\frac{2x}{5} = \frac{x^2 - 5x}{5x}$

d) $x + \frac{x}{x-2} = 0$

e) $\frac{1}{x+2} + \frac{24}{x+3} = 13$

f) $\frac{-2}{x-1} = \frac{x-8}{x+1}$

7. Solve each equation using graphing technology. Round your answers to two decimal places, if necessary.

a) $\frac{2}{x+2} = \frac{3}{x+6}$

b) $\frac{2x-5}{x+10} = \frac{1}{x-6}$

c) $\frac{1}{x-3} = \frac{x+2}{7x+14}$

d) $\frac{1}{x} - \frac{1}{45} = \frac{1}{2x-3}$

e) $\frac{2x+3}{3x-1} = \frac{x+2}{4}$

f) $\frac{1}{x} = \frac{2}{x} + 1 + \frac{1}{1-x}$

8. K a) Use algebra to solve $\frac{x+1}{x-2} = \frac{x+3}{x-4}$. Explain your steps.

b) Verify your answer in part a) using substitution.

c) Verify your answer in part a) using a graphing calculator.

9. The Greek mathematician Pythagoras is credited with the discovery of the Golden Rectangle. This is considered to be the rectangle with the dimensions that are the most visually appealing. In a Golden Rectangle, the length and width are related by the proportion $\frac{l}{w} = \frac{w}{l-w}$. A billboard with a length of 15 m is going to be built. What must its width be to form a Golden Rectangle?

10. The Turtledove Chocolate factory has two chocolate machines. Machine A takes s minutes to fill a case with chocolates, and machine B takes $s + 10$ minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

11. Tayla purchased a large box of comic books for \$300. She gave 15 of the comic books to her brother and then sold the rest on an Internet website for \$330, making a profit of \$1.50 on each one. How many comic books were in the box? What was the original price of each comic book?

12. **A** Polluted water flows into a pond. The concentration of pollutant, c, in the pond at time t minutes is modelled by the equation $c(t) = 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)$, where c is measured in kilograms per cubic metre.

a) When will the concentration of pollutant in the pond reach 6 kg/m^3?

b) What will happen to the concentration of pollutant over time?

13. **T** Three employees work at a shipping warehouse. Tom can fill an order in s minutes. Paco can fill an order in $s - 2$ minutes. Carl can fill an order in $s + 1$ minutes. When Tom and Paco work together, they take about 1 minute and 20 seconds to fill an order. When Paco and Carl work together, they take about 1 minute and 30 seconds to fill an order.

a) How long does each person take to fill an order?

b) How long would all three of them, working together, take to fill an order?

14. **C** Compare and contrast the different methods you can use to solve a rational equation. Make a list of the advantages and disadvantages of each method.

Extending

15. Solve $\frac{x^2 - 6x + 5}{x^2 - 2x - 3} = \frac{2 - 3x}{x^2 + 3x + 3}$ correct to two decimal places.

16. Objects A and B move along a straight line. Their positions, s, with respect to an origin, at t seconds, are modelled by the following functions:

Object A: $s(t) = \frac{7t}{t^2 + 1}$

Object B: $s(t) = t + \frac{5}{t + 2}$

a) When are the objects at the same position?

b) When is object A closer to the origin than object B?

5.5 Solving Rational Inequalities

YOU WILL NEED

- graphing calculator

GOAL

Solve rational inequalities using algebraic and graphical approaches.

LEARN ABOUT the Math

rational inequality
a statement that one rational expression is less than or greater than another rational expression $\left(\text{e.g., } \frac{2x}{x+3} > \frac{x-1}{5x}\right)$

The function $P(t) = \frac{20t}{t+1}$ models the population, in thousands, of Nickelford, t years after 1997. The population, in thousands, of nearby New Ironfield is modelled by $Q(t) = \frac{240}{t+8}$.

? How can you determine the time period when the population of New Ironfield exceeded the population of Nickelford?

EXAMPLE 1 Selecting a strategy to solve a problem

Determine the interval(s) of t where the values of $Q(t)$ are greater than the values of $P(t)$.

Solution A: Using an algebraic strategy to solve an inequality

$$\frac{240}{t+8} > \frac{20t}{t+1}$$

The population of New Ironfield exceeds the population of Nickelford when $Q(t) > P(t)$. $t \geq 0$ in the context of this problem. There are no other restrictions on the expressions in the rational inequality since the values that make both expressions undefined are negative numbers.

$$(t+8)(t+1)\left(\frac{240}{t+8}\right) > (t+8)(t+1)\left(\frac{20t}{t+1}\right)$$

$$\overset{1}{\cancel{(t+8)}}(t+1)\left(\frac{240}{\underset{1}{\cancel{t+8}}}\right) > (t+8)\overset{1}{\cancel{(t+1)}}\left(\frac{20t}{\underset{1}{\cancel{t+1}}}\right)$$

Multiply both sides of the inequality by the LCD. The value of the LCD is always positive, since $t \geq 0$, so the inequality sign is unchanged.

$$\begin{aligned} 240(t+1) &> 20t(t+8) \\ 240t + 240 &> 20t^2 + 160t \\ 0 &> 20t^2 + 160t - 240t - 240 \\ 0 &> 20t^2 - 80t - 240 \\ 0 &> 20(t^2 - 4t - 12) \\ 0 &> 20(t-6)(t+2) \end{aligned}$$

Expand and simplify both sides. Then subtract $240t$ and 240 from both sides.

Factor the resulting quadratic expression.

Examine the sign of the factored polynomial expression on the right side of the inequality.

	$t < -2$	$-2 < t < 6$	$t > 6$
$20(t-6)$	$-$	$-$	$+$
$t+2$	$-$	$+$	$+$
$20(t-6)(t+2)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$

The inequality $0 > 20(t-6)(t+2)$ is true when the expression on the right side is negative. The sign of the factored quadratic expression changes when $t = -2$ and when $t = 6$, because the expression is zero at these values. Use a table to determine when the sign of the expression is negative on each side of these values.

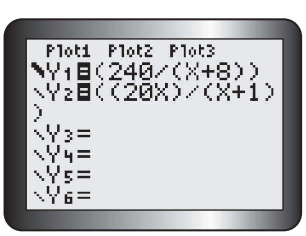
The inequality $0 > 20(t-6)(t+2)$ is true when $-2 < t < 6$.

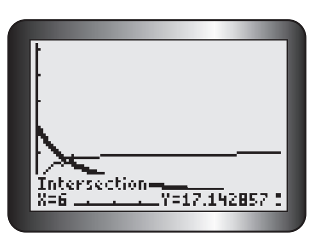
The population of New Ironfield exceeded the population of Nickelford for six years after 1997, until 2003.

Since the domain is $t \geq 0$, however, numbers that are negative cannot be included. Therefore, the solution is $0 \leq t < 6$.

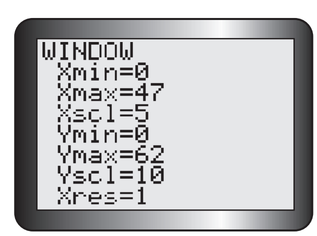
Solution B: Solving a rational inequality by graphing two rational functions

To solve $Q(t) > P(t)$, graph $Q(t) = \frac{240}{t+8}$ and $P(t) = \frac{20t}{t+1}$ using graphing technology, and determine the value of t at the intersection point(s).





It helps to bold the graph of $Q(t)$ so you can remember which graph is which. Use window settings that reflect the domain of the functions.



There is only one intersection within the domain of the functions.

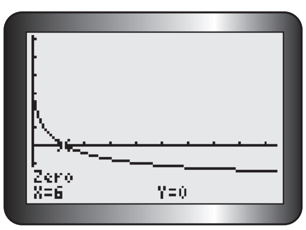
From the graphs, $Q(t) > P(t)$ for $0 \leq t < 6$.
The population of New Ironfield exceeded the population of Nickelford until 2003.

If $Q(t) > P(t)$, the graph of $Q(t)$ lies above the graph of $P(t)$. Looking at the graphs, this is true for the parts of the graph of $Q(t)$ up to the intersection point at $t = 6$. The graphs will not intersect again because each graph is approaching a different horizontal asymptote. From the defining equations, the graph of $Q(t)$ is approaching the line $Q = 0$ while the graph of $P(t)$ is approaching the line $P = 20$.

Solution C: Solving a rational inequality by determining the zeros of a combined function

When $Q(t) > P(t)$, $Q(t) - P(t) > 0$.

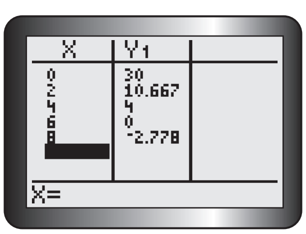
Graph $f(t) = Q(t) - P(t) = \dfrac{240}{t + 8} - \dfrac{20t}{t + 1}$ and use the zero operation to locate the zero.



Combine the two population functions into a single function, $f(t) = Q(t) - P(t)$. When $Q(t) > P(t)$, $f(t)$ will have positive values.

When a function has positive values, its graph lies above the x-axis.

The graph is above the x-axis for $0 \leq t < 6$.



By examining the values of $f(t)$ in a table, you can verify that the function continues to decrease but remains positive when $0 \leq t < 6$.

$f(t)$ has positive values for $0 \leq t < 6$.
For the six years after 1997, the population of New Ironfield exceeded the population of Nickelford.

Reflecting

A. How is the solution to an inequality different from the solution to an equation?

B. In Solution A, how was the rational inequality manipulated to obtain a simpler quadratic inequality?

C. In Solution B, how were the graphs of the related rational functions used to find the solution to an inequality?

D. In Solution C, how did creating a new function help to solve the inequality?

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an inequality that involves a linear function and a reciprocal function

Solve $x - 2 < \frac{8}{x}$.

Solution A: Using an algebraic strategy and a sign chart

$$x - 2 < \frac{8}{x}, x \neq 0$$

$$x - 2 - \frac{8}{x} < 0$$

$$\frac{x^2}{x} - \frac{2x}{x} - \frac{8}{x} < 0$$

$$\frac{x^2 - 2x - 8}{x} < 0$$

$$\frac{(x - 4)(x + 2)}{x} < 0$$

Determine any restrictions on x. Subtract $\frac{8}{x}$ from both sides.

x is the LCD and it can be positive or negative. Multiplying both sides by x would require that two cases be considered, since the inequality sign must be reversed when multiplying by a negative. The alternative is to create an expression with a common denominator, x.

Combine the terms to create a single rational expression.

Factor the numerator.

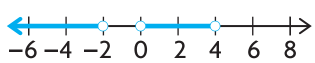
Examine the sign of the rational expression.

	$x < -2$	$-2 < x < 0$	$0 < x < 4$	$x > 4$
$x - 4$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$\dfrac{(x-4)(x+2)}{x}$	$\dfrac{(-)(-)}{-} = -$	$\dfrac{(-)(+)}{-} = +$	$\dfrac{(-)(+)}{+} = -$	$\dfrac{(+)(+)}{+} = +$

The sign of a rational expression changes each time the sign of one of its factors changes. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is negative.

The overall expression is negative when $x < -2$ or when $0 < x < 4$.

The inequality is true when $x \in (-\infty, -2)$ or $x \in (0, 4)$.



Write the solution in interval or set notation, and draw the solution set on a number line.

Solution B: Using graphing technology

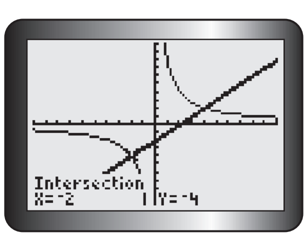
$x - 2 < \frac{8}{x}, x \neq 0$

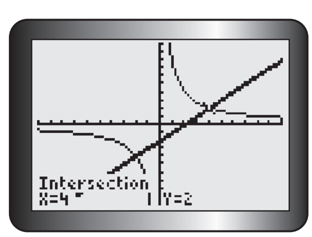
Let $f(x) = x - 2$ and $g(x) = \frac{8}{x}$.

The solution set for the inequality will be all x-values for which $f(x) < g(x)$.

Write each side of the inequality as its own function. Enter both functions in the equation editor, using a bold line for $f(x)$.

Graph $f(x)$ and $g(x)$ on the same axes, and use the intersect operation to determine the intersection points.





$f(x) < g(x)$ where the bold graph of $f(x)$ lies beneath the graph of $g(x)$. Notice that the bold linear function is above the reciprocal function on the left side and close to the vertical asymptote, $x = 0$. It is below the reciprocal function on the right side and close to this asymptote.

$f(x) < g(x)$ when $x < -2$ or when $0 < x < 4$.

The solution set is $\{x \in \mathbf{R} | x < -2 \text{ or } 0 < x < 4\}$.

You can also use interval notation or a number line to describe the solution set, as in Solution A.

EXAMPLE 3 Determining the solution set for an inequality that involves two rational functions

Determine the solution set for the inequality $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$.

Solution A: Using algebra and a sign chart

Rewrite $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$, $x \neq -1, 3$,

as $\frac{x+3}{x+1} - \frac{x-2}{x-3} \geq 0$.

Note the restrictions on x.

Subtract $\frac{x-2}{x-3}$ from both sides to create an inequality with zero on the right side.

$$\frac{(x-3)(x+3)}{(x-3)(x+1)} - \frac{(x-2)(x+1)}{(x-3)(x+1)} \geq 0$$

$$\frac{x^2 - 9 - (x^2 - x - 2)}{(x-3)(x+1)} \geq 0$$

Subtract the rational expressions on the left side using a common denominator.

$$\frac{x^2 - 9 - x^2 + x + 2}{(x-3)(x+1)} \geq 0$$

Expand and simplify the numerator.

$$\frac{x-7}{(x-3)(x+1)} \geq 0$$

A rational expression is zero when its numerator is zero.

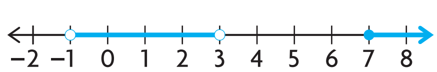
The rational expression is equal to zero when $x = 7$, so 7 is included in the solution set.

Examine the sign of the simplified rational expression on the intervals shown to determine where the rational expression is greater than zero.

	$x < -1$	$-1 < x < 3$	$3 < x < 7$	$x > 7$
$x - 7$	$-$	$-$	$-$	$+$
$x - 3$	$-$	$-$	$+$	$+$
$x + 1$	$-$	$+$	$+$	$+$
$\frac{(x-7)}{(x-3)(x+1)}$	$\frac{-}{(-)(-)} = -$	$\frac{-}{(-)(+)} = +$	$\frac{-}{(+)(+)} = -$	$\frac{+}{(+)(+)} = +$

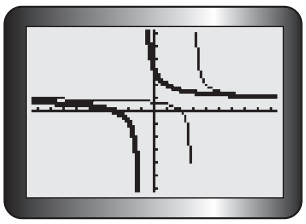
The expression is undefined at $x = -1$ and $x = 3$. It is equal to 0 at $x = 7$. These numbers create four intervals to consider. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is positive.

