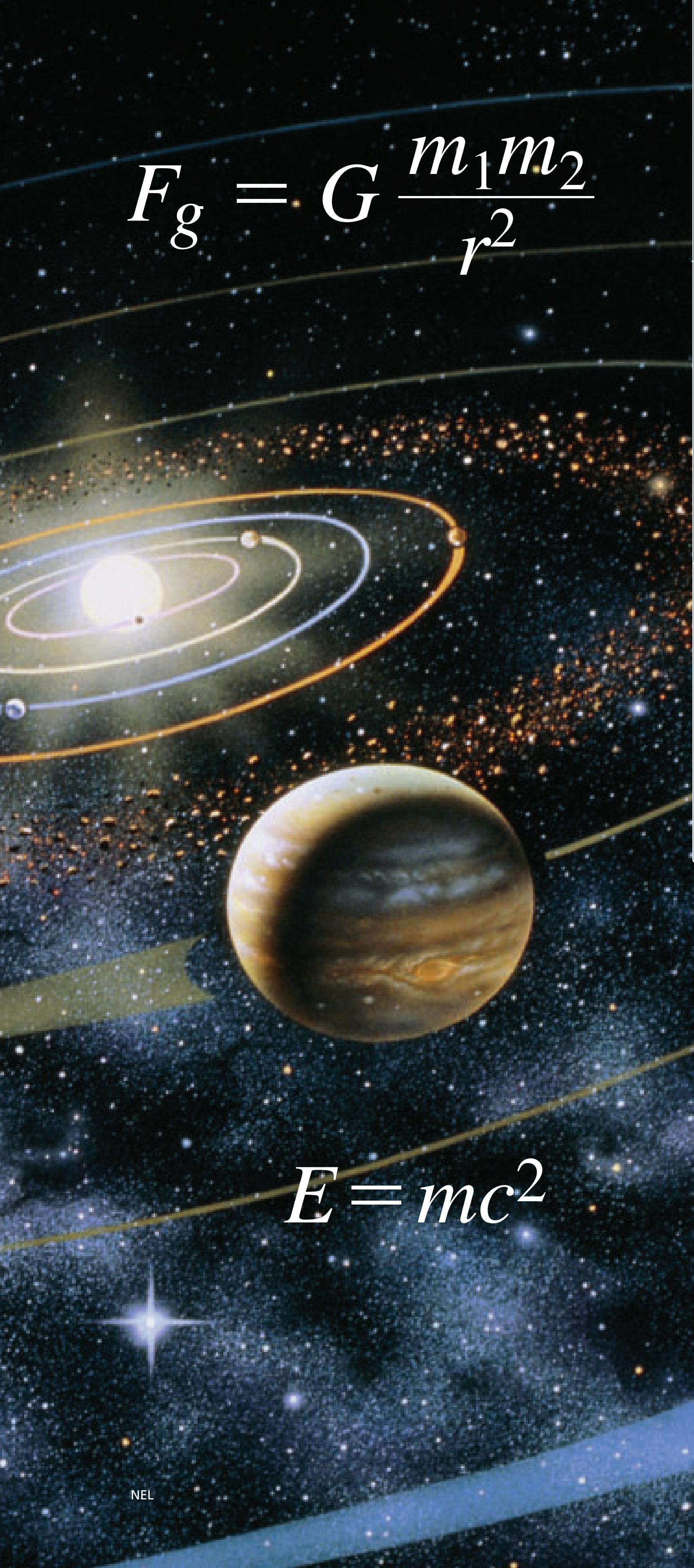

$E = M + \sum_{x=1}^{\infty} a_n e^n$
$s = \frac{1}{2} a t^2$
$F = ma$



$$F_g = G\frac{m_1 m_2}{r^2}$$

$$E = mc^2$$

Chapter 2

Equivalent Algebraic Expressions

GOALS

You will be able to

- Determine whether algebraic expressions are equivalent
- Add, subtract, multiply, and factor polynomials
- Simplify rational expressions
- Add, subtract, multiply, and divide rational expressions

? Many great scientists and mathematicians, such as Hypatia, Nicolaus Copernicus, Johannes Kepler, Isaac Newton, Caroline Herschel, Albert Einstein, and Steven Hawking, have spent their lives trying to explain the nature of the universe. What role does mathematics play in this quest for knowledge?

2 Getting Started

SKILLS AND CONCEPTS You Need

	Polynomial	Type	Degree
	$3x^2 + 2x - 1$	trinomial	2
a)	$4x - 7$		
b)	3		
c)	$7 + 2x^2$		
d)	$2xy$		
e)	$x^2 - 3xy + y^2$		

1. For each polynomial in the table at the left, state the type and the degree. The first one has been done for you.

2. Expand, where necessary, and then simplify.

a) $(2x + 3) + (7x - 5)$

b) $(4x^2 - 7x + 1) - (2x^2 - 3x + 10)$

c) $(2x - 3)(4x + 5)$

d) $(2x - 1)^2$

3. How are expanding and factoring related to each other? Use an example in your explanation.

4. Factor, where possible.

a) $6xy^3 - 8x^2y^3$

b) $a^2 - 7a + 10$

c) $12n^2 + 7n - 10$

d) $9 - 25x^2$

e) $x^2 + 5x + 8$

f) $y^2 - 5y - 36$

5. Simplify.

a) $\frac{3}{4} + \frac{1}{6}$

b) $\frac{-2}{5} - \frac{1}{10}$

c) $\left(\frac{-12}{25}\right)\left(\frac{-10}{9}\right)$

d) $\left(\frac{4}{3}\right) \div \left(\frac{-2}{15}\right)$

6. Simplify.

a) $\left(\frac{2x^2}{3}\right)\left(\frac{5x^3}{4}\right)$

b) $\left(\frac{3x}{2}\right) \div \left(\frac{x^3}{5}\right)$

c) $(2x^2y^3)(4xy^2)$

d) $(25x^5y^3) \div (5x^2y)$

7. Determine where each function is undefined, then state its domain.

a) $f(x) = x$

b) $g(x) = 2x^2$

c) $m(x) = \sqrt{x}$

d) $h(x) = \frac{1}{x}$

e) $j(x) = \frac{3}{x - 4}$

f) $n(x) = \sqrt{x + 10}$

8. Complete the chart to show what you know about polynomials.

Definition:	Characteristics:
Polynomials	
Examples:	Non-examples:

Study *Aid*

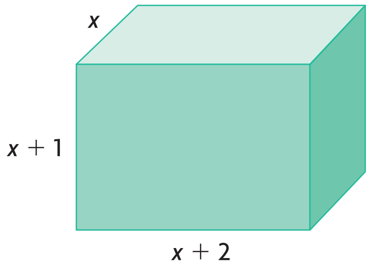
For help, see Essential Skills Appendix.

Question	Appendix
2	A-8
4	A-9
5	A-2
6	A-3

APPLYING What You Know

Doubling Dimensions

A rectangular box has dimensions x, $x + 1$, and $x + 2$.



? Suppose each dimension of the box is doubled. By what factors will the surface area and volume of the box increase?

A. What are the formulas for the volume and surface area of a rectangular box of width w, length l, and height h?

B. Use the given dimensions to write expressions for the volume and surface area of the box.

C. Write expressions for the volume and surface area of the new box after the dimensions are doubled.

D. By what factor has the volume increased? By what factor has the surface area increased? Explain how you know.

2.1 Adding and Subtracting Polynomials

YOU WILL NEED

- graphing calculator

Communication Tip

The numbers 1 and 2 in $a_1(t)$ and $a_2(t)$ are called subscripts. In this case, they are used to distinguish one function from the other. This distinction is necessary because both functions are named with the letter *a*.

GOAL

Determine whether polynomial expressions are equivalent.

LEARN ABOUT the Math

Fred enjoys working with model rockets. He wants to determine the difference in altitude of two different rockets when their fuel burns out and they begin to coast.

The altitudes, in metres, are given by these equations:

$$a_1(t) = -5t^2 + 100t + 1000$$

and

$$a_2(t) = -5t^2 + 75t + 1200$$

where t is the elapsed time, in seconds.

The difference in altitude, $f(t)$, is given by

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

Fred simplified $f(t)$ to $g(t) = 175t + 2200$.

? Are the functions $f(t)$ and $g(t)$ equivalent?

EXAMPLE 1 Selecting a strategy to determine equivalence

Determine if $f(t)$ and $g(t)$ are equivalent functions.

Anita's Solution: Simplifying the Polynomial in $f(t)$

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

Polynomials behave like numbers because, for any value of the variable, the result is a number. I know that with numbers, subtraction is equivalent to adding the opposite, so I subtracted the polynomials by adding the opposite of the second expression.

$$= -5t^2 + 100t + 1000 + 5t^2 - 75t - 1200$$

$$= 25t - 200$$

Then I collected like terms.

Since $g(t) \neq 25t - 200$, the functions are not equivalent.

If two expressions are not equivalent, then, for most values of t, their function values are different. The exception is when the functions intersect.

Maria's Solution: Evaluating the Functions for the Same Value of the Variable

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

$$f(0) = (-5(0)^2 + 100(0) + 1000) - (-5(0)^2 + 75(0) + 1200)$$

I used $t = 0$ because it makes the calculations to find $f(0)$ and $g(0)$ easier.

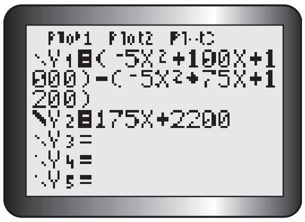
$$= 1000 - 1200$$
$$= -200$$

$$g(0) = 175(0) + 2200$$
$$= 2200$$

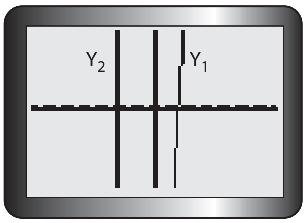
Since $f(0) \neq g(0)$, the functions are not equivalent.

If two functions are equivalent, then their graphs should be identical.

Sam's Solution: Graphing Both Functions



I graphed Fred's original function, $f(t)$, in Y_1 and his final function, $g(t)$, in Y_2. I zoomed out until I could see both functions.



Since the graphs are different, the functions must be different.

The functions are not equivalent.

Reflecting

A. How is subtracting two polynomials like subtracting integers? How is it different?

B. If Fred had not made an error when he simplified, whose method would have shown that his original and final expressions are equivalent?

C. What are the advantages and disadvantages of the three methods used to determine whether two polynomials are equivalent?

APPLY the Math

EXAMPLE 2 Reasoning whether two polynomials are equivalent

Nigel and Petra are hosting a dinner for 300 guests. Cheers banquet hall has quoted these charges:

- \$500, plus \$10 per person, for food,
- \$200, plus \$20 per person, for drinks, and
- a discount of \$5 per person if the number of guests exceeds 200.

Nigel and Petra have created two different functions for the total cost, where n represents the number of guests and $n > 200$.

Nigel's cost function: $C_1(n) = (10n + 500) + (20n + 200) - 5n$

Petra's cost function: $C_2(n) = (10n + 20n - 5n) + (500 + 200)$

Are the functions equivalent?

Lee's Solution

$$C_1(n) = (10n + 500) + (20n + 200) - 5n$$
$$= 10n + 20n - 5n + 500 + 200$$

Nigel's cost function was developed using the cost of each item.

The cost of the food is $(10n + 500)$. The cost of the drinks is $(20n + 200)$. The discount is $5n$.

$$= 25n + 700$$

I simplified by collecting like terms.

$$C_2(n) = (10n + 20n - 5n) + (500 + 200)$$
$$= 25n + 700$$

Petra's cost function was developed by adding the variable costs $(10n + 20n - 5n)$ and the fixed costs $(500 + 200)$. The result is two groups of like terms, which I then simplified.

$$C_1(n) = 25n + 700$$
$$C_2(n) = 25n + 700$$

Both cost functions simplify to the same function.

The two cost functions are equivalent.

EXAMPLE 3 Reasoning about the equivalence of expressions

Are the expressions $xy + xz + yz$ and $x^2 + y^2 + z^2$ equivalent?

Dwayne's Solution

To check for non-equivalence, I substituted some values for x, y, and z.

The values 0, 1, and -1 are often good choices, since they usually make expressions easy to evaluate.

$$xy + xz + yz = 0(0) + 0(1) + 0(1) = 0$$

I tried $x = 0$, $y = 0$, and $z = 1$ and evaluated the first expression.

$$x^2 + y^2 + z^2 = 0^2 + 0^2 + 1^2 = 1$$

I evaluated the second expression, using the same values for x, y, and z.

The expressions are not equivalent.

The expressions result in different values.

In Summary

Key Ideas

- Two polynomial functions or expressions are equivalent if
  - they simplify algebraically to give the same function or expression
  - they produce the same graph
- Two polynomial functions or expressions are not equivalent if
  - they result in different values when they are evaluated with the same numbers substituted for the variable(s)

Need to Know

- If you notice that two functions are equivalent at one value of a variable, it does not necessarily mean they are equivalent at *all* values of the variable. Evaluating both functions at a single value is sufficient to demonstrate non-equivalence, but it isn't enough to demonstrate equivalence.
- The sum of two or more polynomial functions or expressions can be determined by writing an expression for the sum of the polynomials and collecting like terms.
- The difference of two polynomial functions or expressions can be determined by adding the opposite of one polynomial and collecting like terms.

CHECK *Your Understanding*

1. Simplify.

a) $(3x^2 - 7x + 5) + (x^2 - x + 3)$

b) $(x^2 - 6x + 1) - (-x^2 - 6x + 5)$

c) $(2x^2 - 4x + 3) - (x^2 - 3x + 2) + (x^2 - 1)$

2. Show that $f(x)$ and $g(x)$ are equivalent by simplifying each.

$$f(x) = (2x - 1) - (3 - 4x) + (x + 2)$$
$$g(x) = (-x + 6) + (6x - 9) - (-2x - 1)$$

3. Show that $f(x)$ and $g(x)$ are not equivalent by evaluating each function at a suitable value of x.

$$f(x) = 2(x - 3) + 3(x - 3)$$
$$g(x) = 5(2x - 6)$$

PRACTISING

4. Simplify.

a) $(2a + 4c + 8) + (7a - 9c - 3)$

b) $(3x + 4y - 5z) + (2x^2 + 6z)$

c) $(6x + 2y + 9) + (-3x - 5y - 8)$

d) $(2x^2 - 7x + 6) + (x^2 - 2x - 9)$

e) $(-4x^2 - 2xy) + (6x^2 - 3xy + 2y^2)$

f) $(x^2 + y^2 + 8) + (4x^2 - 2y^2 - 9)$

5. Simplify.

a) $(m - n + 2p) - (3n + p - 7)$

b) $(-6m - 2q + 8) - (2m + 2q + 7)$

c) $(4a^2 - 9) - (a^3 + 2a - 9)$

d) $(2m^2 - 6mn + 8n^2) - (4m^2 - mn - 7n^2)$

e) $(3x^2 + 2y^2 + 7) - (4x^2 - 2y^2 - 8)$

f) $5x^2 - (2x^2 - 30) - (-20)$

6. Simplify.

a) $(2x - y) - (-3x + 4y) + (6x - 2y)$

b) $(3x^2 - 2x) + (x^2 - 7x) - (7x + 3)$

c) $(2x^2 + xy - y^2) - (x^2 - 4xy - y^2) + (3x^2 - 5xy)$

d) $(xy - xz + 4yz) + (2x - 3yz) - (4y - xz)$

e) $\left(\frac{1}{2}x + \frac{1}{3}y\right) - \left(\frac{1}{5}x - y\right)$

f) $\left(\frac{3}{4}x + \frac{1}{2}y\right) - \left(\frac{2}{3}x + \frac{1}{4}y - 1\right)$

7. Use two different methods to show that the expressions
K $(3x^2 - x) - (5x^2 - x)$ and $-2x^2 - 2x$ are not equivalent.

8. Determine whether each pair of functions is equivalent.

a) $f(x) = (2x^2 + 7x - 2) - (3x + 7)$ and
$g(x) = (x^2 + 12) + (x^2 + 4x - 17)$

b) $s_1(t) = (t + 2)^3$ and $s_2(t) = t^3 + 8$

c) $y_1 = (x - 1)(x)(x + 2)$ and $y_2 = 3x(x^2 - 1)$

d) $f(n) = 0.5n^2 + 2n - 3 + (1.5n^2 - 6)$ and
$g(n) = n^2 - n + 1 - (-n^2 - 3n + 10)$

e) $y_1 = 3p(q - 2) + 2p(q + 5)$ and $y_2 = p(q + 4)$

f) $f(m) = m(5 - m) - 2(2m - m^2)$ and
$g(m) = 4m^2(m - 1) - 3m^2 + 5m$

9. Determine two non-equivalent polynomials, $f(x)$ and $g(x)$, such that $f(0) = g(0)$ and $f(2) = g(2)$.

10. Kosuke wrote a mathematics contest consisting of 25 multiple-choice questions. The scoring system gave 6 points for a correct answer, 2 points for not answering a question, and 1 point for an incorrect answer. Kosuke got x correct answers and left y questions unanswered.

a) Write an expression for the number of questions he answered incorrectly.

b) Write an expression, in simplified form, for Kosuke's score.

c) Use the expressions you wrote in parts (a) and (b) to determine Kosuke's score if he answered 13 questions correctly and 7 incorrectly.

11. **A** The two equal sides of an isosceles triangle each have a length of $2x + 3y - 1$. The perimeter of the triangle is $7x + 9y$. Determine the length of the third side.

12. Tino owns a small company that produces and sells cellphone cases. The revenue and cost functions for Tino's company are shown below, where x represents the selling price in dollars.

$$\text{Revenue: } R(x) = -50x^2 + 2500x$$

$$\text{Cost: } C(x) = 150x + 9500$$

a) Write the simplified form of the profit function, $P(x) = R(x) - C(x)$.

b) What profit will the company make if it sells the cases for \$12 each?

13. **T** For each pair of functions, label the pairs as equivalent, non-equivalent, or cannot be determined.

a) $f(2) = g(2)$

b) $h(3) = g(4)$

c) $j(8) \neq k(8)$

d) $l(5) \neq m(7)$

e) $n(x) = p(x)$ for all values of x in their domain

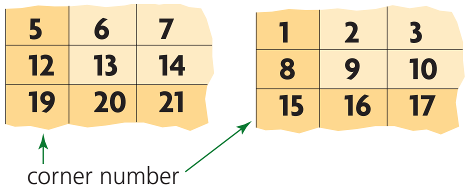
14. **C** Ramy used his graphing calculator to graph three different polynomial functions on the same axes. The equations of the functions all appeared to be different, but his calculator showed only two different graphs. He concluded that two of his functions were equivalent.

a) Is his conclusion correct? Explain.

b) How could he determine which, if any, of the functions were equivalent without using his graphing calculator?

Extending

15. Sanya noticed an interesting property of numbers that form a five-square capital-L pattern on a calendar page:



In each case that she checked, the sum of the five numbers was 18 less than five times the value of the number in the corner of the L. For example, in the calendars shown,

$$5 + 12 + 19 + 20 + 21 = 5(19) - 18$$

$$1 + 8 + 15 + 16 + 17 = 5(15) - 18$$

a) Show that this pattern holds for any numbers on the calendar page.

b) The sum of certain numbers in this pattern is 112. Determine the value of the corner number.

c) Write expressions for the sum of the five numbers, for the other three orientations of the L, when x is the corner number.

16. Since $70 = 5 \times 14$, 70 has 5 as a divisor. The number 70 can also be expressed as the sum of five consecutive natural numbers:

$$70 = 12 + 13 + 14 + 15 + 16$$

a) $105 = 5 \times 21$. Express 105 as the sum of five consecutive natural numbers.

b) Suppose m is a natural number that is greater than 2 and $n = 5m$. Express n as the sum of five consecutive natural numbers.

c) Express 91 as a sum of seven consecutive natural numbers.

17. **a)** Consider the linear functions $f(x) = ax + b$ and $g(x) = cx + d$. Suppose that $f(2) = g(2)$ and $f(5) = g(5)$. Show that the functions must be equivalent.

b) Consider the two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$. Suppose that $f(2) = g(2)$, $f(3) = g(3)$, and $f(4) = g(4)$. Show that the functions must be equivalent.

2.2 Multiplying Polynomials

GOAL

Simplify polynomials by multiplying.

LEARN ABOUT the Math

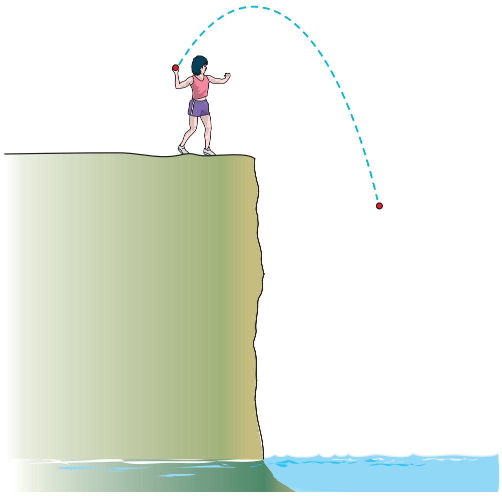
In a physics textbook, Kristina reads about an experiment in which a ball is thrown upward from the top of a cliff, ultimately landing in the water below the cliff. The height of the ball above the cliff, $h(t)$, and its velocity, $v(t)$, at time t are respectively given by

$$h(t) = -5t^2 + 5t + 2.5$$

and

$$v(t) = -10t + 5.$$

Kristina learns that the product of the two functions allows her to determine when the ball moves away from, and when the ball moves toward, the top of the cliff.