The solution set is $\{x \in \mathbf{R} \mid -1 < x < 3 \text{ or } x \geq 7\}$.



Solution B: Using graphing technology

$\frac{x+3}{x+1} \geq \frac{x-2}{x-3}, x \neq -1, 3$

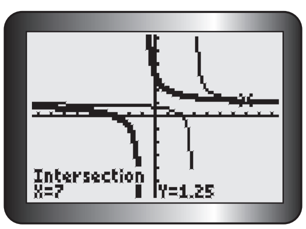


Use each side of the inequality to define a function. Graph $f(x) = \frac{x+3}{x+1}$ with a bold line and $g(x) = \frac{x-2}{x-3}$ with a regular line.

The graph of $f(x)$ has a vertical asymptote at $x = -1$. The graph for $g(x)$ has a vertical asymptote at $x = 3$. Both graphs have $y = 1$ as a horizontal asymptote.

Determine the equations of the asymptotes from the equations of the functions.

Use the intersect operation to locate any intersection points.



It looks as though the graphs might intersect on the left side of the screen, as well as on the right side. No matter how far you trace along the left branches, however, you never reach a point where the y-value is the same on both curves.

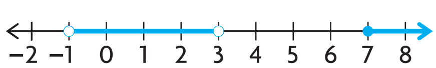
The functions are equal when $x = 7$.

$f(x) > g(x)$ between the asymptotes at $x = -1$ and $x = 3$, and for $x > 7$.

$f(x) = g(x)$ when $x = 7$.

The bold graph of $f(x)$ is above the graph of $g(x)$ between the two vertical asymptotes and then after the intersection point.

The solution set for $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$ is



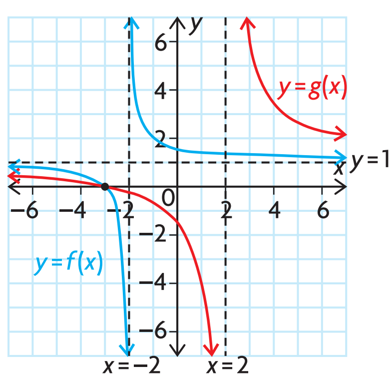
In Summary

Key Ideas

- Solving an inequality means finding all the possible values of the variable that satisfy the inequality.
- To solve a rational inequality algebraically, rearrange the inequality so that one side is zero. Combine the expressions on the no-zero side using a common denominator. Make a table to examine the sign of each factor and the sign of the entire expression on the intervals created by the zeros of the numerator and the denominator.
- Only when you are certain that each denominator is positive can you multiply both sides by the lowest common denominator to make the inequality easier to solve.
- You can always solve a rational inequality using graphing technology.

Need to Know

- When multiplying or dividing both sides of an inequality by a negative it is necessary to reverse the inequality sign to maintain equivalence.
- You can solve an inequality using graphing technology by graphing the functions on each side of the inequality sign and then identifying all the intervals created by the vertical asymptotes and points of intersection. For x-values that satisfy $f(x) > g(x)$, identify the specific intervals where the graph of $f(x)$ is above the graph of $g(x)$. For x-values that satisfy $f(x) < g(x)$, identify the specific intervals where the graph of $f(x)$ is below the graph of $g(x)$.
  Consider the following graph:



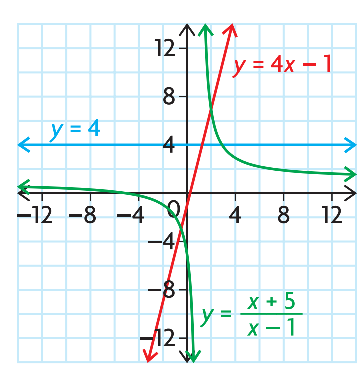
In this graph, there are four intervals to consider: $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$ and $(2, \infty)$. In these intervals, $f(x) > g(x)$ when $x \in (-\infty, -3)$ or $(-2, 2)$, and $f(x) < g(x)$ when $x \in (-3, -2)$ or $(2, \infty)$.

- You can also solve an inequality using graphing technology by creating an equivalent inequality with zero on one side and then identifying the intervals created by the zeros on the graph of the new function. Finding where the graph lies above the x-axis (where $f(x) > 0$) or below the x-axis (where $f(x) < 0$) defines the solutions to the inequality.

CHECK Your Understanding

1. Use the graph shown to determine the solution set for each of the following inequalities.

a) $\dfrac{x + 5}{x - 1} < 4$ **b)** $4x - 1 > \dfrac{x + 5}{x - 1}$



2. **a)** Show that the inequality $\dfrac{6x}{x + 3} \le 4$ is equivalent to the inequality $\dfrac{2(x - 6)}{(x + 3)} \le 0$.

b) Sketch the solution on a number line.

c) Write the solution using interval notation.

3. a) Show that the inequality $x + 2 > \frac{15}{x}$ is equivalent to the inequality $\frac{(x + 5)(x - 3)}{x} > 0$.

b) Use a table to determine the positive/negative intervals for $f(x) = \frac{(x + 5)(x - 3)}{x}$.

c) State the solution to the inequality using both set notation and interval notation.

PRACTISING

4. Use algebra to find the solution set for each inequality. Verify your answer using graphing technology.

a) $\frac{1}{x + 5} > 2$

b) $\frac{1}{2x + 10} < \frac{1}{x + 3}$

c) $\frac{3}{x - 2} < \frac{4}{x}$

d) $\frac{7}{x - 3} \geq \frac{2}{x + 4}$

e) $\frac{-6}{x + 1} > \frac{1}{x}$

f) $\frac{-5}{x - 4} < \frac{3}{x + 1}$

5. Use algebra to obtain a factorable expression from each inequality, if necessary. Then use a table to determine interval(s) in which the inequality is true.

a) $\frac{t^2 - t - 12}{t - 1} < 0$

b) $\frac{t^2 + t - 6}{t - 4} \geq 0$

c) $\frac{6t^2 - 5t + 1}{2t + 1} > 0$

d) $t - 1 < \frac{30}{5t}$

e) $\frac{2t - 10}{t} > t + 5$

f) $\frac{-t}{4t - 1} \geq \frac{2}{t - 9}$

6. Use graphing technology to solve each inequality.

a) $\frac{x + 3}{x - 4} \geq \frac{x - 1}{x + 6}$

b) $x + 5 < \frac{x}{2x + 6}$

c) $\frac{x}{x + 4} \leq \frac{1}{x + 1}$

d) $\frac{x}{x + 9} \geq \frac{1}{x + 1}$

e) $\frac{x - 8}{x} > 3 - x$

f) $\frac{x^2 - 16}{(x - 1)^2} \geq 0$

7. K a) Find all the values of x that make the following inequality true: $\frac{3x - 8}{2x - 1} > \frac{x - 4}{x + 1}$

b) Graph the solution set on a number line. Write the solution set using interval notation and set notation.

8. **a)** Use an algebraic strategy to solve the inequality $\frac{-6t}{t-2} < \frac{-30}{t-2}$.
b) Graph both inequalities to verify your solution.
c) Can these rational expressions be used to model a real-world situation? Explain.

9. A The equation $f(t) = \frac{5t}{t^2 + 3t + 2}$ models the bacteria count, in thousands, for a sample of tap water that is left to sit over time, t, in days. The equation $g(t) = \frac{15t}{t^2 + 9}$ models the bacteria count, in thousands, for a sample of pond water that is also left to sit over several days. In both models, $t > 0$. Will the bacteria count for the tap water sample ever exceed the bacteria count for the pond water? Justify your answer.

10. Consider the inequality $0.5x - 2 < \frac{5}{2x}$.
a) Rewrite the inequality so that there is a single, simplified expression on one side and a zero on the other side.
b) List all the factors of the rational expression in a table, and determine on which intervals the inequality is true.

11. T An economist for a sporting goods company estimates the revenue and cost functions for the production of a new snowboard. These functions are $R(x) = -x^2 + 10x$ and $C(x) = 4x + 5$, respectively, where x is the number of snowboards produced, in thousands. The average profit is defined by the function $AP(x) = \frac{P(x)}{x}$, where $P(x)$ is the profit function. Determine the production levels that make $AP(x) > 0$.

12. C **a)** Explain why the inequalities $\frac{x+1}{x-1} < \frac{x+3}{x+2}$ and $\frac{x+5}{(x-1)(x+2)} < 0$ are equivalent.
b) Describe how you would use a graphing calculator to solve these inequalities.
c) Explain how you would use a table to solve these inequalities.

Extending

13. Solve $\left|\frac{x}{x-4}\right| \geq 1$.

14. Solve $\frac{1}{\sin x} < 4$, $0° \leq x \leq 360°$.

15. Solve $\frac{\cos(x)}{x} > 0.5$, $0° < x < 90°$.

5.6 Rates of Change in Rational Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Determine average rates of change, and estimate instantaneous rates of change for rational functions.

LEARN ABOUT the Math

The instantaneous rate of change at a point on a revenue function is called the *marginal revenue*. It is a measure of the estimated additional revenue from selling one more item.

For example, the demand equation for a toothbrush is $p(x) = \frac{5}{2 + x}$, where x is the number of toothbrushes sold, in thousands, and p is the price, in dollars.

? What is the marginal revenue when 1500 toothbrushes are sold? When is the marginal revenue the greatest? When is it the least?

EXAMPLE 1 Selecting a strategy to determine instantaneous rates of change

Determine the marginal revenue when 1500 toothbrushes are sold and when it is the greatest and the least.

Solution A: Calculating the average rate of change by squeezing centred intervals around $x = 1.5$

Revenue $R(x) = xp(x)$

$$= \frac{5x}{2 + x}$$

Revenue = Number of items sold × Price

The average rate of change close to $x = 1.5$ is shown in the following table.

Centred Intervals	Average Rate of Change $\frac{R(x_2) - R(x_1)}{x_2 - x_1}$
$1.4 \le x \le 1.6$	0.817
$1.45 \le x \le 1.55$	0.816
$1.49 \le x \le 1.51$	0.816
$1.499 \le x \le 1.501$	0.816

x is measured in thousands, so when 1500 toothbrushes are sold, $x = 1.5$.

The average rate of change from $(x_1, R(x_1))$ to $(x_2, R(x_2))$ is the slope of the secant that joins each pair of endpoints.

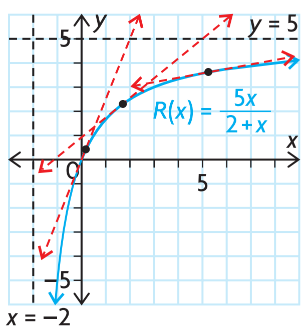
The average rate of change approaches 0.816. The marginal revenue when 1500 toothbrushes are sold is \$0.82 per toothbrush.

When x_1 and x_2 are very close to each other, the slope of the secant is approximately the same as the slope of the tangent. The slopes of the secants near the point where $x = 1.5$ approach 0.816.

Sketch the graph of $R(x) = \frac{5x}{2 + x}$.

The graph starts at (0, 0) and has a horizontal asymptote at $y = 5$.

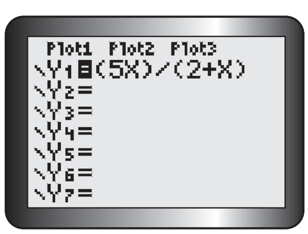
The vertical asymptote at $x = -2$ is not in the domain of $R(x)$, since $x \geq 0$, so it can be ignored.



In the context of the problem, x and $R(x)$ have only positive values. Examine the slope of the tangent lines at various points along the domain of the revenue graph. The slope is the greatest at the beginning of the graph and then decreases as x increases.

The marginal revenue is the greatest when $x = 0$ and then decreases from there, approaching zero, as the graph gets closer to the horizontal asymptote.

Solution B: Using the difference quotient and graphing technology to analyze the revenue function



Enter the revenue function into a graphing calculator.

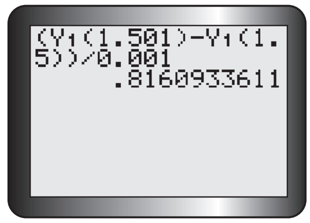
$$\text{Average rate of change} = \frac{R(a + h) - R(a)}{h}$$

Let $h = 0.001$

$$= \frac{R(1.5 + 0.001) - R(1.5)}{0.001}$$

$$= \frac{R(1.501) - R(1.5)}{0.001}$$

Use the difference quotient and a very small value for h, where $a = 1.5$, to estimate the instantaneous rate of change in revenue when 1500 toothbrushes are sold.

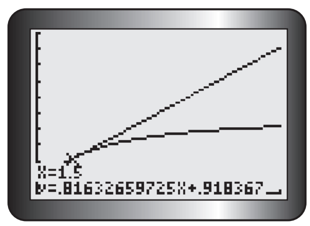


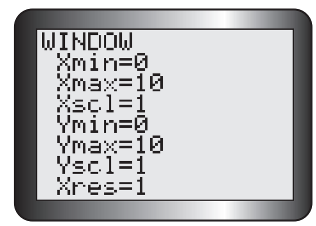
Enter the rate of change expression into the graphing calculator to determine its value, using the equation entered into Y1.

The average rate of change is about 0.816. The marginal revenue when 1500 toothbrushes are sold is \$0.82 per toothbrush.

To verify, graph the revenue function $R(x) = xp(x) = \frac{5x}{2 + x}$ and draw a tangent line at $x = 1.5$.

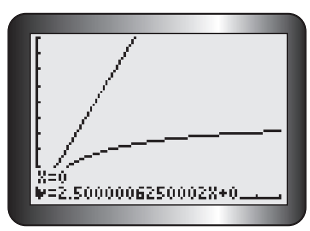
Since x and $R(x)$ only have positive values, graph the function in the first quadrant.



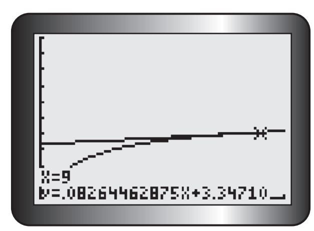


When 1500 toothbrushes are sold, the marginal revenue is \$0.82 per toothbrush.

Use the DRAW feature of the graphing calculator to draw a tangent line where $x = 1.5$.



The marginal revenue is the greatest when $x = 0$.



The marginal revenue decreases to very small values as x increases.

The tangent lines to this curve are steepest at the beginning of the curve. Their slopes decrease as x increases.

Reflecting

A. In Solution A, how were average rates of change used to estimate the instantaneous rate of change at a point?

B. In Solutions A and B, how were graphs used to estimate the instantaneous rate of change at a point?

C. In each solution, how was it determined where the marginal revenue was the greatest? Why was it not possible to determine the least marginal revenue?

D. What are the advantages and disadvantages of each method to determine the instantaneous rate of change?

APPLY the Math

EXAMPLE 2 Connecting the instantaneous rate of change to the slope of a tangent

a) Estimate the slope of the tangent to the graph of $f(x) = \frac{x}{x+3}$ at the point where $x = -5$.

b) Why can there not be a tangent line where $x = -3$?

Solution

a) $f(x) = \dfrac{x}{x+3}$

average rate of change $= \dfrac{f(a+h) - f(a)}{h}$

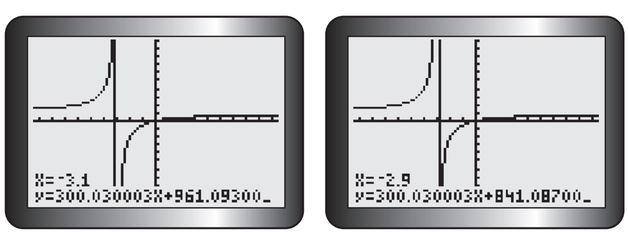
Let $h = 0.001$

$$= \frac{f(-5 + 0.001) - f(-5)}{0.001}$$
$$= \frac{f(-4.999) - f(-5)}{0.001}$$
$$= \frac{\left(\frac{-4.999}{-4.999 + 3}\right) - \left(\frac{-5}{-5+3}\right)}{0.001}$$
$$= \frac{2.500\,750\,375 - 2.5}{0.001}$$
$$\doteq 0.7504$$

Use the difference quotient and a very small value for h close to $a = -5$ to estimate the slope of the tangent where $x = -5$.

The slope of the tangent at $x = -5$ is 0.75.

b) The value -3 is not in the domain of $f(x)$, so no tangent line is possible there. The graph of $f(x)$ has a vertical asymptote at $x = -3$.



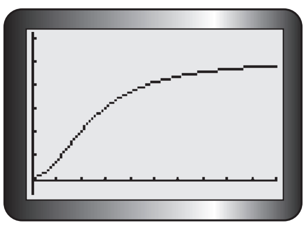
As x approaches -3 from the left and from the right, the tangent lines are very steep. The tangent lines approach a vertical line, but are never actually vertical. There is no point on the graph with an x-coordinate of -3, so there is no tangent line there.

EXAMPLE 3 Selecting a graphing strategy to solve a problem that involves average and instantaneous rates of change

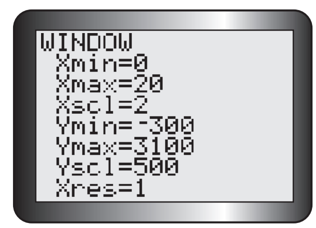
The snowshoe hare population in a newly created conservation area can be predicted over time by the model $p(t) = 50 + \frac{2500t^2}{25 + t^2}$, where p represents the population size and t is the time in years since the opening of the conservation area. Determine when the hare population will increase most rapidly, and estimate the instantaneous rate of change in population at this time.

Solution

Graph $p(t) = 50 + \frac{2500t^2}{25 + t^2}$ for $0 \leq t \leq 20$.

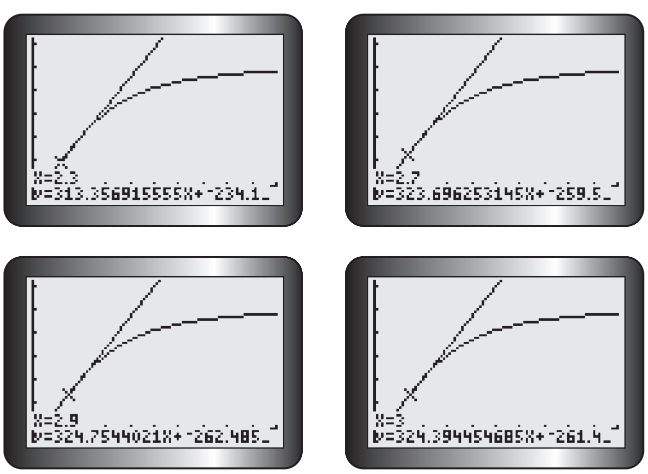


$p(t) \geq 0$, but you can set the minimum y-value to a negative number to allow some space for displayed values.



The slopes of the tangent lines increase slowly at the beginning of the graph. The slopes start to increase more rapidly around $t = 2$. They begin to decrease after $t = 3$.

Draw tangent lines between 2 and 3, and look for the tangent line that has the greatest slope.



The slopes of the tangent lines increase until 2.9, and then decrease. The tangent line at $t = 2.9$ has the greatest slope.

The average rate of change is greatest when t is close to 2.9. The hare population will increase most rapidly about 2 years and 11 months after the conservation area is opened. The instantaneous rate of change in population at this time is approximately 325 hares per year.

0.9×12 months is approximately 11 months.

In Summary

Key Ideas

- The methods that were previously used to calculate the average rate of change and estimate the instantaneous rate of change can be used for rational functions.
- You cannot determine the average and instantaneous rates of change of a rational function at a point where the graph is discontinuous (that is, where there is a hole or a vertical asymptote).

Need to Know

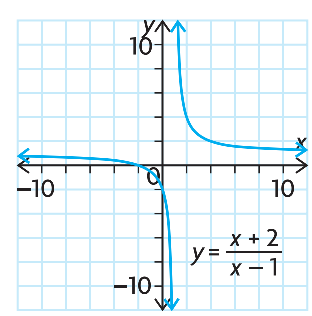
- The average rate of change of a rational function, $y = f(x)$, on the interval from $x_1 \le x \le x_2$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Graphically, this is equivalent to the slope of the secant line that passes through the points (x_1, y_1) and (x_2, y_2) on the graph of $y = f(x)$.
- The instantaneous rate of change of a rational function, $y = f(x)$, at $x = a$ can be approximated using the difference quotient $\frac{f(a + h) - f(a)}{h}$ and a very small value of h. Graphically, this is equivalent to estimating the slope of the tangent line that passes through the point $(a, f(a))$ on the graph of $y = f(x)$.
- The instantaneous rate of change at a vertical asymptote is undefined. The instantaneous rates of change at points that are approaching a vertical asymptote become very large positive or very large negative values. The instantaneous rate of change near a horizontal asymptote approaches zero.

CHECK Your Understanding

1. The graph of a rational function is shown.

a) Determine the average rate of change of the function over the interval $2 \le x \le 7$.

b) Copy the graph, and draw a tangent line at the point where $x = 2$. Determine the slope of the tangent line to estimate the instantaneous rate of change at this point.



2. Estimate the instantaneous rate of change of the function in question 1 at $x = 2$ by determining the slope of a secant line from the point where $x = 2$ to the point where $x = 2.01$. Compare your answer with your answer for question 1, part b).

3. Use graphing technology to estimate the instantaneous rate of change of the function in question 1 at $x = 2$.

PRACTISING

4. Estimate the instantaneous rate of change of $f(x) = \frac{x}{x - 4}$ at the point $(2, -1)$.

5. Select a strategy to estimate the instantaneous rate of change of each function at the given point.

K

a) $y = \frac{1}{25 - x}$, where $x = 13$

b) $y = \frac{17x + 3}{x^2 + 6}$, where $x = -5$

c) $y = \frac{x + 3}{x - 2}$, where $x = 4$

d) $y = \frac{-3x^2 + 5x + 6}{x + 6}$, where $x = -3$

6. Determine the slope of the line that is tangent to the graph of each function at the given point. Then determine the value of x at which there is no tangent line.

a) $f(x) = \frac{-5x}{2x + 3}$, where $x = 2$

b) $f(x) = \frac{x - 6}{x + 5}$, where $x = -7$

c) $f(x) = \frac{2x^2 - 6x}{3x + 5}$, where $x = -2$

d) $f(x) = \frac{5}{x - 6}$, where $x = 4$

7. When polluted water begins to flow into an unpolluted pond, the concentration of pollutant, c, in the pond at t minutes is modelled by $c(t) = \frac{27t}{10\,000 + 3t}$, where c is measured in kilograms per cubic metre. Determine the rate at which the concentration is changing after

A

a) 1 h **b)** one week

8. The demand function for snack cakes at a large bakery is given by the function $p(x) = \frac{15}{2x^2 + 11x + 5}$. The x-units are given in thousands of cakes, and the price per snack cake, $p(x)$, is in dollars.

a) Find the revenue function for the cakes.

b) Estimate the marginal revenue for $x = 0.75$. What is the marginal revenue for $x = 2.00$?

9. At a small clothing company, the estimated average cost function for producing a new line of T-shirts is $C(x) = \frac{x^2 - 4x + 20}{x}$, where x is the number of T-shirts produced, in thousands. $C(x)$ is measured in dollars.

a) Calculate the average cost of a T-shirt at a production level of 3000 pairs.

b) Estimate the rate at which the average cost is changing at a production level of 3000 T-shirts.

10. Suppose that the number of houses in a new subdivision after t months of development is modelled by $N(t) = \frac{100t^3}{100 + t^3}$, where N is the number of houses and $0 \leq t \leq 12$.

a) Calculate the average rate of change in the number of houses built over the first 6 months.

b) Calculate the instantaneous rate of change in the number of houses built at the end of the first year.

c) Graph the function using a graphing calculator. Discuss what happens to the rate at which houses were built in this subdivision during the first year of development.

11. T Given the function $f(x) = \frac{x - 2}{x - 5}$, determine an interval and a point where the average rate of change and the instantaneous rate of change are equal.

12. C **a)** The position of an object that is moving along a straight line at t seconds is given by $s(t) = \frac{3t}{t + 4}$, where s is measured in metres. Explain how you would determine the average rate of change of $s(t)$ over the first 6 s.

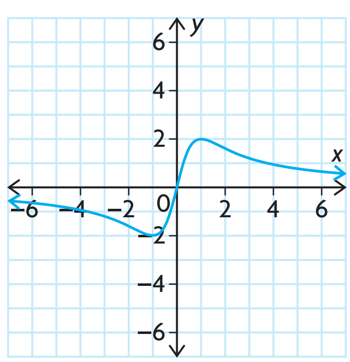
b) What does the average rate of change mean in this context?

c) Compare two ways that you could determine the instantaneous rate of change when $t = 6$. Which method is easier? Explain. Which method is more accurate? Explain.

d) What does the instantaneous rate of change mean in this context?

Extending

13. The graph of the rational function $f(x) = \frac{4x}{x^2 + 1}$ has been given the name Newton's Serpentine. Determine the equations for the tangents at the points where $x = -\sqrt{3}$, 0, and $\sqrt{3}$.



14. Determine the instantaneous rate of change of Newton's Serpentine at points around the point (0, 0). Then determine the instantaneous rate of change of this instantaneous rate of change.

5 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

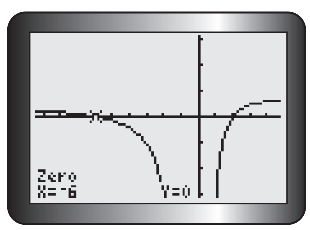
- See Lesson 5.4, Examples 1, 2, and 3.
- Try Chapter Review Questions 7, 8, and 9.

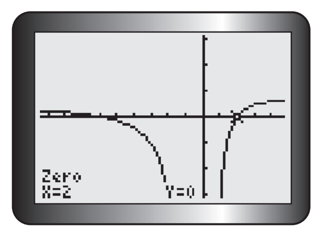
Q: How do you solve and verify a rational equation such as $\frac{3x - 8}{2x - 1} = \frac{x - 4}{x + 1}$?

A: You can solve a simple rational equation algebraically by multiplying each term in the equation by the lowest common denominator and then solving the resulting polynomial equation.

For example, to solve $\frac{3x - 8}{2x - 1} = \frac{x - 4}{x + 1}$, multiply the equation by $(2x - 1)(x + 1)$, where $x \neq -1$ or $\frac{1}{2}$. Then solve the resulting polynomial equation.

To verify your solutions, you can graph the corresponding function, $f(x) = \frac{3x - 8}{2x - 1} - \frac{x - 4}{x + 1}$, using graphing technology and determine the zeros of f.





The zeros are -6 and 2, so the solution to the equation is $x = -6$ or 2.

Study Aid

- See Lesson 5.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 10 and 11.

Q: How do you solve a rational inequality, such as $\frac{x - 2}{x + 1} > \frac{x - 6}{x - 2}$?

A1: You can solve a rational inequality algebraically by creating and solving an equivalent linear or polynomial inequality with zero on one side. For factorable polynomial inequalities of degree 2 or more, use a table to identify the positive/negative intervals created by the zeros and vertical asymptotes of the rational expression.

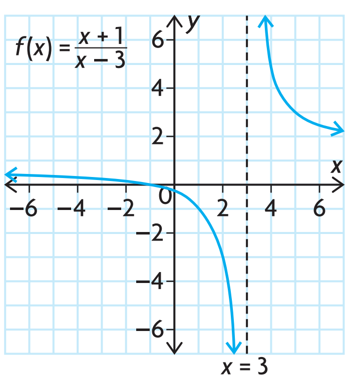
A2: You can use graphing technology to graph the functions on both sides of the inequality, determine their intersection and the locations of all vertical asymptotes, and then note the intervals of x that satisfy the inequality.

Q: How do you determine the average or instantaneous rate of change of a rational function?

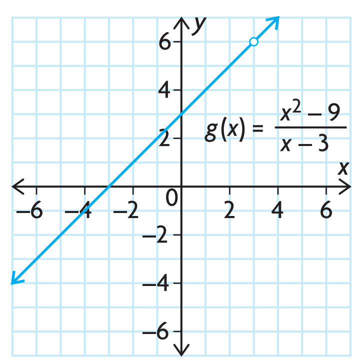
A: You can determine average and instantaneous rates of change of a rational function at points within the domain of the function using the same methods that are used for polynomial functions.

Q: When is it not possible to determine the average or instantaneous rate of change of a rational function?

A: You cannot determine the average and instantaneous rates of change of a rational function at a point where the graph has a hole or a vertical asymptote. You can only calculate the instantaneous rate of change at a point where the rational function is defined and where a tangent line can be drawn. A rational function is not defined at a point where there is a hole or a vertical asymptote. For example, $f(x) = \frac{x+1}{x-3}$ and $g(x) = \frac{x^2-9}{x-3}$ are rational functions that are not defined at $x = 3$.



The graph of $f(x)$ has a vertical asymptote at $x = 3$.



The graph of $g(x)$ has a hole at $x = 3$.

You cannot draw a tangent line on either graph at $x = 3$, so you cannot determine an instantaneous rate of change at this point.

Study Aid

- See Lesson 5.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 12, 13, and 14.

PRACTICE Questions

Lesson 5.1

1. For each function, determine the domain and range, intercepts, positive/negative intervals, and increasing and decreasing intervals. Use this information to sketch a graph of the reciprocal function.