? How can she simplify the expression for $v(t) \times h(t)$?

EXAMPLE 1 Selecting a strategy to simplify a product: The distributive property

Simplify the expression $v(t) \times h(t) = (-10t + 5)(-5t^2 + 5t + 2.5)$.

Sam's Solution

$$\begin{aligned} v(t)h(t) &= (-10t + 5)(-5t^2 + 5t + 2.5) \\ &= (-10t)(-5t^2 + 5t + 2.5) + (5)(-5t^2 + 5t + 2.5) \\ &= (50t^3 - 50t^2 - 25t) + (-25t^2 + 25t + 12.5) \\ &= 50t^3 - 50t^2 - 25t^2 - 25t + 25t + 12.5 \\ &= 50t^3 - 75t^2 + 12.5 \end{aligned}$$

I used the **commutative property** for multiplication to create an equivalent expression.

I used the **distributive property** to expand the product of the polynomials. I multiplied each of the three terms in the trinomial by each of the terms in the binomial.

Then I grouped and collected like terms.

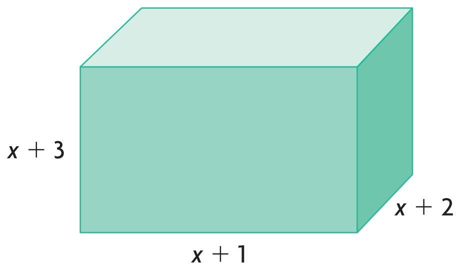
Reflecting

1. How does the simplified expression differ from the original?
2. Sam grouped together like terms in order to simplify. Is this always necessary? Explain.
3. Would Sam have gotten a different answer if he multiplied $(-5t^2 + 5t + 2.5)(-10t + 5)$ by multiplying each term in the second factor by each term in the first factor? Explain.

WORK WITH the Math

EXAMPLE 2 Selecting a strategy to multiply three binomials

Determine a simplified function that represents the volume of the given box.



Fred's Solution: Starting with the First Two Binomials

$V = lwh$

$V = (x + 1)(x + 2)(x + 3)$

$= (x^2 + 3x + 2)(x + 3)$

I know that multiplication is **associative,** so I can multiply in any order. I multiplied the first two binomials together and got a trinomial.

$= (x^2 + 3x + 2)(x) + (x^2 + 3x + 2)(3)$

Then I took the x-term from $(x + 3)$ and multiplied it by the trinomial. Next, I took the 3 term from $(x + 3)$ and multiplied it by the trinomial.

$= x^3 + 3x^2 + 2x + 3x^2 + 9x + 6$

$= x^3 + 6x^2 + 11x + 6$

Finally, I simplified by collecting like terms and arranged the terms in descending order.

Atish's Solution: Starting with the Last Two Binomials

$V = (x + 1)(x + 2)(x + 3)$ ← I multiplied the last two binomials together and got a trinomial.

$= (x + 1)(x^2 + 5x + 6)$ ← Then I took the x-term from $(x + 1)$ and multiplied it by the trinomial. Next, I took the 1 term from $(x + 1)$ and multiplied it by the trinomial.

$= x^3 + 5x^2 + 6x + x^2 + 5x + 6$

$= x^3 + 6x^2 + 11x + 6$ ← Finally, I simplified by collecting like terms and arranged the terms in descending order.

EXAMPLE 3 Selecting a strategy to determine non-equivalence

Is $(2x + 3y + 4z)^2 = 4x^2 + 9y^2 + 16z^2$?

Mathias's Solution: Using Substitution and then Evaluating

Let $x = y = z = 1$

$$\begin{aligned} \text{L.S.} &= (2x + 3y + 4z)^2 \\ &= (2 + 3 + 4)^2 \\ &= 9^2 \\ &= 81 \end{aligned} \qquad \begin{aligned} \text{R.S.} &= 4x^2 + 9y^2 + 16z^2 \\ &= 4 + 9 + 16 \\ &= 29 \end{aligned}$$

← I substituted 1 for each of the variables in each expression to see if the results would be different.

Since the left side did not equal the right side, the expressions are not equivalent.

Lee's Solution: Expanding and Simplifying

$(2x + 3y + 4z)^2 = (2x + 3y + 4z)(2x + 3y + 4z)$ ← I wrote the left side of the equation as the product of two identical factors. I multiplied directly, multiplying each term in one polynomial by each term in the other.

$= 4x^2 + 6xy + 8xz + 6xy + 9y^2 + 12yz + 8xz + 12yz + 16z^2$

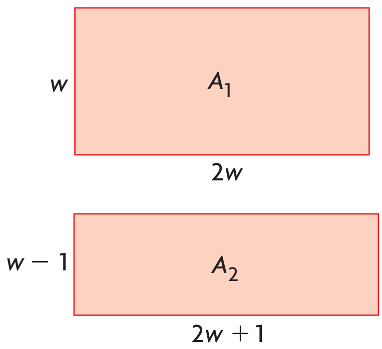
$= 4x^2 + 12xy + 16xz + 9y^2 + 24yz + 16z^2$ ← I simplified by collecting like terms.

The expressions are not equivalent. ← The simplified expression does not result in $4x^2 + 9y^2 + 16z^2$.

EXAMPLE 4 Representing changes in area as a polynomial

A rectangle is twice as long as it is wide. Predict how the area will change if the length of the rectangle is increased by 1 and the width is decreased by 1. Write an expression for the change in area and interpret the result.

Kim's Solution



Since we are increasing the length and decreasing the width by the same amount, I predict that there will be no change in the area.

$$A_1 = (2w)w$$
$$= 2w^2$$

To check my prediction, I let w represent the width and $2w$ the length. Their product gives the original area, A_1.

$$A_2 = (2w + 1)(w - 1)$$
$$= 2w^2 - w - 1$$

I increased the length by 1 and decreased the width by 1 by adding and subtracting 1 to my previous expressions. The product gives the area of the new rectangle, A_2.

$$\text{change in area} = A_2 - A_1$$
$$= (2w^2 - w - 1) - (2w^2)$$
$$= -w - 1$$

I took the difference of the new area, A_2, and the original area, A_1, to represent the change in area.

My prediction was wrong.

w represents width, which is always positive. Substituting any positive value for w into $-w - 1$ results in a negative number. This means that the new rectangle must have a smaller area than the original one.

In Summary

Key Idea

- The product of two or more expressions, one of which contains at least two terms, can be found by using the distributive property (often called expanding) and then collecting like terms.

Need to Know

- Since polynomials behave like numbers, the multiplication of polynomials has the same properties:
  For any polynomials *a*, *b*, and *c*:

$$ab = ba \text{ (commutative property)}$$
$$(ab)c = a(bc) \text{ (associative property)}$$
$$a(b + c) = ab + ac \text{ (distributive property)}$$

  With the use of the distributive property, the product of two polynomials can be found by multiplying each term in one polynomial by each term in the other.

CHECK Your Understanding

1. Expand and simplify.

a) $2x(3x - 5x^2 + 4y)$ **c)** $(x + 4)^2$

b) $(3x - 4)(2x + 5)$ **d)** $(x + 1)(x^2 + 2x - 3)$

2. a) Is $(3x + 2)^2 = 9x^2 + 4$? Justify your decision.

b) Write the simplified expression that is equivalent to $(3x + 2)^2$.

3. Expand and simplify $(2x + 4)(3x^2 + 6x - 5)$ by

a) multiplying from left to right

b) multiplying from right to left

PRACTISING

4. Expand and simplify.

a) $5x(5x^2 + 3x - 4)$

b) $(x - 6)(2x + 5)$

c) $(x + 3)(x - 3) + (5x - 6)(3x - 7)$

d) $4(n - 4)(3 + n) - 3(n - 5)(n + 8)$

e) $3(2x - 1)^2 - 5(4x + 1)^2$

f) $2(3a + 4)(a - 6) - (3 - a)^2 + 4(5 - a)$

5. Expand and simplify.

a) $4x(x + 5)(x - 5)$
b) $-2a(a + 4)^2$
c) $(x + 2)(x - 5)(x - 2)$
d) $(2x + 1)(3x - 5)(4 - x)$
e) $(9a - 5)^3$
f) $(a - b + c - d)(a + b - c - d)$

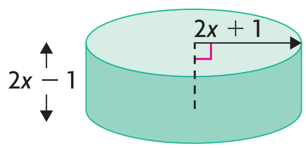
6. Determine whether each pair of expressions is equivalent.

a) $(3x - 2)(2x - 1)$ and $3x(2x - 1) - 2(2x - 1)$
b) $(x - 4)(2x^2 + 5x - 6)$ and
$2x^2(x - 4) + 5x(x - 4) - 6(x - 4)$
c) $(x + 2)(3x - 1) - (1 - 2x)^2$ and $x^2 + 9x - 3$
d) $2(x - 3)(2x^2 - 4x + 5)$ and $4x^3 - 20x^2 + 34x - 30$
e) $(4x + y - 3)^2$ and $16x^2 - 8xy + 24x + y^2 - 6y + 9$
f) $3(y - 2x)^3$ and $-24x^3 + 36x^2y - 18xy^2 + 3y^3$

7. **T** Is the equation $(x - 1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1$ true for all, some, or no real numbers? Explain.

8. **K** Recall that the associative property of multiplication states that $(ab)c = a(bc)$.

a) Verify this property for the product $19(5x + 7)(3x - 2)$ by expanding and simplifying in two different ways.
b) Which method did you find easier?



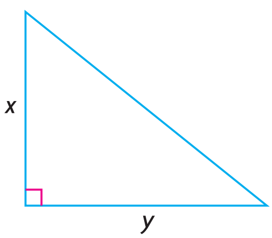
9. **A** A cylinder with a top and bottom has radius $2x + 1$ and height $2x - 1$. Write a simplified expressions for its

a) surface area, where $SA = 2\pi r^2 + 2\pi rh$
b) volume, where $V = \pi r^2 h$

10. a) Is $(x - 3)^2 = (3 - x)^2$? Explain.
b) Is $(x - 3)^3 = (3 - x)^3$? Explain.