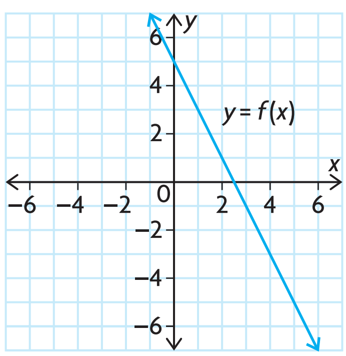
a) $f(x) = 3x + 2$

b) $f(x) = 2x^2 + 7x - 4$

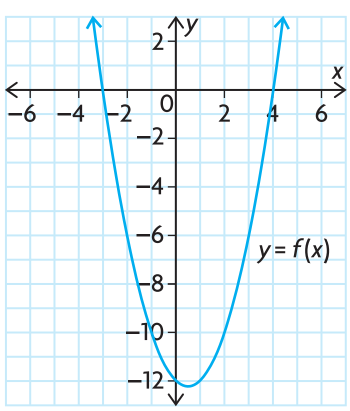
c) $f(x) = 2x^2 + 2$

2. Given the graphs of $f(x)$ below, sketch the graphs of $y = \frac{1}{f(x)}$.

a)



b)



Lesson 5.2

3. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal or oblique asymptotes.

a) $y = \frac{1}{x + 17}$

b) $y = \frac{2x}{5x + 3}$

c) $y = \frac{3x + 33}{-4x^2 - 42x + 22}$

d) $y = \frac{3x^2 - 2}{x - 1}$

Lesson 5.3

4. The population of locusts in a Prairie a town over the last 50 years is modelled by the function $f(x) = \frac{75x}{x^2 + 3x + 2}$. The locust population is given in hundreds of thousands. Describe the locust population in the town over time, where x is time in years.

5. For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \frac{2}{x + 5}$

b) $f(x) = \frac{4x - 8}{x - 2}$

c) $f(x) = \frac{x - 6}{3x - 18}$

d) $f(x) = \frac{4x}{2x + 1}$

6. Describe how you can determine the behaviour of the values of a rational function on either side of a vertical asymptote.

Lesson 5.4

7. Solve each equation algebraically, and verify your solution using a graphing calculator.

a) $\frac{x-6}{x+2} = 0$

b) $15x + 7 = \frac{2}{x}$

c) $\frac{2x}{x-12} = \frac{-2}{x+3}$

d) $\frac{x+3}{-4x} = \frac{x-1}{-4}$

8. A group of students have volunteered for the student council car wash. Janet can wash a car in m minutes. Rodriguez can wash a car in $m - 5$ minutes, while Nick needs the same amount of time as Janet. If they all work together, they can wash a car in about 3.23 minutes. How long does Janet take to wash a car?

9. The concentration of a toxic chemical in a spring-fed lake is given by the equation $c(x) = \frac{50x}{x^2 + 3x + 6}$, where c is given in grams per litre and x is the time in days. Determine when the concentration of the chemical is 6.16 g/L.

Lesson 5.5

10. Use an algebraic process to find the solution set of each inequality. Verify your answers using graphing technology.

a) $-x + 5 < \frac{1}{x+3}$

b) $\frac{55}{x+16} > -x$

c) $\frac{2x}{3x+4} > \frac{x}{x+1}$

d) $\frac{x}{6x-9} \leq \frac{1}{x}$

11. A biologist predicted that the population of tadpoles in a pond could be modelled by the function $f(t) = \frac{40t}{t^2+1}$, where t is given in days. The function that actually models the tadpole population is $g(t) = \frac{45t}{t^2 + 8t + 7}$. Determine where $g(t) > f(t)$.

Lesson 5.6

12. Estimate the slope of the line that is tangent to each function at the given point. At what point(s) is it not possible to draw a tangent line?

a) $f(x) = \frac{x+3}{x-3}$, where $x = 4$

b) $f(x) = \frac{2x-1}{x^2+3x+2}$, where $x = 1$

13. The concentration, c, of a drug in the bloodstream t hours after the drug was taken orally is given by $c(t) = \frac{5t}{t^2+7}$, where c is measured in milligrams per litre.

a) Calculate the average rate of change in the drug's concentration during the first 2 h since ingestion.

b) Estimate the rate at which the concentration of the drug is changing after exactly 3 h.

c) Graph $c(t)$ on a graphing calculator. When is the concentration of the drug increasing the fastest in the bloodstream? Explain.

14. Given the function $f(x) = \frac{2x}{x-4}$, determine the coordinates of a point on $f(x)$ where the slope of the tangent line equals the slope of the secant line that passes through $A(5, 10)$ and $B(8, 4)$.

15. Describe what happens to the slope of a tangent line on the graph of a rational function as the x-coordinate of the point of tangency

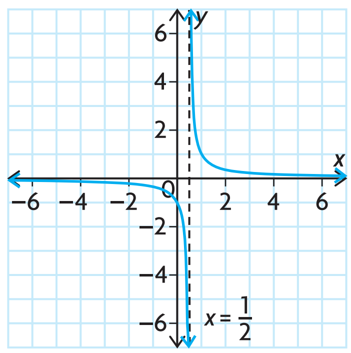
a) gets closer and closer to the vertical asymptote.

b) grows larger in both the positive and negative direction.

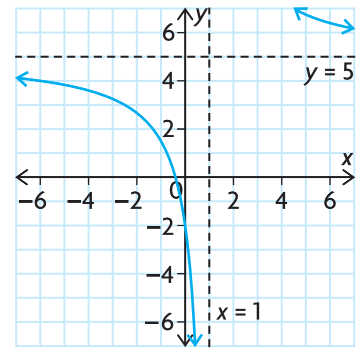
5 Chapter Self-Test

1. Match each graph with the equation of its corresponding function.

a)



b)



A $y = \dfrac{5x + 2}{x - 1}$ **B** $y = \dfrac{1}{2x - 1}$

2. Suppose that n is a constant and that $f(x)$ is a linear or quadratic function defined when $x = n$. Complete the following sentences.

a) If $f(n)$ is large, then $\dfrac{1}{f(n)}$ is....

b) If $f(n)$ is small, then $\dfrac{1}{f(n)}$ is....

c) If $f(n) = 0$, then $\dfrac{1}{f(n)}$ is....

d) If $f(n)$ is positive, then $\dfrac{1}{f(n)}$ is....

3. Without using graphing technology, sketch the graph of $y = \dfrac{2x + 6}{x - 2}$.

4. A company purchases x kilograms of steel for \$2249.52. The company processes the steel and turns it into parts that can be used in other factories. After this process, the total mass of the steel has dropped by 25 kg (due to trimmings, scrap, and so on), but the value of the steel has increased to \$10 838.52. The company has made a profit of \$2/kg. What was the original mass of the steel? What is the original cost per kilogram?

5. Select a strategy to solve each of the following.

a) $\dfrac{-x}{x - 1} = \dfrac{-3}{x + 7}$ **b)** $\dfrac{2}{x + 5} > \dfrac{3x}{x + 10}$

6. If you are given the equation of a rational function of the form $f(x) = \dfrac{ax + b}{cx + d}$, explain

a) how you can determine the equations of all vertical and horizontal asymptotes without graphing the function

b) when this type of function would have a hole instead of a vertical asymptote

5 Chapter Task

A New School

Researchers at a school board have developed models to predict population changes in the three areas they service. The models are $A(t) = \frac{360}{t + 6}$ for area A, $B(t) = \frac{30t}{t + 1}$ for area B, and $C(t) = \frac{50}{41 - 2t}$ for area C, where the population is measured in thousands and t is the time, in years, since 2007. The existing schools are full, and the board has agreed that a new school should be built.

? In which area should the new school be built, and when will the new school be needed?

A. Graph each population function for the 20 years following 2007. Use your graphs to describe the population trends in each area between 2007 and 2027.

B. Describe the intervals of increase or decrease for each function.

C. Determine which area will have the greatest population in 2010, 2017, 2022, and 2027.

D. Determine the intervals over which
- the population of area A is greater than the population of area B
- the population of area A is greater than the population of area C
- the population of area B is greater than the population of area C

E. Determine when the population of area B will be increasing most rapidly and when the population of area C will be increasing most rapidly.

F. What will happen to the population in each area over time?

G. Decide where and when the school should be built. Compile your results into a recommendation letter to the school board.

Task Checklist

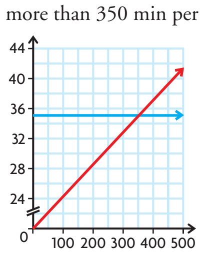
- ✔ Did you show all your steps?
- ✔ Did you draw and label your graphs accurately?
- ✔ Did you support your choice of location for the school?
- ✔ Did you explain your thinking clearly?

6. a) Answers may vary. For example, $2x + 1 > 17$
b) Answers may vary. For example, $3x - 4 \geq -16$
c) Answers may vary. For example, $2x + 3 \leq -21$
d) Answers may vary. For example, $-19 < 2x - 1 < -3$

7. a) $x \in \left(\frac{25}{2}, \infty\right)$
b) $x \in \left[-\frac{23}{8}, \infty\right)$
c) $x \in (-\infty, 2)$
d) $x \in (-\infty, 3]$

8. a) $\{x \in \mathbf{R} \mid -2 < x < 4\}$
b) $\{x \in \mathbf{R} \mid -1 \leq x \leq 0\}$
c) $\{x \in \mathbf{R} \mid -3 \leq x \leq 5\}$
d) $\{x \in \mathbf{R} \mid -6 < x < -2\}$

9. a) The second plan is better if one calls more than 350 min per month.
b)



10. a) $-1 < x < 2$
b) $x \leq -\frac{3}{2}$ or $x \geq 5$
c) $x < -\frac{5}{2}$ or $1 < x < 7$
d) $x \leq -4$ or $1 \leq x \leq 5$

11. negative when $x \in (0, 5)$, positive when $x \in (-\infty, -2), (-2, 0), (5, \infty)$

12. $x \leq -3.81$

13. between January 1993 and March 1994 and between October 1995 and October 1996

14. a) average $= 7$, instantaneous $\doteq 8$
b) average $= 13$, instantaneous $\doteq 15$
c) average $= 129$, instantaneous $\doteq 145$
d) average $= -464$, instantaneous $\doteq -485$

15. positive when $-1 < x < 1$, negative when $x < -1$ or $x > 1$, and zero at $x = -1, 1$

16. a) $t \doteq 2.2$ s
b) -11 m/s
c) about -22 m/s

17. a) about 57.002
b) about 56.998
c) Both approximate the instantaneous rate of change at $x = 3$.

18. a) male:
$f(x) = 0.001x^3 - 0.162x^2 + 3.394x + 72.365$;
female:
$g(x) = 0.0002x^3 - 0.026x^2 + 1.801x + 14.369$
b) More females than males will have lung cancer in 2006.
c) The rate was changing faster for females, on average. Looking only at 1975 and 2000, the incidence among males increased only 5.5 per 100 000, while the incidence among females increased by 31.7.
d) Between 1995 and 2000, the incidence among males decreased by 6.1 while the incidence among females increased by 5.6. Since 1998 is about halfway between 1995 and 2000, an estimate for the instantaneous rate of change in 1998 is the average rate of change from 1995 to 2000. The two rates of change are about the same in magnitude, but the rate for females is positive, while the rate for males is negative.

Chapter Self-Test, p. 242

1. $1, \frac{3}{2}, -2$

2. a) positive when $x < -2$ and $0 < x < 2$, negative when $-2 < x < 0$ and $x > 2$, and zero at $-2, 0, 2$
b) positive when $-1 < x < 1$, negative when $x < -1$ or $1 < x$, and zero at $x = -1, 1$
c) -1

3. a) Cost with card: $50 + 5n$; Cost without card: $12n$
b) at least 8 pizzas

4. a) $x < \frac{1}{2}$
b) $-2 \leq x \leq 1$
c) $-2 < x < -1$ or $x > 5$
d) $x \geq -3$

5. a) 15 m
b) 4.6 s
c) -3 m/s

6. a) about 5 b) $(1, 3)$ c) $y = 5x - 2$

7. Since all the exponents are even and all the coefficients are positive, all values of the function are positive and greater than or equal to 4 for all real numbers x.

8. a) $\{x \in \mathbf{R} \mid -2 \leq x \leq 7\}$
b) $-2 < x < 7$

9. 2 cm by 2 cm by 15 cm

Chapter 5

Getting Started, pp. 246–247

1. a) $(x - 5)(x + 2)$
b) $3(x + 5)(x - 1)$
c) $(4x - 7)(4x + 7)$
d) $(3x - 2)(3x - 2)$
e) $(a - 3)(3a + 10)$
f) $(2x + 3y)(3x - 7y)$

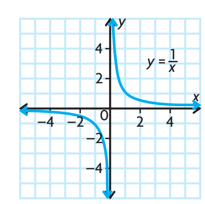
2. a) $3 - 2s$
b) $\frac{n^3}{3m}, m, n \neq 0$
c) $3x^2 - 4x - 1, x \neq 0$
d) $\frac{1}{5x - 2}, x \neq \frac{2}{5}$
e) $-\frac{x + 6}{3 + x}, x \neq -3, 3$
f) $\frac{a - b}{a - 3b}, a \neq -5b, \frac{3b}{2}$

3. a) $\frac{7}{15}$
b) $\frac{6}{x}, x \neq 0$
c) $\frac{-4x^2 + 20x - 6}{x - 3}, x \neq -2, 3$
d) $\frac{x^3 + 2x - 8x}{x^2 - 1}, x \neq -1, 0, 1, 3$

4. a) $1\frac{11}{21}$
b) $\frac{19x}{12}$
c) $\frac{4 + x}{x^2}, x \neq 0$
d) $\frac{3x - 6}{x^2 - 3x}, x \neq 0, 3$
e) $\frac{2x + 10 + y}{x^2 - 25}, x \neq 5, -5$
f) $\frac{-2a + 50}{(a + 3)(a - 5)(a + 3)}, x \neq -3, 4, 5$

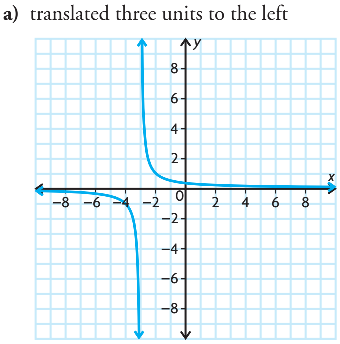
5. a) $x = 6$
b) $x = 2$
c) $x = 3$
d) $x = \frac{-12}{7}$

6.

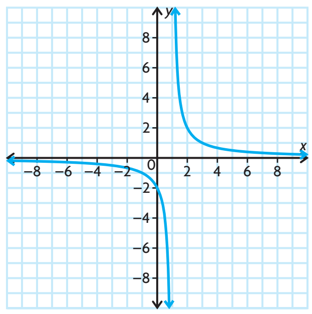


vertical: $x = 0$; horizontal: $y = 0$;
$D = \{x \in \mathbf{R} \mid x \neq 0\}$;
$R = \{y \in \mathbf{R} \mid y \neq 0\}$

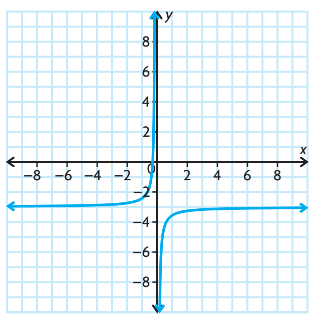
7. a) translated three units to the left



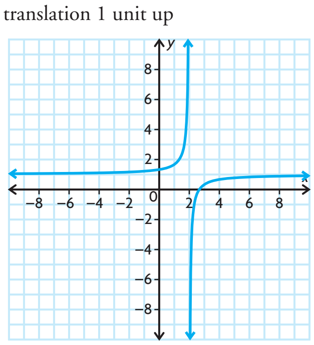
b) vertical stretch by a factor of 2 and a horizontal translation 1 unit to the right



c) reflection in the x-axis, vertical compression by a factor of $\frac{1}{2}$, and a vertical translation 3 units down



d) reflection in the x-axis, vertical compression by a factor of $\frac{2}{3}$, horizontal translation 2 units right, and a vertical translation 1 unit up



8. Factor the expressions in the numerator and the denominator. Simplify each expression as necessary. Multiply the first expression by the reciprocal of the second.

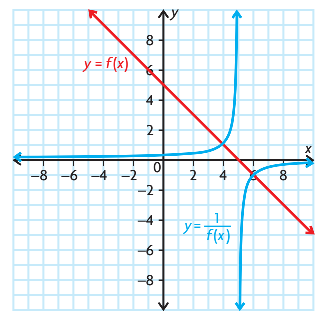
$$\frac{-3(3y-2)}{2(3y+2)}$$

Lesson 5.1, pp. 254–257

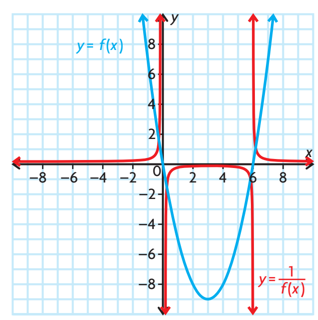
1. a) C; The reciprocal function is F.
b) A; The reciprocal function is E.
c) D; The reciprocal function is B.
d) F; The reciprocal function is C.
e) B; The reciprocal function is D.
f) E; The reciprocal function is A.

2. a) $x = 6$
b) $x = -\frac{4}{3}$
c) $x = 5$ and $x = -3$
d) $x = -\frac{5}{2}$ and $x = \frac{5}{2}$
e) no asymptotes
f) $x = -1.5$ and $x = -1$

3. a)



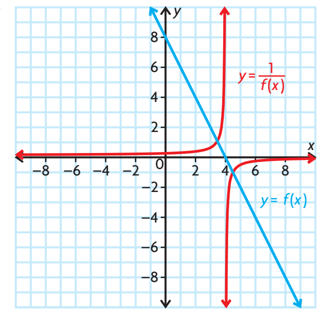
b)



4. a)

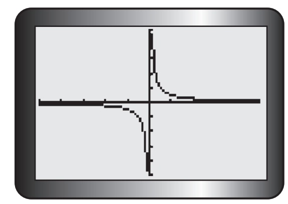
x	$f(x)$	$\frac{1}{f(x)}$
-4	16	$\frac{1}{16}$
-3	14	$\frac{1}{14}$
-2	12	$\frac{1}{12}$
-1	10	$\frac{1}{10}$
0	8	$\frac{1}{8}$
1	6	$\frac{1}{6}$
2	4	$\frac{1}{4}$
3	2	$\frac{1}{2}$
4	0	undefined
5	-2	$-\frac{1}{2}$
6	-4	$-\frac{1}{4}$
7	-6	$-\frac{1}{6}$

b)

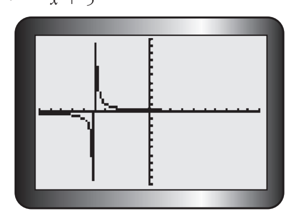


c) $f(x) = -2x + 8$, $y = \frac{1}{-2x+8}$

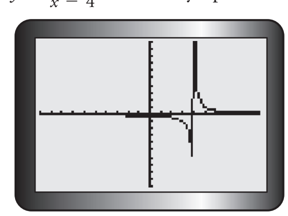
5. a) $y = \frac{1}{2x}$; vertical asymptote at $x = 0$



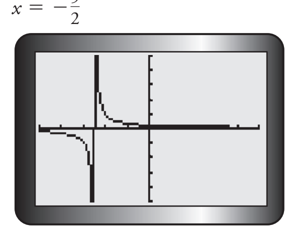
b) $y = \frac{1}{x+5}$; vertical asymptote at $x = -5$



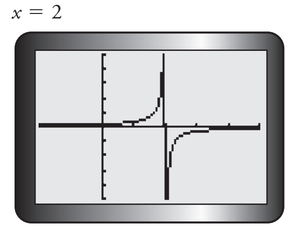
c) $y = \frac{1}{x-4}$; vertical asymptote at $x = 4$



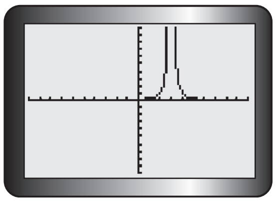
d) $y = \frac{1}{2x+5}$; vertical asymptote at $x = -\frac{5}{2}$



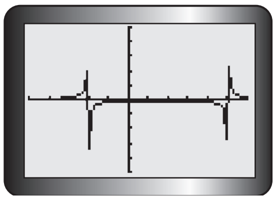
e) $y = \frac{1}{-3x+6}$; vertical asymptote at $x = 2$



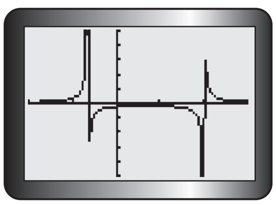
f) $y = \frac{1}{(x-3)^2}$; vertical asymptote at $x = 3$



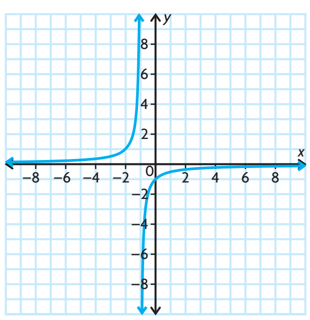
g) $y = \frac{1}{x^2 - 3x - 10}$; vertical asymptotes at $x = -2$ and $x = 5$



h) $y = \frac{1}{3x^2 - 4x - 4}$; vertical asymptotes at $x = -\frac{2}{3}$ and $x = 2$



6. a)



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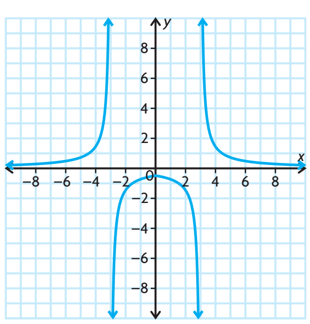
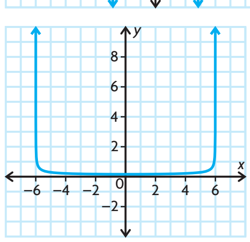
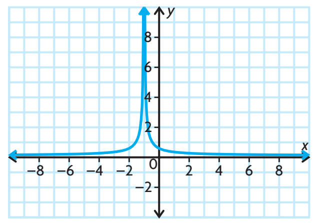


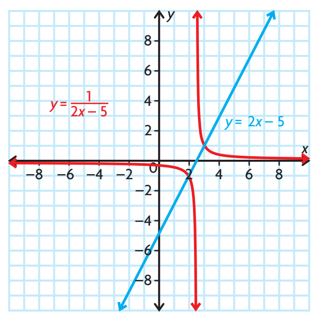
c)



d)

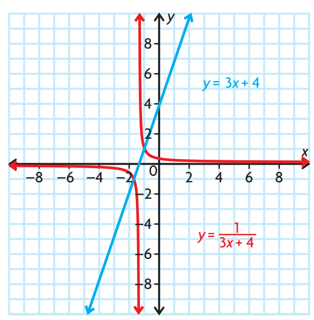


7. a)



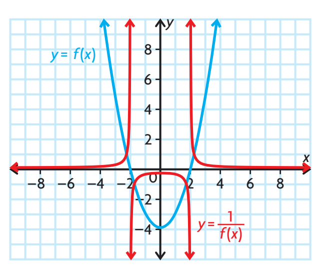
$D = \left\{x \in \mathbf{R} \mid x \neq \frac{5}{2}\right\}$,
$R = \{y \in \mathbf{R} \mid y \neq 0\}$

b)

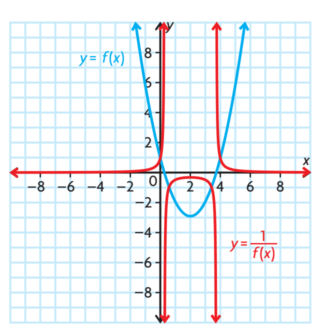


$D = \left\{x \in \mathbf{R} \mid x \neq -\frac{4}{3}\right\}$,
$R = \{y \in \mathbf{R} \mid y \neq 0\}$

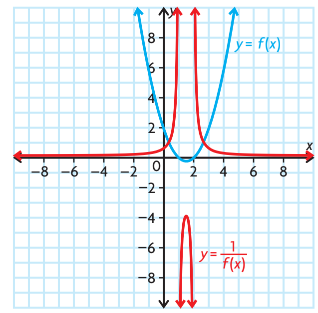
8. a)



b)



c)



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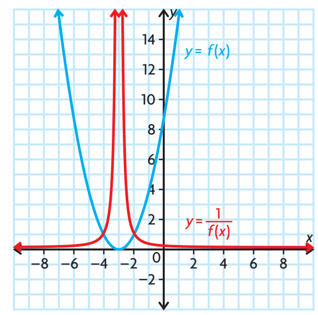
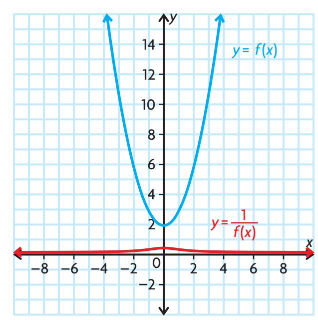
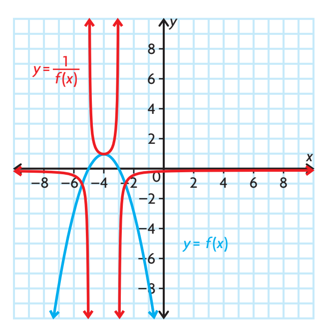


e)

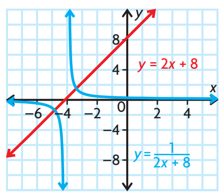


f)

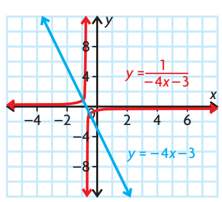


Answers

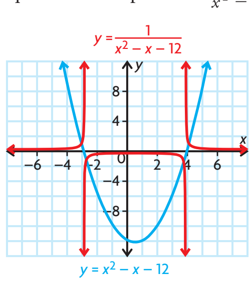
9. a) D = $\{x \in \mathbf{R}\}$
R = $\{y \in \mathbf{R}\}$
y-intercept = 8
x-intercept = -4
negative on $(-\infty, -4)$
positive on $(-4, -\infty)$
increasing on $(-\infty, \infty)$
equation of reciprocal = $\frac{1}{2x + 8}$



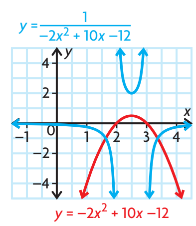
b) D = $\{x \in \mathbf{R}\}$
R = $\{y \in \mathbf{R}\}$
y-intercept = -3
x-intercept = $-\frac{3}{4}$
positive on $\left(-\infty, -\frac{3}{4}\right)$
negative on $\left(-\frac{3}{4}, \infty\right)$
decreasing on $(-\infty, \infty)$
equation of reciprocal = $\frac{1}{-4x - 3}$



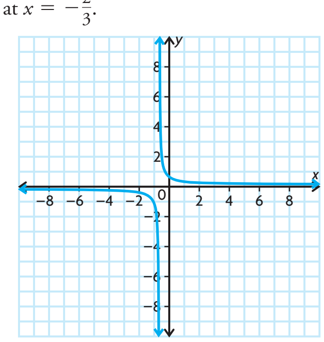
c) D = $\{x \in \mathbf{R}\}$
R = $\{y \in \mathbf{R} \mid y \geq -12.25\}$
y-intercept = 12
x-intercepts = 4, -3
decreasing on $(-\infty, 0.5)$
increasing on $(0.5, \infty)$
positive on $(-\infty, -3)$ and $(4, \infty)$
negative on $(-3, 4)$
equation of reciprocal = $\frac{1}{x^2 - x - 12}$



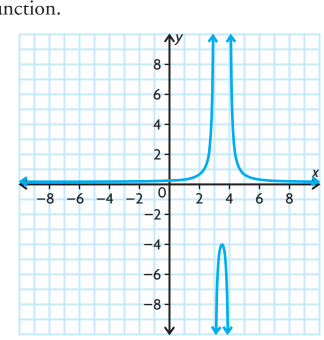
d) D = $\{x \in \mathbf{R}\}$
R = $\{y \in \mathbf{R} \mid y \leq 2.5\}$
y-intercept = -12
x-intercepts = 3, 2
increasing on $(-\infty, 2.5)$
decreasing on $(2.5, \infty)$
negative on $(-\infty, 2)$ and $(3, \infty)$
positive on $(2, 3)$
equation of reciprocal = $\frac{1}{-2x^2 + 10x - 12}$



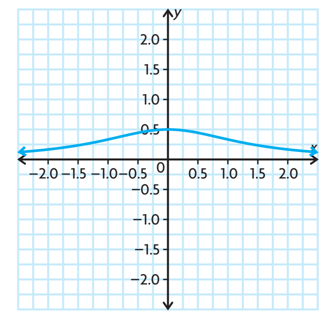
10. Answers may vary. For example, a reciprocal function creates a vertical asymptote when the denominator is equal to 0 for a specific value of x. Consider $\frac{1}{ax + b}$. For this expression, there is always some value of x that is $\frac{-b}{a}$ that will result in a vertical asymptote for the function. This is a graph of $y = \frac{1}{3x + 2}$ and the vertical asymptote is at $x = -\frac{2}{3}$.



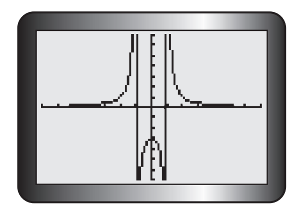
Consider the function $\frac{1}{(x - 3)(x - 4)}$. The graph of the quadratic function in the denominator crosses the x-axis at 3 and 4 and therefore will have vertical asymptotes at 3 and 4 in the graph of the reciprocal function.