11. Expand and simplify.

a) $(x^2 + 2x - 1)^2$
b) $(2 - a)^3$
c) $(x^3 + x^2 + x + 1)(x^3 - x^2 - x - 1)$
d) $2(x + 1)^2 - 3(2x - 1)(3x - 5)$



12. The two sides of the right triangle shown at the left have lengths x and y. Represent the change in the triangle's area if the length of one side is doubled and the length of the other side is halved.

13. The kinetic energy of an object is given by $E = \frac{1}{2}mv^2$, where m represents the mass of the object and v represents its speed. Write a simplified expression for the kinetic energy of the object if

a) its mass is increased by x
b) its speed is increased by y

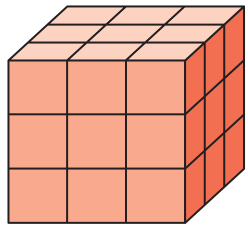
14. **a)** If $f(x)$ has two terms and $g(x)$ has three terms, how many terms will the product of $f(x)$ and $g(x)$ have before like terms are collected? Explain and illustrate with an example.

C

b) In general, if two or more polynomials are to be multiplied, how can you determine how many terms the product will have before like terms are collected? Explain and illustrate with an example.

Extending

15. Suppose a $3 \times 3 \times 3$ cube is painted red and then divided into twenty-seven $1 \times 1 \times 1$ cubes.



a) How many of the 27 smaller cubes are coloured red on

- **i)** three faces?
- **ii)** two faces?
- **iii)** one face?
- **iv)** no faces?

b) Answer (i) to (iv) from part (a) for a $10 \times 10 \times 10$ cube divided into one thousand $1 \times 1 \times 1$ cubes.

c) Answer (i) to (iv) from part (a) for an $n \times n \times n$ cube.

d) Check your results for parts (a) and (b) and by substituting 3 and then 10 into the expressions obtained in your answers to part (c).

16. Many tricks in mental arithmetic involve algebra. For instance, Cynthie claims to have an easy method for squaring any two-digit number whose last digit is 5; for example, 75. Here are her steps:

i) Remove the last digit from the number you wish to square.
ii) Add the resulting number from part (i) to its square.
iii) Write the digits 25 at the end of the number you obtained in part (ii). The number that results will be the answer you want.

a) Choose a two-digit number whose last digit is 5, and determine whether Cynthie's method for squaring works for that number.

b) Show algebraically that Cynthie's method always works.

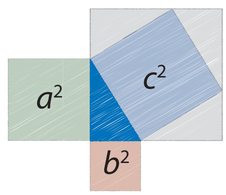
Cynthie's steps for 75^2

75
7
$7 + 7^2 = 56$
$5625 = 75^2$

Curious *Math*

Pythagorean Triples

$a^2 + b^2 = c^2$



A Pythagorean triple is three natural numbers that satisfy the equation of the Pythagorean theorem, that is, $a^2 + b^2 = c^2$.

An example of a Pythagorean triple is 3, 4, 5, since $3^2 + 4^2 = 5^2$. The numbers 6, 8, 10 also work, since each number is just twice the corresponding number in the example 3, 4, 5.

1. Show that 5, 12, 13 and 8, 15, 17 are Pythagorean triples.
2. Show that, for any value of p, $p^2 + \left(\frac{1}{2}p^2 - \frac{1}{2}\right)^2 = \left(\frac{1}{2}p^2 + \frac{1}{2}\right)^2$.
3. Use the relationship in question 2 to produce three new Pythagorean triples.
4. Show that, for any values of p and q, $(2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2$. This relationship was known to the Babylonians about 4000 years ago!
5. Use the relationship in question 4 to identify two more new Pythagorean triples.

2.3 Factoring Polynomials

GOAL

Review and extend factoring skills.

LEARN ABOUT the Math

Mai claims that, for any natural number n, the function $f(n) = n^3 + 3n^2 + 2n + 6$ always generates values that are not **prime**.

? Is Mai's claim true?

EXAMPLE 1 Selecting a factoring strategy: Testing values of n to determine a pattern

Show that $f(n) = n^3 + 3n^2 + 2n + 6$ can be factored for any natural number, n.

Sally's Solution

$f(1) = 12 = 4 \times 3$

$f(2) = 30 = 5 \times 6$

$f(3) = 66 = 6 \times 11$

$f(4) = 126 = 7 \times 18$

$f(5) = 216 = 8 \times 27$

After some calculations and guess and check, I found a pattern. The first factor was of the form $n + 3$ and the second factor was of the form $n^2 + 2$.

$f(n) = (n + 3)(n^2 + 2)$

$= n^3 + 3n^2 + 2n + 6$

To confirm the pattern, I multiplied $(n + 3)$ by $(n^2 + 2)$.

Since both factors produce numbers greater than 1, $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Sometimes an expression that doesn't appear to be factorable directly can be factored by grouping terms of the expression and dividing out common factors.

EXAMPLE 2 Selecting a factoring strategy: Grouping

Factor $f(n) = n^3 + 3n^2 + 2n + 6$ by grouping.

Noah's Solution

$$f(n) = n^3 + 3n^2 + 2n + 6$$
$$= (n^3 + 3n^2) + (2n + 6)$$

I separated $f(n)$ into two groups: the first two terms and the last two terms.

$$= n^2(n + 3) + 2(n + 3)$$

I factored each group by dividing by its common factor.

$$= (n + 3)(n^2 + 2)$$

Then I factored by dividing each term by the common factor $n + 3$.

Both factors produce numbers greater than 1, so $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Reflecting

A. Why is Noah's method called factoring by grouping?

B. What are the advantages and disadvantages of Sally's and Noah's methods of factoring?

APPLY the Math

EXAMPLE 3 Selecting factoring strategies: Quadratic expressions

Factor.

a) $x^2 - x - 30$
b) $18x^2 - 50$
c) $10x^2 - x - 3$
d) $9x^2 + 30x + 25$
e) $2x^2 + x + 3$

Winnie's Solution

a) $x^2 - x - 30$
$$= (x + 5)(x - 6)$$

This is a trinomial of the form $ax^2 + bx + c$, where $a = 1$. I can factor it by finding two numbers whose sum is -1 and whose product is -30. These numbers are 5 and -6.

Check: $(x + 5)(x - 6)$

$= (x + 5)x - (x + 5)6$

$= x^2 + 5x - 6x - 30$

$= x^2 - x - 30$

I checked the answer by multiplying the two factors.

b) $18x^2 - 50$

$= 2(9x^2 - 25)$

First I divided each term by the common factor, 2. This left a difference of squares.

$= 2(3x + 5)(3x - 5)$

I took the square root of $9x^2$ and 25 to get $3x$ and 5, respectively.

c) $10x^2 - x - 3$

This is a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, and it has no common factor.

$= 10x^2 + 5x - 6x - 3$

I used decomposition by finding two numbers whose sum is -1 and whose product is $(10)(-3) = -30$. These numbers are 5 and -6. I used them to "decompose" the middle term.

$= 5x(2x + 1) - 3(2x + 1)$

I factored the group consisting of the first two terms and the group consisting of the last two terms by dividing each group by its common factor.

$= (2x + 1)(5x - 3)$

I divided out the common factor of $2x + 1$ from each term.

d) $9x^2 + 30x + 25$

$= (3x + 5)^2$

I noticed that the first and last terms are perfect squares. The square roots are $3x$ and 5, respectively. The middle term is double the product of the two square roots, $2(3x)(5) = 30x$. So this trinomial is a perfect square, namely, the square of a binomial.

e) $2x^2 + x + 3$

Trinomials of this form may be factored by decomposition. I tried to come up with two integers whose sum is 1 and whose product is 6. There were no such integers, so the trinomial cannot be factored.

EXAMPLE 4 Selecting a factoring strategy: Grouping

Factor $f(x) = x^3 + x^2 + x + 1$.

Fred's Solution

$$f(x) = x^3 + x^2 + x + 1$$
$$= (x^3 + x^2) + (x + 1)$$
$$= x^2(x + 1) + (x + 1)$$

I grouped pairs of terms. Then I factored the **greatest common factor** (GCF) from each pair.

$$= (x + 1)(x^2 + 1)$$

Then I factored out the greatest common factor, $(x + 1)$, to complete the factoring.

EXAMPLE 5 Selecting a factoring strategy: Grouping as a difference of squares

Factor $g(x) = x^2 - 6x + 9 - 4y^2$.

Fran's Solution

$$g(x) = x^2 - 6x + 9 - 4y^2$$
$$= (x - 3)^2 - (2y)^2$$

I recognized that the group consisting of the first three terms was the square of the binomial $x - 3$ and the last term was the square of $2y$.

$$= (x - 3 - 2y)(x - 3 + 2y)$$

I factored the resulting expression by using a difference of squares.

In Summary

Key Ideas

- Factoring a polynomial means writing it as a product. So factoring is the opposite of expanding.

factoring

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

expanding

(continued)

- If a polynomial has more than three terms, you may be able to factor it by grouping. This is only possible if the grouping of terms allows you to divide out the same common factor from each group.

Need to Know

- To factor a polynomial fully means that only 1 and -1 remain as common factors in the factored expression.
- To factor polynomials fully, you can use factoring strategies that include
  - dividing by the greatest common factor (GCF)
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a = 1$
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a \neq 1$
  - recognizing a polynomial that can be factored as a difference of squares: $a^2 - b^2 = (a + b)(a - b)$
  - recognizing a polynomial that can be factored as a perfect square: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$
  - factoring by grouping

CHECK *Your Understanding*

1. Factor.

a) $x^2 - 6x - 27$
b) $25x^2 - 49$
c) $4x^2 + 20x + 25$
d) $6x^2 - x - 2$

2. Each expression given can be factored by grouping. Describe how you would group the terms to factor each.

a) $ac + bc - ad - bd$
b) $x^2 + 2x + 1 - y^2$
c) $x^2 - y^2 - 10y - 25$

3. Factor.

a) $x^2 - 3x - 28$
b) $36x^2 - 25$
c) $9x^2 - 42x + 49$
d) $2x^2 - 7x - 15$

PRACTISING

4. Factor.

a) $4x^3 - 6x^2 + 2x$
b) $3x^3y^2 - 9x^2y^4 + 3xy^3$
c) $4a(a + 1) - 3(a + 1)$
d) $7x^2(x + 1) - x(x + 1) + 6(x + 1)$
e) $5x(2 - x) + 4x(2x - 5) - (3x - 4)$
f) $4t(t^2 + 4t + 2) - 2t(3t^2 - 6t + 17)$

5. Factor.

a) $x^2 - 5x - 14$
b) $x^2 + 4xy - 5y^2$
c) $6m^2 - 90m + 324$
d) $2y^2 + 5y - 7$
e) $8a^2 - 2ab - 21b^2$
f) $16x^2 + 76x + 90$

6. Factor.

a) $x^2 - 9$

b) $4n^2 - 49$

c) $x^8 - 1$

d) $9(y - 1)^2 - 25$

e) $3x^2 - 27(2 - x)^2$

f) $-p^2q^2 + 81$

7. Factor.

a) $ax + ay + bx + by$

b) $2ab + 2a - 3b - 3$

c) $x^3 + x^2 - x - 1$

d) $1 - x^2 + 6x - 9$

e) $a^2 - b^2 + 25 + 10a$

f) $2m^2 + 10m + 10n - 2n^2$

8. **K** Andrij claims that the following statement is true:
$x^3 - y^3 = (x - y)(x^2 + y^2)$
Is Andrij correct? Justify your decision.