However, a quadratic function, such as $x^2 + c$, which has no real zeros, will not have a vertical asymptote in the graph of its reciprocal function. For example, this is the graph of $y = \frac{1}{x^2 + 2}$.

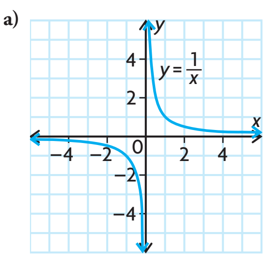


11. $y = \frac{3}{x^2 - 1}$



12. a) 500
b) $t = 2$
c) $t = 10\,000$
d) If you were to use a value of t that was less than one, the equation would tell you that the number of bacteria was increasing as opposed to decreasing. Also, after time $t = 10\,000$, the formula indicates that there is a smaller and smaller fraction of 1 bacteria left.
e) D = $\{x \in \mathbf{R} \mid 1 < x < 10\,000\}$,
R = $\{y \in \mathbf{R} \mid 1 < y < 10\,000\}$

13. a)

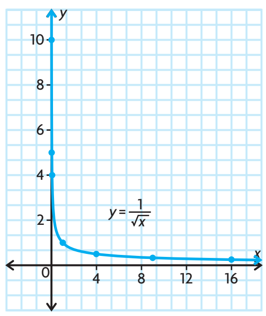


D = $\{x \in \mathbf{R} \mid x \neq -n\}$,
R = $\{y \in \mathbf{R} \mid y \neq 0\}$

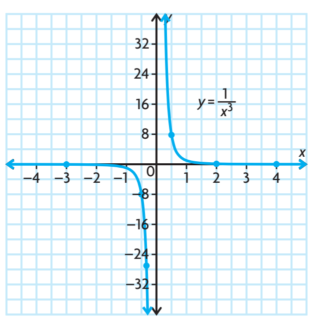
b) The vertical asymptote occurs at $x = -n$. Changes in n in the $f(x)$ family cause changes in the y-intercept—an increase in n causes the intercept to move up the y-axis and a decrease causes it to move down the y-axis. Changes in n in the $g(x)$ family cause changes in the vertical asymptote of the function—an increase in n causes the asymptote to move down the x-axis and a decrease in n causes it to move up the x-axis.
c) $x = 1 - n$ and $x = -1 - n$

14. Answers may vary. For example:
1) Determine the zero(s) of the function $f(x)$—these will be the asymptote(s) for the reciprocal function $g(x)$.
2) Determine where the function $f(x)$ is positive and where it is negative—the reciprocal function $g(x)$ will have the same characteristics.
3) Determine where the function $f(x)$ is increasing and where it is decreasing—the reciprocal function $g(x)$ will have opposite characteristics.

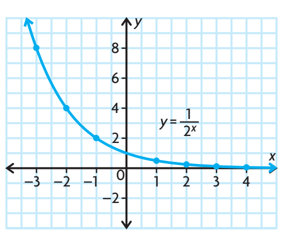
15. **a)**



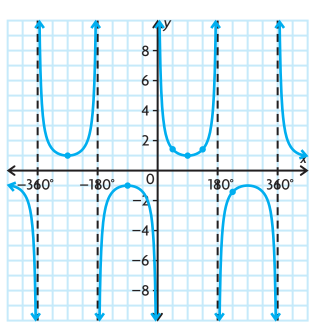
b)



c)



d)



16. $y = \frac{1}{x + 4} - 1$

Lesson 5.2, p. 262

1. **a)** A; The function has a zero at 3 and the reciprocal function has a vertical asymptote at $x = 3$. The function is positive for $x < 3$ and negative for $x > 3$.
b) C; The function in the numerator factors to $(x + 3)(x - 3)$. $(x - 3)$ factors out of both the numerator and the denominator. The equation simplifies to $y = x + 3$, but has a hole at $x = 3$.
c) F; The function in the denominator has a zero at $x = -3$, so there is a vertical asymptote at $x = -3$. The function is always positive.
d) D; The function in the denominator has zeros at $y = 1$ and $y = -3$. The rational function has vertical asymptotes at $x = 1$ and $x = -3$.
e) B; The function has no zeros and no vertical asymptotes or holes.
f) E; The function in the denominator has a zero at $x = 3$ and the rational function has a vertical asymptote at $x = 3$. The degree of the numerator is exactly 1 more than the degree of the denominator, so the graph has an oblique asymptote.

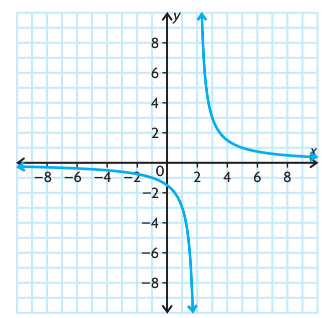
2. **a)** vertical asymptote at $x = -4$; horizontal asymptote at $y = 1$
b) vertical asymptote at $x = -\frac{3}{2}$; horizontal asymptote at $y = 0$
c) vertical asymptote at $x = 6$; horizontal asymptote at $y = 2$
d) hole at $x = -3$
e) vertical asymptotes at $x = -3$ and 5; horizontal asymptote at $y = 0$
f) vertical asymptote at $x = -1$; horizontal asymptote at $y = -1$
g) hole at $x = 2$
h) vertical asymptote at $x = \frac{5}{2}$; horizontal asymptote at $y = -2$
i) vertical asymptote at $x = -\frac{1}{4}$; horizontal asymptote at $y = 1$
j) vertical asymptote at $x = 4$; hole at $x = -4$; horizontal asymptote at $y = 0$
k) vertical asymptote at $x = \frac{3}{5}$; horizontal asymptote at $y = \frac{1}{5}$
l) vertical asymptote at $x = 4$; horizontal asymptote at $y = -\frac{3}{2}$

3. Answers may vary. For example:
a) $y = \frac{x - 1}{x^2 + x - 2}$
b) $y = \frac{1}{x^2 - 4}$
c) $y = \frac{x^2 - 4}{x^2 + 3x + 2}$
d) $y = \frac{2x}{x + 1}$
e) $y = \frac{x^3}{x^2 + 5}$

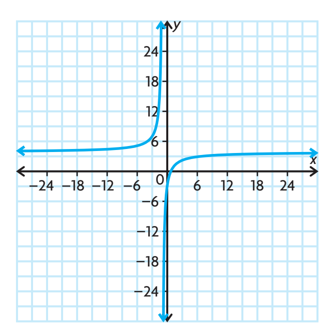
Lesson 5.3, pp. 272–274

1. **a)** A **b)** C **c)** D **d)** B

2. **a)** $x = 2$
b) As $x \to 2$ from the right, the values of $f(x)$ get larger. As $x \to 2$ from the left, the values become larger in magnitude but are negative.
c) $y = 0$
d) As $x \to -\infty$ and as $x \to \infty$, $f(x) \to 0$.
e) $D = \{x \in \mathbf{R} \mid x \neq 3\}$
$R = \{y \in \mathbf{R} \mid y \neq 0\}$
f) positive: $(2, \infty)$
negative: $(-\infty, 2)$
g)



3. **a)** $x = -1$
b) As $x \to -1$ from the left, $y \to \infty$. As $x \to -1$ from the right, $y \to -\infty$.
c) $y = 4$
d) As $x \to \pm\infty$, $f(x)$ gets closer and closer to 4.
e) $D = \{x \in \mathbf{R} \mid x \neq -1\}$
$R = \{y \in \mathbf{R} \mid y \neq 4\}$
f) positive: $(-\infty, -1)$ and $\left(\frac{3}{4}, -\infty\right)$
negative: $\left(-1, \frac{3}{4}\right)$
g)



Answers

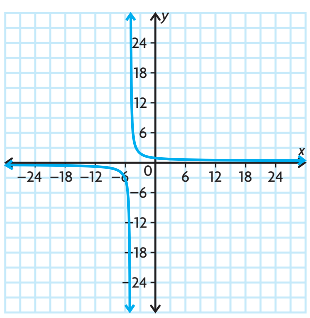
4. **a)** $x = -3$; As $x = -3$, $y = -\infty$ on the left. As $x = -3$, $y = \infty$ on the right.

b) $x = 5$; As $x = 5$, $y = -\infty$ on the left. As $x = 5$, $y = \infty$ on the right.

c) $x = \frac{1}{2}$; As $x = \frac{1}{2}$, $y = -\infty$ on the left. As $x = \frac{1}{2}$, $y = \infty$ on the right.

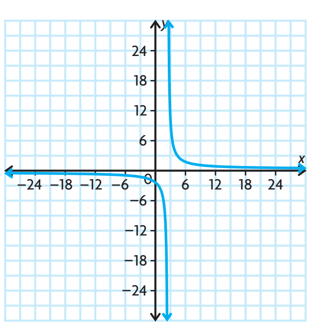
d) $x = -\frac{1}{4}$; As $x = -\frac{1}{4}$, $y = -\infty$ on the left. As $x = -\frac{1}{4}$, $y = \infty$ on the right.

5. **a)** vertical asymptote at $x = -5$
horizontal asymptote at $y = 0$
$D = \{x \in \mathbf{R} \mid x \neq -5\}$
$R = \{y \in \mathbf{R} \mid y \neq 0\}$
y-intercept $= \frac{3}{5}$
$f(x)$ is negative on $(-\infty, -5)$ and positive on $(-5, \infty)$.



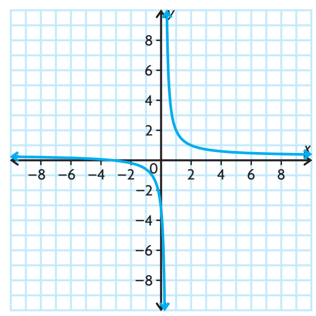
The function is decreasing on $(-\infty, -5)$ and on $(-5, \infty)$. The function is never increasing.

b) vertical asymptote at $x = \frac{5}{2}$
horizontal asymptote at $y = 0$
$D = \left\{x \in \mathbf{R} \mid x \neq \frac{5}{2}\right\}$
$R = \{y \in \mathbf{R} \mid y \neq 0\}$
y-intercept $= -2$
$f(x)$ is negative on $\left(-\infty, \frac{5}{2}\right)$ and positive on $\left(\frac{5}{2}, \infty\right)$.



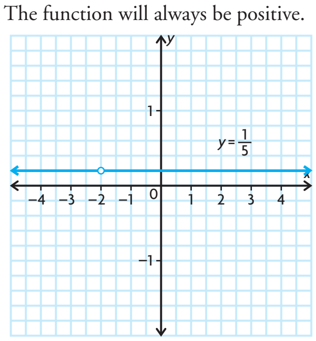
The function is decreasing on $\left(-\infty, \frac{5}{2}\right)$ and on $\left(\frac{5}{2}, \infty\right)$. The function is never increasing.

c) vertical asymptote at $x = \frac{1}{4}$
horizontal asymptote at $y = \frac{1}{4}$
$D = \left\{x \in \mathbf{R} \mid x \neq \frac{1}{4}\right\}$
$R = \left\{y \in \mathbf{R} \mid y \neq \frac{1}{4}\right\}$
x-intercept $= -5$
y-intercept $= -1$
$f(x)$ is positive on $(-\infty, -5)$ and $\left(\frac{1}{4}, \infty\right)$ and negative on $\left(-5, \frac{1}{4}\right)$.



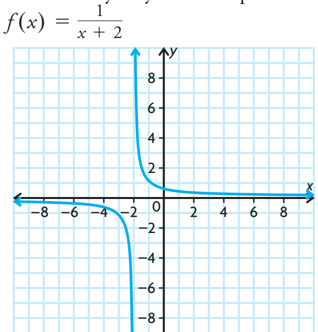
The function is decreasing on $\left(-\infty, \frac{1}{4}\right)$ and on $\left(\frac{1}{4}, \infty\right)$. The function is never increasing.

d) hole $x = -2$
$D = \{x \in \mathbf{R} \mid x \neq -2\}$
$R = \left\{y = \frac{1}{5}\right\}$
y-intercept $= \frac{1}{5}$
The function will always be positive.

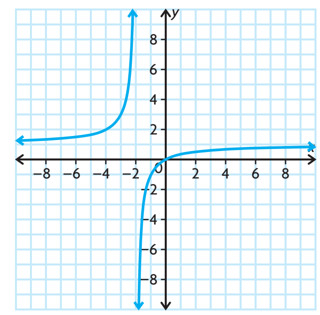


The function is neither increasing nor decreasing; it is constant.

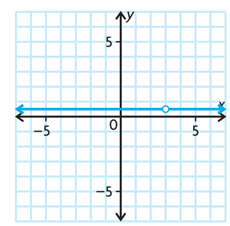
6. **a)** Answers may vary. For example:
$f(x) = \frac{1}{x + 2}$



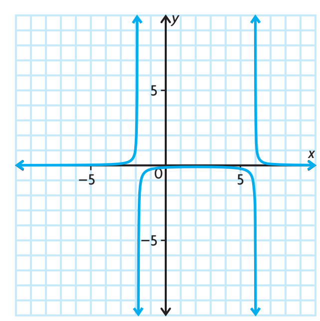
b) Answers may vary. For example:
$y = \frac{x}{x + 2}$



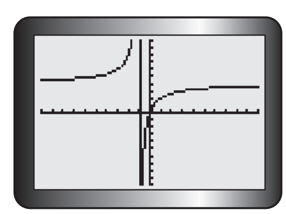
c) Answers may vary. For example:
$f(x) = \frac{x - 3}{2x - 6}$

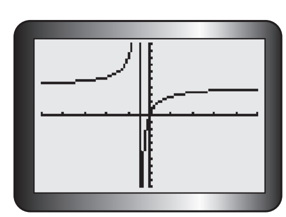


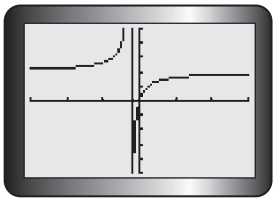
d) Answers may vary. For example:
$f(x) = \frac{1}{x^2 - 4x - 12}$

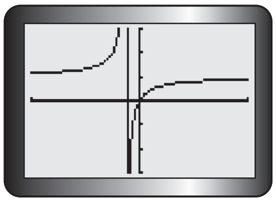


7. **a)**









The equation has a general vertical asymptote at $x = -\frac{1}{n}$. The function has a general horizontal asymptote at $y = \frac{8}{n}$. The vertical asymptotes are $-\frac{1}{8}$, $-\frac{1}{4}$, $-\frac{1}{2}$, and -1. The horizontal asymptotes are 8, 4, 2, and 1. The function contracts as n increases. The function is always increasing. The function is positive on $\left(-\infty, -\frac{17}{n}\right)$ and $\left(\frac{3}{10}, \infty\right)$. The function is negative on $\left(-\frac{17}{n}, \frac{3}{10}\right)$.

b) The horizontal and vertical asymptotes both approach 0 as the value of n increases; the x- and y-intercepts do not change, nor do the positive and negative characteristics or the increasing and decreasing characteristics.

c) The vertical asymptote becomes $x = \frac{17}{n}$ and the horizontal becomes $x = -\frac{10}{n}$. The function is always increasing. The function is positive on $\left(-\infty, \frac{3}{10}\right)$ and $\left(\frac{17}{n}, \infty\right)$. The function is negative on $\left(\frac{3}{10}, \frac{17}{n}\right)$. The rest of the characteristics do not change.

8. $f(x)$ will have a vertical asymptote at $x = 1$; $g(x)$ will have a vertical asymptote at $x = -\frac{3}{2}$. $f(x)$ will have a horizontal asymptote at $x = 3$; $g(x)$ will have a vertical asymptote at $x = \frac{1}{2}$.

9. **a)** \$27 500
b) \$40 000
c) \$65 000
d) No, the value of the investment at $t = 0$ should be the original value invested.
e) The function is probably not accurate at very small values of t because as $t \to 0$ from the right, $x \to \infty$.
f) \$15 000

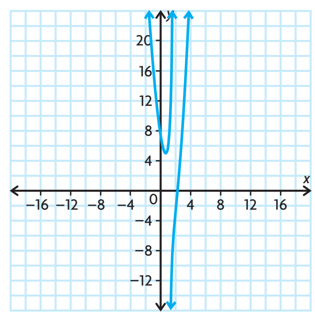
10. The concentration increases over the 24 h period and approaches approximately 1.89 mg/L.

11. Answers may vary. For example, the rational functions will all have vertical asymptotes at $x = -\frac{d}{c}$. They will all have horizontal asymptotes at $y = \frac{a}{c}$. They will intersect the y-axis at $y = \frac{b}{d}$. The rational functions will have an x-intercept at $x = -\frac{b}{a}$.

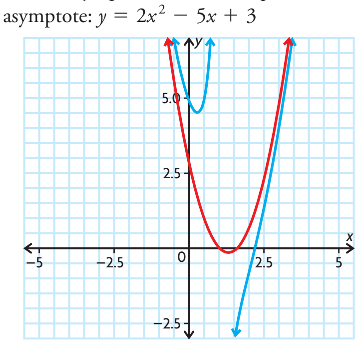
12. Answers may vary. For example, $f(x) = \frac{2x^2}{2 + x}$.

13. $f(x) = 2x^2 - 5x + 3 - \dfrac{2}{x - 1}$

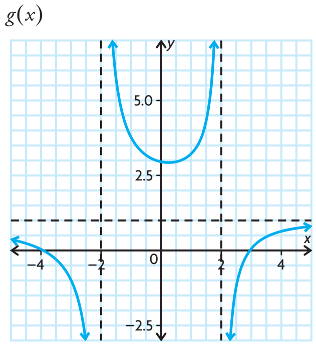
As $x \to \pm\infty$, $f(x) \to \infty$.



vertical asymptote: $x = 1$; oblique asymptote: $y = 2x^2 - 5x + 3$



14. **a)** $f(x)$
b) $g(x)$ and $h(x)$
c) $g(x)$
d)



Mid-Chapter Review, p. 277

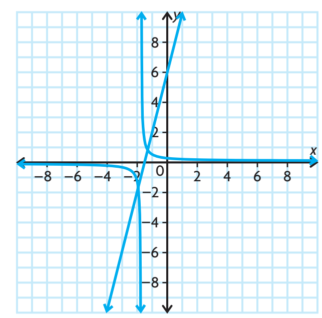
1. **a)** $\dfrac{1}{x - 3}$; $x = 3$

b) $\dfrac{1}{-4q + 6}$; $q = \dfrac{3}{2}$

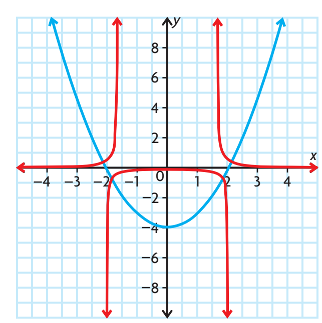
c) $\dfrac{1}{z^2 + 4z - 5}$; $z = -5$ and 1

d) $\dfrac{1}{6d^2 + 7d - 3}$; $d = \dfrac{1}{3}$ and $-\dfrac{3}{2}$

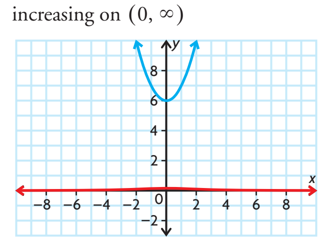
2. **a)** $D = \{x \in \mathbf{R}\}$; $R = \{x \in \mathbf{R}\}$; y-intercept $= 6$; x-intercept $= -\frac{3}{2}$; negative on $\left(-\infty, -\frac{3}{2}\right)$; positive on $\left(-\frac{3}{2}, \infty\right)$; increasing on $(-\infty, \infty)$



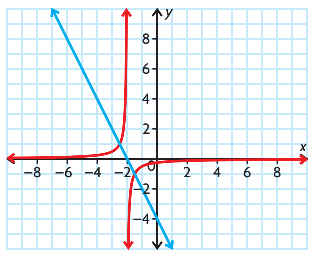
b) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y > -4\}$; y-intercept $= -4$; x-intercepts are 2 and -2; decreasing on $(-\infty, 0)$; increasing $(0, \infty)$; positive on $(-\infty, -2)$ and $(2, \infty)$; negative on $(-2, 2)$



c) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y > 6\}$; no x-intercepts; function will never be negative; decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$



d) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$; x-intercept $= -2$; function is always decreasing; positive on $(-\infty, -2)$; negative on $(-2, \infty)$

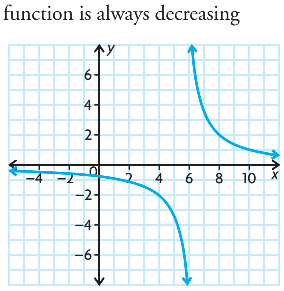


3. Answers may vary. For example: (1) Hole: Both the numerator and the denominator contain a common factor, resulting in $\frac{0}{0}$ for a specific value of x. (2) Vertical asymptote: A value of x causes the denominator of a rational function to be 0. (3) Horizontal asymptote: A horizontal asymptote is created by the ratio between the numerator and the denominator of a rational function as the function $\to \infty$ and $-\infty$. A continuous rational function is created when the denominator of the rational function has no zeros.

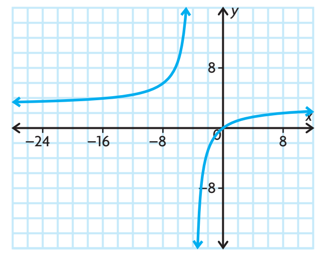
4. **a)** $x = 2$; vertical asymptote
b) hole at $x = 1$
c) $x = -\frac{1}{2}$; horizontal asymptote
d) $x = 6$; oblique asymptote
e) $x = -5$ and $x = 3$; vertical asymptotes

5. $y = \frac{x}{x-2}$, $y = 1$; $y = \frac{-7x}{4x+2}$, $y = \frac{-7}{4}$; $y = \frac{1}{x^2 + 2x - 15}$, $x = 0$

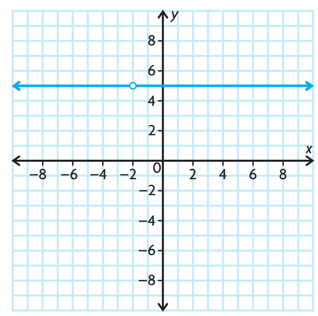
6. **a)** vertical asymptote: $x = 6$; horizontal asymptote: $y = 0$; no x-intercept; y-intercept: $-\frac{5}{6}$; negative when the denominator is negative; positive when the numerator is positive; $x - 6$ is negative on $x < 6$; $f(x)$ is negative on $(-\infty, 6)$ and positive on $(6, \infty)$; function is always decreasing



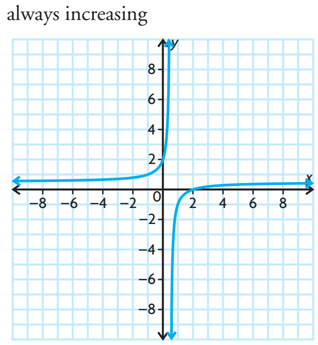
b) vertical asymptote: $x = -4$; horizontal asymptote: $y = 3$; x-intercept: $x = 0$; y-intercept: $f(0) = 0$; function is always increasing; positive on $(-\infty, -4)$ and $(0, \infty)$; negative on $(-4, 0)$



c) straight, horizontal line with a hole at $x = -2$; always positive and never increases or decreases



d) vertical asymptote: $x = \frac{1}{2}$; horizontal asymptote: $y = \frac{1}{2}$; x-intercept: $x = 2$; y-intercept: $f(0) = 5$; function is always increasing



7. Answers may vary. For example: Changing the function to $y = \frac{7x+6}{x+1}$ changes the graph. The function now has a vertical asymptote at $x = -1$ and still has a horizontal asymptote at $y = 7$. However, the function is now constantly increasing instead of decreasing. The new function still has an x-intercept at $x = -\frac{6}{7}$, but now has a y-intercept at $y = 6$.

8. $n = \frac{1}{3}$; $m = 35$

9. Answers may vary. For example, $f(x) = \frac{4x+8}{x+2}$.
The graph of the function will be a horizontal line at $y = 4$ with a hole at $x = -2$.

Lesson 5.4, pp. 285–287

1. 3; -2; Answers may vary. For example, substituting each value for x in the equation produces the same value on each side of the equation, so both are solutions.

2. **a)** $x = -3$ **b)** $x = 5$ **c)** $x = -1$ and 2 **d)** $x = -4$

3. **a)** $f(x) = \frac{x-3}{x+3} - 2$
b) $f(x) = \frac{3x-1}{x} - \frac{5}{2}$
c) $f(x) = \frac{x-1}{x} - \frac{x+1}{x+3}$
d) $f(x) = \frac{x-2}{x+3} - \frac{x-4}{x+5}$

4. **a)** $x = -9$ **b)** $x = 2$ **c)** $x = 3$ **d)** $x = -\frac{1}{2}$

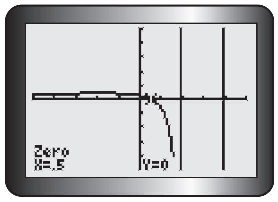
5. **a)** $x = 3$ **b)** $x = \frac{3}{4}$ **c)** $x = -9$ **d)** $x = 0$ **e)** $x = \frac{1}{4}$ **f)** $x = -23$

6. **a)** The function will have no real solutions.
b) $x = 3$ and $x = -0.5$
c) $x = -5$
d) $x = 0$ and $x = -1$
e) The original equation has no real solutions.
f) $x = 5$ and $x = 2$

7. **a)** $x = 6$ **b)** $x = 1.30, 7.70$ **c)** $x = 10$ **d)** $x = 3.25, 20.75$ **e)** $x = -1.71, 2.71$ **f)** $x = -0.62, 1.62$

8. **a)** $\frac{x+1}{x-2} = \frac{x+3}{x-4}$
Multiply both sides of the equation by the LCD, $(x-2)(x-4)$.
$$(x-2)(x-4)\left(\frac{x+1}{x-2}\right) = (x-2)(x-4)\left(\frac{x+3}{x-4}\right)$$
$(x-4)(x+1) = (x-2)(x+3)$
Simplify. $x^2 - 3x - 4 = x^2 + x - 6$
Simplify the equation so that 0 is on one side of the equation.
$$x^2 - x^2 - 3x - x - 4 + 6 = x^2 - x^2 + x - x - 6 + 6$$
$-4x + 2 = 0$
$-2(2x - 1) = 0$
Since the product is equal to 0, one of the factors must be equal to 0. It must be $2x - 1$ because 2 is a constant.
$$2x - 1 = 0$$
$$2x - 1 + 1 = 0 + 1$$
$$2x = 1$$
$$\frac{2x}{2} = \frac{1}{2}$$
$$x = \frac{1}{2}$$

b) $\dfrac{\frac{1}{2} + 1}{\frac{1}{2} - 2} = -1$ and $\dfrac{\frac{1}{2} + 3}{\frac{1}{2} - 4} = -1$

c) 

9. $w = 9.271$

10. Machine A $= 25.8$ min;
Machine B $= 35.8$ min

11. 75; \$4.00

12. **a)** After 6666.67 s
b) The function appears to approach 9 kg/m^3 as time increases.

13. **a)** Tom $= 4$ min; Carl $= 5$ min; Paco $= 2$ min
b) 6.4 min

14. Answers may vary. For example, you can use either algebra or graphing technology to solve a rational equation. With algebra, solving the equation takes more time, but you get an exact answer. With graphing technology, you can solve the equation quickly, but you do not always get an exact answer.

15. $x = -3.80, -1.42, 0.90, 4.33$

16. **a)** $x = 0.438$ and 1.712
b) $(0, 0.438)$ and $(1.712, \infty)$

Lesson 5.5, pp. 295–297

1. **a)** $(\infty, 1)$ and $(3, \infty)$
b) $(-0.5, 1)$ and $(2, \infty)$

2. **a)** Solve the inequality for x.

$$\frac{6x}{x+3} \le 4$$
$$\frac{6x}{x+3} - 4 \le 0$$
$$\frac{6x}{x+3} - 4\frac{x+3}{x+3} \le 0$$
$$\frac{6x - 4x - 12}{x+3} \le 0$$
$$\frac{2x - 12}{x+3} \le 0$$
$$\frac{2(x-6)}{x+3} \le 0$$

b) 

c) $(-3, 6]$

3. **a)**
$$x + 2 > \frac{15}{x}$$
$$x + 2 - \frac{15}{x} > 0$$
$$\frac{x^2}{x} + \frac{2x}{x} - \frac{15}{x} > 0$$
$$\frac{x^2 + 2x - 15}{x} > 0$$
$$\frac{(x+5)(x-3)}{x} > 0$$

b) negative: $x < -5$ and $0 < x < 3$;
positive: $-5 < x < 0$, $x > 3$

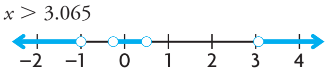
c) $\{x \in \mathbf{R} \mid -5 < x < 0 \text{ or } x > 3\}$ or $(-5, 0)$ or $(3, \infty)$

4. **a)** $5 < x < -4.5$
b) $-7 < x < -5$ and $x > -3$
c) $0 < x < 2$ and $x > 8$
d) $-6.8 \le x < -4$ and $x > 3$
e) $x < -1$ and $-\frac{1}{7} < x < 0$
f) $-1 < x < \frac{7}{8}$ and $x < 4$

5. **a)** $t < -3$ or $1 < t < 4$
b) $-3 \le t \le 2$ or $t > 4$
c) $-\frac{1}{2} < t < \frac{1}{3}$ or $t > \frac{1}{2}$
d) $t < -2$ and $-2 < t < 3$
e) $t < -5$ and $-2 < t < 0$
f) $-1 \le t < 0.25$ and $2 \le t < 9$

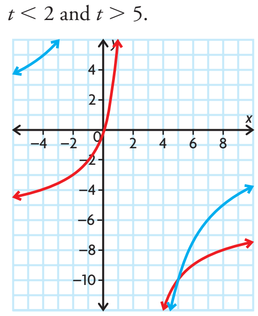
6. **a)** $x \in (-\infty, -6)$ or $x \in (-1, 4)$
b) $x \in (3, \infty)$
c) $x \in (-4, -2)$ or $x \in (-1, 2)$
d) $x \in (-\infty, -9)$ or $x \in [-3, -1)$ or $x \in [3, \infty)$
e) $x \in (-2, 0)$ or $x \in (4, \infty)$
f) $x \in (-\infty, -4)$ or $x \in (4, \infty)$

7. **a)** $x < -1$, $-0.2614 < x < 0.5$, $x > 3.065$

b) 

c) Interval notation: $(-\infty, -1)$, $(-0.2614, 0.5)$, $(3.065, \infty)$
Set notation: $\{x \in \mathbf{R} \mid x < -1, -0.2614 < x < 0.5, \text{ or } x > 3.065\}$

8. **a)** $t < 2$ and $t > 5$.

b) 

c) It would be difficult to find a situation that could be represented by these rational expressions because very few positive values of x yield a positive value of y.