9. Factor.

a) $2x(x - 3) + 7(3 - x)$

b) $xy + 6x + 5y + 30$

c) $x^3 - x^2 - 4x + 4$

d) $y^2 - 49 + 14x - x^2$

e) $6x^2 - 21x - 12x + 42$

f) $12m^3 - 14m^2 - 30m + 35$

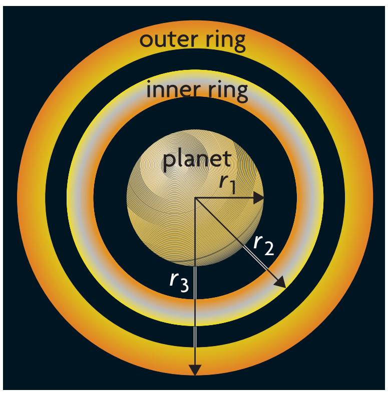
10. **T** Show that the function $f(n) = 2n^3 + n^2 + 6n + 3$ always produces a number that is divisible by an odd number greater than 1, for any natural number, n.

11. Sedna has designed a fishpond in the shape of a right triangle with two sides of length a and b and hypotenuse of length c.

a) Write an expression in factored form for a^2.

b) The hypotenuse is 3 m longer than b, and the sum of the lengths of the hypotenuse and b is 11 m. What are the lengths of the sides of the pond?

12. **A** Saturn is the ringed planet most people think of, but Uranus and Neptune also have rings. In addition, there are ringed planets outside our solar system. Consider the cross-section of the ringed planet shown.



a) Write factored expressions for

i) the area of the region between the planet and the inner ring

ii) the area of the region between the planet and the outer ring

iii) the difference of the areas from parts (i) and (ii)

b) What does the quantity in part (iii) represent?

13. Create a flow chart that will describe which strategies you would use to try to factor a polynomial. For each path through the flow chart, give an example of a polynomial that would follow that path and show its factored form. Explain how your flow chart could describe how to factor or show the non-factorability of any polynomial in this chapter.

C

Extending

14. The polynomial $x^4 - 5x^2 + 4$ is not factorable, but it can be factored by a form of completing the square:

$$\begin{aligned} & x^4 - 5x^2 + 4 \\ &= x^4 + 4x^2 + 4 - 4x^2 - 5x^2 \\ &= (x^2 + 2)^2 - 9x^2 \\ &= (x^2 + 2 - 3x)(x^2 + 2 + 3x) \\ &= (x^2 - 3x + 2)(x^2 + 3x + 2) \\ &= (x - 2)(x - 1)(x + 2)(x + 1) \end{aligned}$$

The first three terms form a perfect square.

This is now a difference of squares.

Use this strategy to factor each polynomial by creating a perfect square.

a) $x^4 + 3x^2 + 36$
b) $x^4 - 23x^2 + 49$

15. Expanding confirms that $x^2 - 1 = (x - 1)(x + 1)$ and also that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.
Make conjectures and determine similar factorings for each expression.

a) $x^4 - 1$
b) $x^5 - 1$
c) $x^n - 1$
d) $x^n - y^n$

16. Mersenne numbers are numbers of the form $n = 2^m - 1$, where m is a natural number. For example, if $m = 6$, then $2^6 - 1 = 64 - 1 = 63$, and if $m = 5$, then $2^5 - 1 = 32 - 1 = 31$. $63 = 3 \times 21$ is a Mersenne number that is composite, and $31 = 1 \times 31$ is a Mersenne number that is prime. The French mathematician Mersenne was interested in finding the values of m that produced prime numbers, n.

a) $63 = 3(21)$ can also be expressed as $(2^2 - 1)(2^4 + 2^2 + 2^0)$, and $63 = 7(9)$ can also be expressed as $(2^3 - 1)(2^3 + 2^0)$. Expand the expressions that contain powers, treating them like polynomials, to show that you get $2^6 - 1$.
b) If $m = 9$, then $n = 2^9 - 1 = 511 = 7(73) = (2^3 - 1)(2^6 + 2^3 + 2^0)$. Using these types of patterns, show that $n = 2^{35} - 1$ is composite.
c) If m is composite, will the Mersenne number $n = 2^m - 1$ always be composite? Explain.

Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you determine whether two polynomials are equivalent? For example, suppose that $f(x) = (x - 2)^2$, $g(x) = (2 - x)^2$, and $h(x) = (x + 2)(x - 2)$.

A1: You can simplify both polynomials. If their simplified versions are the same, the polynomials are equivalent; otherwise, they are not.

Simplifying yields

$f(x) = x^2 - 4x + 4, \quad g(x) = 4 - 4x + x^2, \quad \text{and} \quad h(x) = x^2 - 4.$

So $f(x)$ and $g(x)$ are equivalent, but $h(x)$ is not equivalent to either of them.

A2: If the domains of two functions differ in value for any number in both domains, then the functions are not equivalent.

For example, for the functions above, $f(0) = 4$ while $h(0) = -4$. So $f(x)$ and $h(x)$ are not equivalent.

A3: You can graph both functions. If the graphs are exactly the same, then the functions are equivalent; otherwise, they are not.

Study *Aid*

- See Lesson 2.1, Examples 1, 2, and 3 and Lesson 2.2, Example 3.
- Try Mid-Chapter Review Question 2.

Q: How do you add, subtract, and multiply polynomials?

A: When the variables of a polynomial are replaced with numbers, the result is a number. The properties for adding, subtracting, and multiplying polynomials are the same as the properties for the numbers.

For any polynomials a, b, c:

Commutative Property

$a + b = b + a;\ ab = ba$

(*Note:* $a - b \neq b - a$, except in special cases.)

Associative Property

$(a + b) + c = a + (b + c);\ (ab)c = a(bc)$

(*Note:* $(a - b) - c \neq a - (b - c)$, except in special cases.)

Distributive Property

$a(b + c) = ab + ac;\ a(b - c) = ab - ac$

Study *Aid*

- See Lesson 2.1, Example 1 Anita's Solution and Example 2.
- See Lesson 2.2, Examples 1, 2, and 3 Lee's Solution.
- Try Mid-Chapter Review Questions 1 and 3 to 6.

Because of the distributive property, the product of two polynomials can be found by multiplying each term in one polynomial by each term in the other and can be simplified by collecting like terms. For example,

$$(2x + 3y - 5z)(2x + 3y + 4z)$$
$$= 4x^2 + 6xy + 8xz + 6xy + 9y^2 + 12yz - 10xz - 15yz - 20z^2$$
$$= 4x^2 + 12xy - 2xz + 9y^2 - 3yz - 20z^2$$

Study Aid

- See Lesson 2.3, Examples 1 to 5.
- Try Mid-Chapter Review Questions 7 to 10.

Q: What strategies can you use to factor polynomials?

A: The strategies include:

Common factoring

EXAMPLE

$$9x^2 - 18x$$
$$= 9x(x - 2)$$

Decomposition

EXAMPLE

$$6x^2 + 5x - 4$$
$$= 6x^2 - 3x + 8x - 4$$
$$= 3x(2x - 1) + 4(2x - 1)$$
$$= (2x - 1)(3x + 4)$$

Factoring a difference of squares

EXAMPLE

$$9x^2 - 16$$
$$= (3x + 4)(3x - 4)$$

Factoring by grouping

EXAMPLE

$$5b + 2ab + 4a + 10$$
$$= (5b + 2ab) + (4a + 10)$$
$$= b(5 + 2a) + 2(2a + 5)$$
$$= (2a + 5)(b + 2)$$

PRACTICE Questions

Lesson 2.1

1. Simplify.

a) $(4a^2 - 3a + 2) - (-2a^2 - 3a + 9)$
b) $(2x^2 - 4xy + y^2) - (4x^2 + 7xy - 2y^2) + (3x^2 + 6y^2)$
c) $-(3d^2 - 2cd + d) + d(2c - 5d) - 3c(2c + d)$
d) $3x(2x + y) - 4x[5 - (3x + 2)]$
e) $2a(3a - 5b + 4) - 6(3 - 2a - b)$
f) $7x(2x^2 + 3y - 3) - 3x(9 - 2x + 4y)$

2. Determine whether each pair of functions is equivalent.

a) $g(t) = (t - 2)^5$ and $h(t) = (2 - t)^5$
b) $f(x) = (x^2 - 6x) - (x^2 + x - 4) + (2x^2 + 1)$ and $g(x) = (4x^2 - 7x - 3) - (2x^2 - 8)$
c) $h(x) = (x - 4)(x + 7)(x + 4)$ and $d(x) = (x + 7)(x^2 - 16)$
d) $b(t) = (3t + 1)^3$ and $c(t) = 27t^3 + 27t^2 + 9t - 1$

Lesson 2.2

3. If you multiply a linear polynomial by a quadratic one, what is the degree of the product polynomial? Justify your answer.

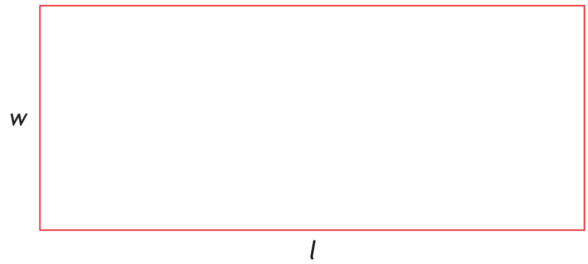
4. The sum of the ages of Pam, Dion, and their three children, in years, is $5x - 99$, where x is Dion's age. Pam is five years younger than Dion. What is the sum of the ages of their children?