9. The only values that make the expression greater than 0 are negative. Because the values of t have to be positive, the bacteria count in the tap water will never be greater than that of the pond water.

10. **a)** $\dfrac{(x^2 - 4x - 5)}{2x} < 0$

b)

	$x < -1$	$-1 < x < 0$	$0 < x < 5$	$x > 5$
$(x - 5)$	−	−	−	+
$(x + 1)$	−	+	+	+
$2x$	−	−	+	+
$\dfrac{(x-5)(x+1)}{2x}$	−	+	−	+

The inequality is true for $x < -1$ and $0 < x < 5$

11. when $x > 5$

12. **a)** The first inequality can be manipulated algebraically to produce the second inequality.
b) Graph the equation $y = \frac{x+1}{x-1} - \frac{x+3}{x+2}$ and determine when it is negative.
c) The values that make the factors of the second inequality zero are -5, -2, and 1. Determine the sign of each factor in the intervals corresponding to the zeros. Determine when the entire expression is negative by examining the signs of the factors.

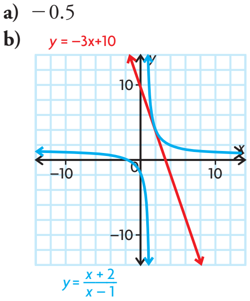
13. $[2, 4)$ and $(4, \infty)$

14. $14.48 < x < 165.52$ and $180 < x < 360$

15. $0 < x < 2$

Lesson 5.6, pp. 303–305

1. **a)** -0.5

b) 

slope $= -3$

2. -3

3. -3

4. -1

5. **a)** 0.01
b) -0.3
c) -1.3
d) 6

6. **a)** slope $= 286.1$; vertical asymptote: $x = -1.5$
b) slope $= -2.74$; vertical asymptote: $x = -5$
c) slope $= 44.65$; vertical asymptote: $x = -\frac{5}{3}$
d) slope $= -1.26$; vertical asymptote: $x = 6$

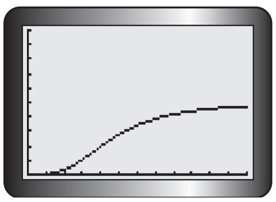
Answers

7. **a)** 0.01
b) 0.34

8. **a)** $R(x) = \dfrac{15x}{2x^2 + 11x + 5}$
b) 0.3, −0.03

9. **a)** $5.67
b) −2

10. **a)** 68.46
b) 94.54
c)



The number of houses that were built increases slowly at first, but rises rapidly between the third and sixth months. During the last six months, the rate at which the houses were built decreases.

11. Answers may vary. For example: $14 \le x \le 15$; $x = 14.5$

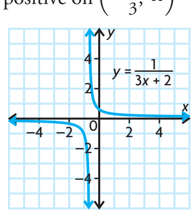
12. **a)** Find $s(0)$ and $s(6)$, and then solve $\dfrac{s(6) - s(0)}{6 - 0}$.
b) The average rate of change over this interval gives the object's speed.
c) To find the instantaneous rate of change at a specific point, you could find the slope of the line that is tangent to the function $s(t)$ at the specific point. You could also find the average rate of change on either side of the point for smaller and smaller intervals until it stabilizes to a constant. It is generally easier to find the instantaneous rate using a graph, but the second method is more accurate.
d) The instantaneous rate of change for a specific time, t, is the acceleration of the object at this time.

13. $y = -0.5x - 2.598$; $y = -0.5x + 2.598$; $y = 4x$

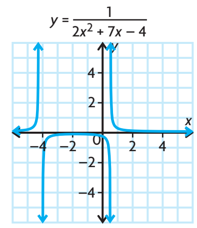
14. The instantaneous rate of change at $(0, 0) = 4$. The rate of change at this rate of change will be 0.

Chapter Review, pp. 308–309

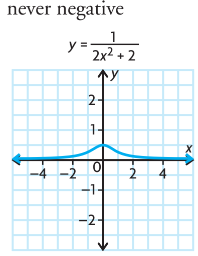
1. **a)** $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$; x-intercept $= -\frac{2}{3}$; y-intercept $= 2$; always increasing; negative on $\left(-\infty, -\frac{2}{3}\right)$; positive on $\left(-\frac{2}{3}, \infty\right)$



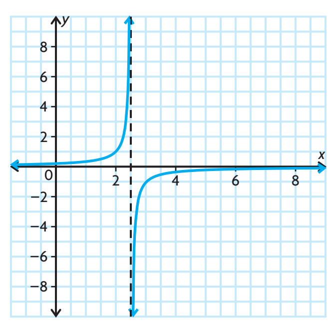
b) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y > -10.125\}$; x-intercept $= 0.5$ and -4; positive on $(-\infty, -4)$ and $(0.5, \infty)$; negative on $(-4, 0.5)$; decreasing on $(-\infty, -10.125)$; increasing on $(-10.125, \infty)$



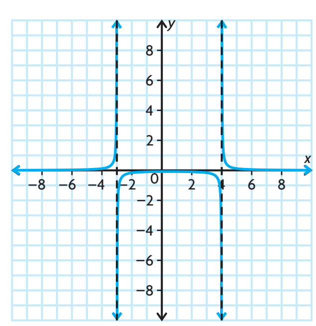
c) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y > 2\}$; no x-intercepts; y-intercept $= 2$; decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$; always positive, never negative



2. **a)**



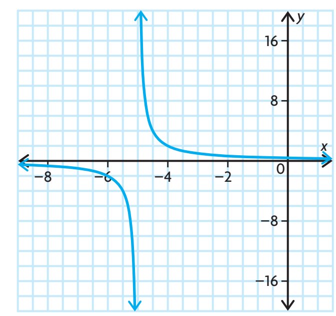
b)



3. **a)** $x = -17$
b) $x = -\frac{3}{5}$; horizontal asymptote; $y = \frac{2}{5}$
c) $x = 0.5$; hole at $x = -11$
d) $x = 1$; oblique asymptote; $y = 3x + 3$

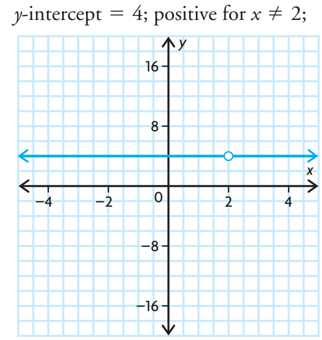
4. The locust population increased during the first 1.75 years, to reach a maximum of 1 248 000. The population gradually decreased until the end of the 50 years, when the population was 128 000.

5. **a)** x-intercept $= 2$; horizontal asymptote: $y = 0$; y-intercept $= \frac{2}{5}$; vertical asymptote: $x = -5$;



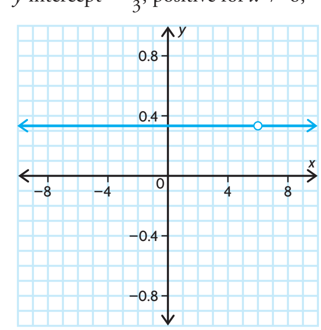
The function is never increasing and is decreasing on $(-\infty, -5)$ and $(-5, \infty)$.
$D = \{x \in \mathbf{R} \mid x \ne -5\}$; negative for $x < -5$; positive for $x > -5$

b) $D = \{x \in \mathbf{R} \mid x \ne 2\}$; no x-intercept; y-intercept $= 4$; positive for $x \ne 2$;



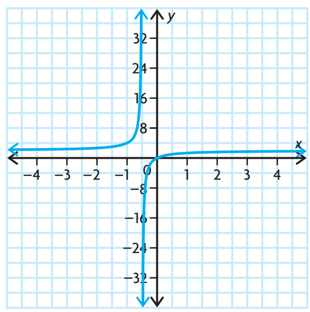
never increasing or decreasing

c) $D = \{x \in \mathbf{R} \mid x \ne 6\}$; no x-intercept; y-intercept $= \frac{1}{3}$; positive for $x \ne 6$;



never increasing or decreasing

d) $x = -0.5$; vertical asymptote: $x = -0.5$; $D = \{x \in \mathbf{R} \mid x \neq -0.5\}$; x-intercept $= 0$; y-intercept $= 0$; horizontal asymptote $= 2$; $R = \{y \in \mathbf{R} \mid x \neq 2\}$; positive on $x < -0.5$ and $x > 0$; negative on $-0.5 < x < 0$



The function is never decreasing and is increasing on $(-\infty, -0.5)$ and $(-0.5, \infty)$.

6. Answers may vary. For example, consider the function $f(x) = \frac{1}{x-6}$. You know that the vertical asymptote would be $x = 6$. If you were to find the value of the function very close to $x = 6$ (say $f(5.99)$ or $f(6.01)$) you would be able to determine the behaviour of the function on either side of the asymptote.

$$f(5.99) = \frac{1}{(5.99) - 6} = -100$$

$$f(6.01) = \frac{1}{(6.01) - 6} = 100$$

To the left of the vertical asymptote, the function moves toward $-\infty$. To the right of the vertical asymptote, the function moves toward ∞.

7. **a)** $x = 6$
b) $x = 0.2$ and $x = -\frac{2}{3}$
c) $x = -6$ or $x = 2$
d) $x = -1$ and $x = 3$

8. about 12 min

9. $x = 1.82$ days and 3.297 days

10. **a)** $x < -3$ and $-2.873 < x < 4.873$
b) $-16 < x < -11$ and $-5 < x$
c) $-2 < x < -1.33$ and $-1 < x < 0$
d) $0 < x < 1.5$

11. $-0.7261 < t < 0$ and $t > 64.73$

12. **a)** -6; $x = 3$
b) 0.2; $x = -2$ and $x = -1$

13. **a)** 0.455 mg/L/h
b) -0.04 mg/L/h
c) The concentration of the drug in the blood stream appears to be increasing most rapidly during the first hour and a half; the graph is steep and increasing during this time.

14. $x = 5$ and $x = 8$; $x = 6.5$

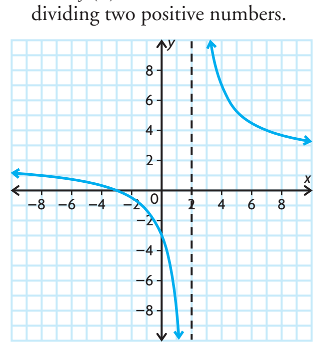
15. **a)** As the x-coordinate approaches the vertical asymptote of a rational function, the line tangent to graph will get closer and closer to being a vertical line. This means that the slope of the line tangent to the graph will get larger and larger, approaching positive or negative infinity depending on the function, as x gets closer to the vertical asymptote.
b) As the x-coordinate grows larger and larger in either direction, the line tangent to the graph will get closer and closer to being a horizontal line. This means that the slope of the line tangent to the graph will always approach zero as x gets larger and larger.

Chapter Self-Test, p. 310

1. **a)** B
b) A

2. **a)** If $f(n)$ is very large, then that would make $\frac{1}{f(n)}$ a very small fraction.
b) If $f(n)$ is very small (less than 1), then that would make $\frac{1}{f(n)}$ very large.
c) If $f(n) = 0$, then that would make $\frac{1}{f(n)}$ undefined at that point because you cannot divide by 0.
d) If $f(n)$ is positive, then that would make $\frac{1}{f(n)}$ also positive because you are dividing two positive numbers.

3.



4. 4326 kg; \$0.52/kg

5. **a)** Algebraic; $x = -1$ and $x = -3$
b) Algebraic with factor table
The inequality is true on $(-10, -5.5)$ and on $(-5, 1.2)$.

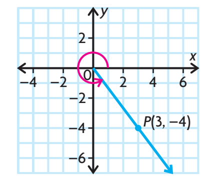
6. **a)** To find the vertical asymptotes of the function, find the zeros of the expression in the denominator. To find the equation of the horizontal asymptotes, divide the first two terms of the expressions in the numerator and denominator.
b) This type of function will have a hole when both the numerator and the denominator share the same factor $(x + a)$.

Chapter 6

Getting Started, p. 314

1. **a)** 28°
b) 332°

2. **a)**



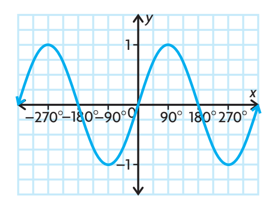
$\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$,
$\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

b) 307°

3. **a)** $\frac{\sqrt{3}}{2}$ **c)** $\frac{\sqrt{3}}{2}$ **e)** $-\sqrt{2}$
b) 0 **d)** $\frac{1}{2}$ **f)** -1

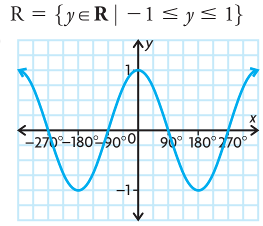
4. **a)** 60°, 300°
b) 30°, 210°
c) 45°, 225°
d) 180°
e) 135°, 315°
f) 90°

5. **a)**



period $= 360°$; amplitude $= 1$; $y = 0$; $R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

b)



period $= 360°$; amplitude $= 1$; $y = 0$; $R = \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

6. **a)** period $= 120°$; $y = 0$; 45° to the left; amplitude $= 2$