5. Expand and simplify.

a) $2(x - 5)(3x - 4)$
b) $(3x - 1)^3$
c) $2(x^2 - 3x + 4)(-x^2 + 3x - 4)$
d) $(5x - 4)(3x - 5) - (2x - 3)^2$
e) $3(2x - 5) - 9(4x - 5)$
f) $-(x - y)^3$

6. If the length of the rectangle shown is increased by 2 and the width is decreased by 1, determine the change in

a) the perimeter **b)** the area



Lesson 2.3

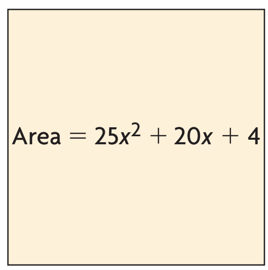
7. Factor.

a) $x(x - 2) - 3(x - 2)$
b) $x^2 - 11x + 28$
c) $3a^2 - 10a - 8$
d) $30x^2 - 9x - 3$
e) $16 - 25x^2$
f) $4(2 - a)^2 - 81$

8. Factor.

a) $2n - 6m + 5n^2 - 15mn$
b) $y^2 + 9 - 6y - x^2$
c) $y - b - (y - b)^2$
d) $2x^2 - 8y^2 + 8x + 8$
e) $w^2 + wb - aw - ab$
f) $ab + b^2 + 6a + 6b$

9. What is the perimeter of the square shown?



10. The expression $3n^2 - 11n + k$ can be factored into two linear polynomials with integer coefficients. Determine the possible values of k.

2.4 Simplifying Rational Functions

GOAL

Define rational functions, and explore methods of simplifying the related rational expression.

LEARN ABOUT the Math

Adonis has designed a game called "2 and 1" to raise money at a charity casino. To start the game, Adonis announces he will draw n numbers from a set that includes all the natural numbers from 1 to $2n$.

The players then pick three numbers.

Adonis draws n numbers and announces them. The players check for matches. Any player who has at least two matches wins.

rational function

any function that is the ratio of two polynomials. A rational function can be expressed as $f(x) = \frac{R(x)}{S(x)}$, where R and S are polynomials and $S \neq 0$; for example,

$$f(x) = \frac{x^2 - 2x + 3}{4x - 1}, x \neq \frac{1}{4}$$

A rational expression is a quotient of polynomials; for example,

$$\frac{2x - 1}{3x}, x \neq 0$$

The probability of a player winning is given by the rational function

$$P(n) = \frac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}$$

For example, if Adonis draws 5 numbers from the set 1 to 10, the probability of winning is

$$P(5) = \frac{3(5)^3 - 3(5)^2}{8(5)^3 - 12(5)^2 + 4(5)}$$
$$= \frac{5}{12}$$

The game is played at a rapid pace, and Adonis needs a fast way to determine the range he should use, based on the number of players and their chances of winning.

? What is the simplified expression for the probability of a player winning at "2 and 1"?

EXAMPLE 1 Simplifying rational functions

Write the simplified expression for the function defined by

$$P(n) = \frac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}.$$

Faez's Solution

$$P(n) = \frac{3n^2(n-1)}{4n(2n^2 - 3n + 1)}$$

I knew that I could simplify rational numbers by first factoring numerators and denominators and dividing each by the common factor $\left(\text{e.g., } \frac{24}{27} = \frac{3(8)}{3(9)} = \frac{8}{9}\right)$.

So I tried the same idea here. I factored the numerator and denominator of $P(n)$.

$$= \frac{3n^2(\cancel{n-1})^{1}}{4n(2n-1)(\cancel{n-1})_{1}}$$

Then I divided by the common factor, $(n - 1)$.

$$= \frac{3n^2}{4n(2n-1)}$$

Restrictions:

When $4n(2n-1)(n-1) = 0$,

$$4n = 0 \quad (2n-1) = 0 \quad (n-1) = 0$$

$$n = 0 \quad n = \frac{1}{2} \quad n = 1$$

Since I cannot divide by zero, I determined the **restrictions** by calculating the values of n that make the factored denominator zero. I solved $4n(2n-1)(n-1) = 0$ by setting each factor equal to 0.

$$P(n) = \frac{3n^2}{4n(2n-1)};\ n \neq 0, \frac{1}{2}, 1.$$

restrictions

the values of the variable(s) in a rational function or rational expression that cause the function to be undefined. These are the zeros of the denominator or, equivalently, the numbers that are not in the domain of the function.

Reflecting

A. How is working with rational expressions like working with rational numbers? How is it different?

B. How do the restrictions on the rational expression in $P(n)$ relate to the domain of this rational function?

C. How does factoring help to simplify and determine the restrictions on the variable?

APPLY the Math

EXAMPLE 2 Selecting a strategy for simplifying the quotient of a monomial and a monomial

Simplify and state any restrictions on the variables.

$$\frac{30x^4y^3}{-6x^7y}$$

Tanya's Solution

$$\frac{30x^4y^3}{-6x^7y} = \frac{\cancel{-6x^4y}^{1}(-5y^2)}{\cancel{-6x^4y}_{1}(x^3)}$$

I factored the numerator and denominator by dividing out the GCF $-6x^4y$. Then I divided the numerator and denominator by the GCF.

$$= \frac{-5y^2}{x^3};\ x, y \neq 0$$

I determined the restrictions by finding the zeros of the *original* denominator by solving $-6x^7y = 0$. This gives the restrictions $x, y \neq 0$.

EXAMPLE 3 Selecting a strategy for simplifying the quotient of a polynomial and a monomial

Simplify and state any restrictions on the variables.

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

Lee's Solution

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

$$= \frac{\cancel{2x}^{1}(5x^3 - 4x + 2)}{\cancel{2x}_{1}(x)}$$

I factored the numerator and denominator by dividing out the GCF $2x$. Then I divided both the numerator and denominator by $2x$.

$$= \frac{5x^3 - 4x + 2}{x};\ x \neq 0$$

I determined the restrictions by solving $2x^2 = 0$ to get the zeros of the original denominator. The only restriction is $x \neq 0$.

EXAMPLE 4 — Selecting a strategy for simplifying a function involving the quotient of a trinomial and a binomial

Simplify $f(x)$ and state the domain, where $f(x) = \frac{x^2 + 7x - 8}{2 - 2x}$.

Michel's Solution

$$f(x) = \frac{x^2 + 7x - 8}{2 - 2x}$$

$$= \frac{(x - 1)(x + 8)}{2(1 - x)}$$

I factored the numerator and denominator and noticed that there were two factors that were similar, but with opposite signs.

$$= \frac{-\cancel{(1 - x)}^{1}(x + 8)}{2\cancel{(1 - x)}_{1}}$$

I divided out the common factor, -1, from $(x - 1)$ in the numerator, so that it became identical to $(1 - x)$ in the denominator. I divided the numerator and denominator by the GCF $1 - x$.

$$= \frac{-(x + 8)}{2}; x \neq 1$$

I determined the restrictions by solving $2(1 - x) = 0$. The only restriction is $x \neq 1$. This means that $f(x)$ is undefined when $x = 1$, so $x = 1$ must be excluded from the domain.

The domain is $\{x \in \mathbf{R} \mid x \neq 1\}$.

EXAMPLE 5 — Selecting a strategy for simplifying the quotient of quadratics in two variables

Simplify and state any restrictions on the variables: $\frac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$.

Hermione's Solution

$$\frac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$$

$$= \frac{4\cancel{(x - 2y)}^{1}(x + 2y)}{(x + 3y)\cancel{(x - 2y)}_{1}}$$

I factored the numerator and denominator and then divided by the GCF $x - 2y$.

$$= \frac{4(x + 2y)}{x + 3y}; x \neq -3y, 2y$$

I determined the restrictions by finding the zeros of the factored denominator by solving $(x + 3y)(x - 2y) = 0$.

So, $(x + 3y) = 0$ and $(x - 2y) = 0$.

The restrictions are $x \neq -3y, 2y$.

In Summary

Key Ideas

- A rational function can be expressed as the ratio of two polynomial functions. For example,

$$f(x) = \frac{6x + 2}{x - 1}; x \neq 1$$

A rational expression is the ratio of two polynomials.
For example,

$$\frac{6x + 2}{x - 1}; x \neq 1$$

- Both rational functions and rational expressions are undefined for numbers that make the denominator zero. These numbers must be excluded or restricted from being possible values for the variables. As a result, for all rational functions, the domain is the set of all real numbers, except those numbers that make the denominator equal zero.

Need to Know

- Rational functions and rational expressions can be simplified by factoring the numerator and denominator and then dividing both by their greatest common factor.
- The restrictions are found by determining all the zeros of the denominator. If the denominator contains two or more terms, the zeros can be determined from its factored form before the function or expression is simplified.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{6 - 4t}{2}$ b) $\frac{9x^2}{6x^3}$ c) $\frac{7a^2b^3}{21a^4b}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5(x + 3)}{(x + 3)(x - 3)}$ b) $\frac{6x - 9}{2x - 3}$ c) $\frac{4a^2b - 2ab^2}{(2a - b)^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{(x - 1)(x - 3)}{(x + 2)(x - 1)}$ b) $\frac{5x^2 + x - 4}{25x^2 - 40x + 16}$ c) $\frac{x^2 - 7xy + 10y^2}{x^2 + xy - 6y^2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\frac{14x^3 - 7x^2 + 21x}{7x}$
b) $\frac{-5x^3y^2}{10xy^3}$
c) $\frac{2t(5 - t)}{5t^2(t - 5)}$
d) $\frac{5ab}{15a^4b - 10a^2b^2}$
e) $\frac{2x^2 + 10x}{-3x - 15}$
f) $\frac{2ab - 6a}{9a - 3ab}$

5. Simplify. State any restrictions on the variables.

a) $\frac{a + 4}{a^2 + 3a - 4}$
b) $\frac{x^2 - 9}{15 - 5x}$
c) $\frac{x^2 - 5x + 6}{x^2 + 3x - 10}$
d) $\frac{10 + 3p - p^2}{25 - p^2}$
e) $\frac{t^2 - 7t + 12}{t^3 - 6t^2 + 9t}$
f) $\frac{6t^2 - t - 2}{2t^2 - t - 1}$

6. State the domain of each function. Explain how you found each answer.

a) $f(x) = \frac{2 + x}{x}$
b) $g(x) = \frac{3}{x(x - 2)}$
c) $h(x) = \frac{-3}{(x + 5)(x - 5)}$
d) $f(x) = \frac{1}{x^2 - 1}$
e) $g(x) = \frac{1}{x^2 + 1}$
f) $h(x) = \frac{x - 1}{x^2 - 1}$

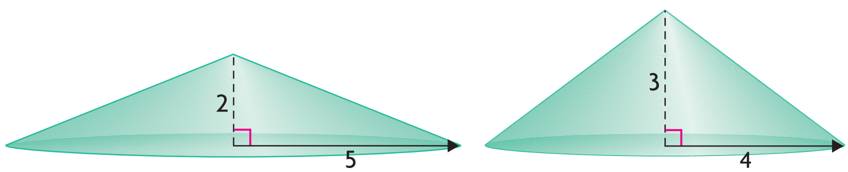
7. Determine which pairs of functions are equivalent. Explain your reasoning.

a) $f(x) = 2x^2 + x - 7$ and $g(x) = \frac{6x^2 + 3x - 21}{3}$
b) $h(x) = 3x^2 + 5x + 1$ and $j(x) = \frac{3x^3 + 5x^2 + x}{x}, x \neq 0$

8. **A** An isosceles triangle has two sides of length $9x + 3$. The perimeter of the triangle is $30x + 10$.

a) Determine the ratio of the base to the perimeter, in simplified form. State the restriction on x.
b) Explain why the restriction on x in part (a) is necessary in this situation.

9. Two cones have radii in the ratio 5:4 and heights in the ratio 2:3. Determine the ratio of their volumes, where $V = \frac{1}{3}\pi r^2 h$.



10. **K** Simplify. State any restrictions on the variables.

a) $\frac{20t^3 + 15t^2 - 5t}{5t}$
b) $\frac{5(4x - 2)}{8(2x - 1)^2}$
c) $\frac{x^2 - 9x + 20}{16 - x^2}$
d) $\frac{2x^2 - xy - y^2}{x^2 - 2xy + y^2}$

11. A rectangle is six times as long as it is wide. Determine the ratio of its area to its perimeter, in simplest form, if its width is w.

12. The quotient of two polynomials is $3x - 2$. Give two examples of a rational expression equivalent to this polynomial that has the restriction $x \neq 4$.

13. Give an example of a rational function that could have three restrictions that are consecutive numbers.

14. Consider the rational expression $\frac{2x + 1}{x - 4}$.

T

a) Identify, if possible, a rational expression with integer coefficients that simplifies to $\frac{2x + 1}{x - 4}$, for each set of restrictions.

i) $x \neq -1, 4$ ii) $x \neq 0, 4$ iii) $x \neq \frac{2}{3}, 4$ iv) $x \neq -\frac{1}{2}, 4$

b) Is there a rational expression with denominator of the form $ax^2 + bx + c, a \neq 0$, that simplifies to $\frac{2x + 1}{x - 4}$, and has only the restriction $x \neq 4$? Explain.

15. Can two different rational expressions simplify to the same polynomial?

C

Explain using examples.

Extending

16. Mathematicians are often interested in the "end behaviour" of functions, that is, the value of the output as the input, x, gets greater and greater and approaches infinity and as the input, x, gets lesser and lesser and approaches negative infinity. For example, the output of $f(x) = \frac{1}{x}$ as x approaches infinity gets closer and closer to 0. By calculating values of the function, make a conjecture about both end behaviours of each rational function.

a) $f(x) = \frac{50x + 73}{x^2 - 10x - 400}$

b) $g(x) = \frac{4x^3 - 100}{5x^3 + 87x + 28}$

c) $h(x) = \frac{-7x^2 + 3x}{200x + 9999}$

Communication *Tip*

"As x approaches positive infinity" is commonly written as $x \to \infty$, and "as x approaches negative infinity" is commonly written as $x \to -\infty$.

The limiting end behaviour of the function $f(x) = \frac{1}{x}$, which approaches zero as x approaches infinity, is written as

$$\lim_{x \to \infty} f(x) = 0$$

17. Simplify. State any restrictions on the variables.

a) $a(t) = \frac{-2(1 + t^2)^2 + 2t(2)(1 + t^2)(2t)}{(1 + t^2)^4}$

b) $f(x) = \frac{2(2x + 1)(2)(3x - 2)^3 - (2x + 1)^2(3)(3x - 2)^2}{(3x - 2)^6}$

2.5 Exploring Graphs of Rational Functions

GOAL

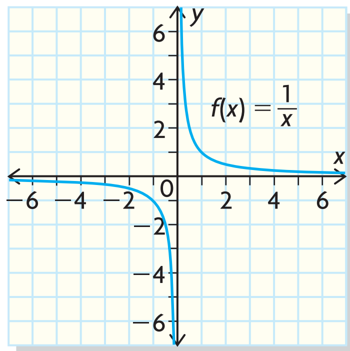
Explore some features of rational functions.

YOU WILL NEED

- graphing calculator

EXPLORE the Math

The graph of the rational function $f(x) = \frac{1}{x}$ is shown at the right. Its domain is $\{x \in \mathbf{R} \mid x \neq 0\}$, and it has a vertical **asymptote** at $x = 0$ and a horizontal asymptote at $y = 0$.



? What are some features of the graphs of rational functions, at or near numbers that are not in their domain?

A. Some rational functions simplify to polynomials. For example, $f(x) = \frac{x^2 - 4}{x - 2}$ can be simplified by factoring from $f(x) = \frac{(x + 2)(x - 2)}{x - 2}$ to $f(x) = x + 2$, where $x \neq 2$. Graph $f(x)$ prior to simplifying it, and zoom in and trace near $x = 2$. Describe what happens to the graph at $x = 2$.

B. Determine another rational function that simplifies to a polynomial with domain $\{x \in \mathbf{R} \mid x \neq 1\}$. Describe what happens to the graph at $x = 1$.

C. Some rational functions cannot be simplified; for example, $g(x) = \frac{1}{x - 3}$. Graph $g(x)$ and zoom in near $x = 3$. Describe what happens to the graph near $x = 3$.

D. Determine another rational function with domain $\{x \in \mathbf{R} \mid x \neq 2\}$ that can't be simplified. Graph your function and describe what happens to the graph at $x = 2$.

E. Determine the equation of a simplified rational function that has two vertical asymptotes: $x = -1$ and $x = 2$. Graph your function.

F. Determine the equation of a rational function that has both a vertical asymptote and a "hole." Graph your function.

G. The rational function $h(x) = \frac{1}{x}$ has a horizontal asymptote $y = 0$. Apply a transformation to $h(x)$ that will result in a rational function that has the horizontal asymptote $y = 2$. Determine the equation of this function and graph it.

H. Determine the equation of a rational function without any "holes," vertical asymptotes, or horizontal asymptotes. Graph your function.

I. Review what you have discovered and summarize your findings.

Reflecting

J. What determines where a rational function has a hole? A vertical asymptote?

K. When does a rational function have the horizontal asymptote $y = 0$? When does a rational function have another horizontal line as a horizontal asymptote?

L. Some rational functions have asymptotes, others have holes, and some have both. Explain how you can identify, without graphing, which graphical features a rational function will have.

In Summary

Key Idea

- The restricted values of rational functions correspond to two different kinds of graphical features: holes and vertical asymptotes.

Need to Know

- Holes occur at restricted values that result from a factor of the denominator that is also a factor of the numerator. For example,

$$g(x) = \frac{x^2 + 7x + 12}{x + 3}$$

has a hole at $x = -3$, since $g(x)$ can be simplified to the polynomial

$$g(x) = \frac{(x + 3)(x + 4)}{(x + 3)} = x + 4$$

- Vertical asymptotes occur at restricted values that are still zeros of the denominator after simplification. For example,

$$h(x) = \frac{5}{x - 8}$$

has a vertical asymptote at $x = 8$.

FURTHER Your Understanding

1. Identify a rational function whose graph is a horizontal line except for two holes. Graph the function.
2. Identify a rational function whose graph lies entirely above the x-axis and has a single vertical asymptote. Graph the function.
3. Identify a rational function whose graph has the horizontal asymptote $y = 2$ and two vertical asymptotes. Graph the function.

2.6 Multiplying and Dividing Rational Expressions

GOAL

Develop strategies for multiplying and dividing rational expressions.

LEARN ABOUT the Math

Tulia is telling Daisy about something that her chemistry teacher was demonstrating. It is about the variables X, Y, and Z.

Tulia: I didn't catch how X, Y, and Z are related.
Daisy: Tell me what the units of the three quantities are.
Tulia: They were $\frac{\text{mol}}{\text{L} \cdot \text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1}, respectively.
Daisy: I have no idea what any of those mean.
Tulia: I guess I'll have to look for help elsewhere.
Daisy: Hold on. All of the units are like rational expressions, so maybe there is some operation that relates them.
Tulia: Like what?

? How are the three quantities related?

EXAMPLE 1 Selecting a strategy for multiplying rational expressions

Use multiplication to show how the expressions $\frac{\text{mol}}{\text{L} \cdot \text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1} are related.

Luke's Solution

$$\frac{\overset{1}{\cancel{\text{mol}}}}{\underset{1}{\cancel{\text{L}}} \cdot \text{s}} \times \frac{\overset{1}{\cancel{\text{L}}}}{\underset{1}{\cancel{\text{mol}}}} = \frac{1}{\text{s}}$$

Daisy suggested that the quantities are related by multiplication. I wrote the quantities as a product and then simplified the expression.

$$\text{s}^{-1} = \frac{1}{\text{s}}$$

$$\text{So, } \frac{\text{mol}}{\text{L} \cdot \text{s}} \times \frac{\text{L}}{\text{mol}} = \text{s}^{-1}$$

I wrote the variable with a negative exponent as a rational expression with a positive exponent. This showed that the quantities are related by multiplication.

Reflecting

A. Was Daisy correct in saying that the units of *X*, *Y*, and *Z* were rational expressions?

B. Explain why Daisy's method for multiplying the rational expressions was correct.

APPLY the Math

EXAMPLE 2 Selecting a strategy for multiplying simple rational expressions

Simplify and state the restrictions.

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

Buzz's Solution

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

When I substituted values for the variables, the result was a fraction, so I multiplied the rational expressions the same way as when I multiply fractions. For $x = 1$, $y = 1$ the expression becomes

$$\frac{6(1)^2}{5(1)(1)} \times \frac{15(1)(1)^3}{8(1)(1)^4} = \frac{6}{5} \times \frac{15}{8} = \frac{9}{4}$$

$$= \frac{90x^3y^3}{40x^2y^5}$$

I multiplied the numerators and then the denominators. I did this by multiplying the coefficients and adding the exponents when the base was the same.

$$= \frac{\overset{1}{\cancel{10x^2y^3}}(9x)}{\underset{1}{\cancel{10x^2y^3}}(4y^2)}$$

I factored the numerator and denominator by dividing out the GCF $10x^2y^3$. Then I divided both the numerator and denominator by the GCF.

$$= \frac{9x}{4y^2};\ x \neq 0, y \neq 0$$

I determined the restrictions by setting the original denominator to zero: $40x^2y^5 = 0$. So, neither x nor y can be zero.

EXAMPLE 3 Selecting a strategy for multiplying more complex rational expressions

Simplify and state the restrictions.

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

Willy's Solution

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{2(2 - x)}$$

I factored the numerators and denominators.

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{-2(-2 + x)}$$

I noticed that $(x - 2)$ in the numerator was the opposite of $(2 - x)$ in the denominator. I divided out the common factor -1 from $(-2 + x)$ to get the signs the same in these factors.

$$= \frac{\overset{1}{\cancel{(x - 2)}}(x + 2)(x + 3)\cancel{(x + 6)}}{-2(x + 6)^{\cancel{2}\,1}\underset{1}{\cancel{(-2 + x)}}}$$

I multiplied the numerators and denominators and then divided out the common factors.

$$= \frac{-(x + 2)(x + 3)}{2(x + 6)};\ x \neq -6, 2$$

I determined the restrictions on the denominators by solving the equations $(x + 6)^2 = 0$ and $(-2 + x) = 0$.

EXAMPLE 4 — Selecting a strategy for dividing rational expressions

Simplify and state the restrictions.

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

Aurora's Solution

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

$$= \frac{21p - 3p^2}{16p + 4p^2} \times \frac{12 + 7p + p^2}{14 - 9p + p^2}$$

When I substituted values for the variables, the result was a fraction, so I divided by multiplying the first rational expression by the reciprocal of the second, just as I would for fractions.

$$= \frac{3p(7 - p)}{4p(4 + p)} \times \frac{(3 + p)(4 + p)}{(7 - p)(2 - p)}$$

I factored the numerators and the denominators.

$$= \frac{3\cancel{p}\overset{1}{\cancel{(7 - p)}}}{4\cancel{p}\underset{1}{\cancel{(4 + p)}}} \times \frac{(3 + p)\overset{1}{\cancel{(4 + p)}}}{\underset{1}{\cancel{(7 - p)}}(2 - p)}$$

I simplified by dividing the numerators and denominators by all of their common factors.

$$= \frac{3(3 + p)}{4(2 - p)};\ p \neq 0, -4, 7, 2, -3$$

I used the factored form of each denominator to determine the zeros by solving for p in $4p = 0$, $(4 + p) = 0$, $(7 - p) = 0$, $(2 - p) = 0$, and $(3 + p) = 0$.

In Summary

Key Idea

- The procedures you use to multiply or divide rational numbers can be used to multiply and divide rational expressions. That is, if A, B, C, and D are polynomials, then

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}, \text{ provided that } B, D \neq 0$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}, \text{ provided that } B, D, \text{ and } C \neq 0$$

(continued)

Need to Know

- To multiply rational expressions,
  - factor the numerators and denominators, if possible
  - divide out any factors that are common to the numerator and denominator
  - multiply the numerators, multiply the denominators, and then write the result as a single rational expression
- To divide two rational expressions,
  - factor the numerators and denominators, if possible
  - multiply by the reciprocal of the divisor
  - divide out any factors common to the numerator and denominator
  - multiply the numerators and then multiply the denominators
  - write the result as a single rational expression
- To determine the restrictions, solve for the zeros of all of the denominators in the factored expression. In the case of division, both the numerator and denominator of the divisor must be used. Both are needed because the reciprocal of this expression is used in the calculation.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{2}{3} \times \frac{5}{8}$

b) $\frac{6x^2y}{5y^3} \times \frac{xy}{8}$

c) $\frac{(x + 1)(x - 5)}{(x + 4)} \times \frac{(x + 4)}{2(x - 5)}$

d) $\frac{x^2}{2x + 1} \times \frac{6x + 3}{5x}$

2. Simplify. State any restrictions on the variables.

a) $\frac{2x}{3} \div \frac{x^2}{5}$

b) $\frac{x - 7}{10} \div \frac{2x - 14}{25}$

c) $\frac{3x(x - 6)}{(x + 2)(x - 7)} \div \frac{(x - 6)}{(x + 2)}$

d) $\frac{x^2 - 1}{x - 2} \div \frac{x + 1}{12 - 6x}$

3. Simplify. State any restrictions on the variables.

a) $\frac{(x + 1)^2}{x^2 + 2x - 3} \times \frac{(x - 1)^2}{x^2 + 4x + 3}$

b) $\frac{2x + 10}{x^2 - 4x + 4} \div \frac{x^2 - 25}{x - 2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\frac{2x^2}{7} \times \frac{21}{x}$

b) $\frac{7a}{3} \div \frac{14a^2}{5}$

c) $\frac{2x^3y}{3xy^2} \times \frac{9x}{4x^2y}$

d) $\frac{3a^2b^3}{2ab^2} \div \frac{9a^2b}{14a^2}$

5. Simplify. State any restrictions on the variables.

a) $\frac{2(x+1)}{3} \times \frac{x-1}{6(x+1)}$

b) $\frac{3a-6}{a+2} \div \frac{a-2}{a+2}$

c) $\frac{2(x-2)}{9x^3} \times \frac{12x^4}{2-x}$

d) $\frac{3(m+4)^2}{2m+1} \div \frac{5(m+4)}{7m+14}$

6. Simplify. State any restrictions on the variables.

a) $\frac{(x+1)(x-3)}{(x+2)^2} \times \frac{2(x+2)}{(x-3)(x+3)}$

b) $\frac{2(n^2-7n+12)}{n^2-n-6} \div \frac{5(n-4)}{n^2-4}$

c) $\frac{2x^2-x-1}{x^2-x-6} \times \frac{6x^2-5x+1}{8x^2+14x+5}$

d) $\frac{9y^2-4}{4y-12} \div \frac{9y^2+12y+4}{18-6y}$

7. Simplify. State any restrictions on the variables.

a) $\frac{x^2-5xy+4y^2}{x^2+3xy-28y^2} \times \frac{x^2+2xy+y^2}{x^2-y^2}$

b) $\frac{2a^2-12ab+18b^2}{a^2-7ab+10b^2} \div \frac{4a^2-12ab}{a^2-7ab+10b^2}$

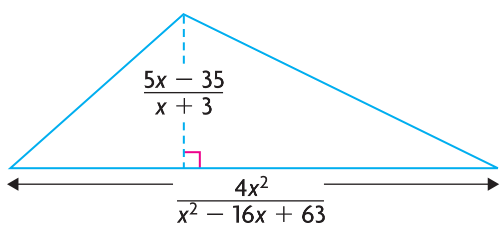
c) $\frac{10x^2+3xy-y^2}{9x^2-y^2} \div \frac{6x^2+3xy}{12x+4y}$

d) $\frac{15m^2+mn-2n^2}{2n-14m} \times \frac{7m^2-8mn+n^2}{5m^2+7mn+2n^2}$

8. **K** Simplify. State any restrictions on the variables.

$$\frac{x^2+x-6}{(2x-1)^2} \times \frac{x(2x-1)^2}{x^2+2x-3} \div \frac{x^2-4}{3x}$$

9. **A** Determine the area of the triangle in simplified form. State the restrictions.



10. An object has mass $m = \dfrac{p + 1}{3p + 1}$ and density $\rho = \dfrac{p^2 - 1}{9p^2 + 6p + 1}$.

Determine its volume v, where $\rho = \dfrac{m}{v}$. State the restrictions on any variables.

11. **T** Liz claims that if $x = y$, she can show that $x + y = 0$ by following these steps:

Since $x = y$,

$x^2 = y^2$ ← I squared both sides of the equation.

So $x^2 - y^2 = 0$. ← I rearranged terms in the equation.

$\dfrac{x^2 - y^2}{x - y} = \dfrac{0}{x - y}$ ← I divided both sides by $x - y$.

$\dfrac{\overset{1}{\cancel{(x - y)}}(x + y)}{\underset{1}{\cancel{x - y}}} = \dfrac{0}{x - y}$ ← I factored and simplified.

$x + y = 0$

Sarit says that's impossible because if $x = 1$, then $y = 1$, since $x = y$. Substituting into Liz's final equation, $x + y = 0$, gives $1 + 1 = 2$, not 0.

Explain the error in Liz's reasoning.

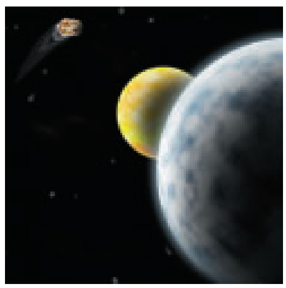
12. **C**
a) Why do you usually factor all numerators and denominators *before* multiplying rational functions?
b) Are there any exceptions to the rule in part (a)? Explain.
c) Sam says that dividing two rational functions and multiplying the first function by the reciprocal of the second will produce the same function. Is this true? Explain.

Extending

13. Simplify. State any restrictions on the variables.

$$\dfrac{\dfrac{m^2 - mn}{6m^2 + 11mn + 3n^2} \div \dfrac{m^2 - n^2}{2m^2 - mn - 6n^2}}{\dfrac{4m^2 - 7mn - 2n^2}{3m^2 + 7mn + 2n^2}}$$

14. Newton's law of gravitation states that any two objects exert a gravitational force on each other due to their masses, $F_g = G\dfrac{m_1 m_2}{r^2}$, where F is the gravitational force, G is a constant (the universal gravitational constant), m_1 and m_2 are the masses of the objects, and r is the separation distance between the centres of objects. The mass of Mercury is 2.2 times greater than the mass of Pluto. Pluto is 102.1 times as far from the Sun as Mercury. How many times greater is the gravitational force between the Sun and Mercury than the gravitational force between the Sun and Pluto?



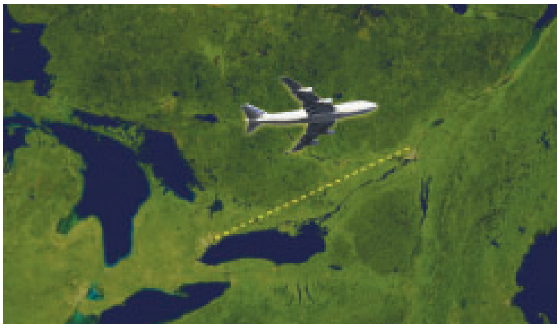
2.7

Adding and Subtracting Rational Expressions

GOAL

Develop strategies for adding and subtracting rational expressions.

LEARN ABOUT the Math



A jet flies along a straight path from Toronto to Montreal and back again. The straight-line distance between these cities is 540 km. On Monday, the jet made the round trip when there was no wind. On Friday, it made the round trip when there was a constant wind blowing from Toronto to Montreal at 80 km/h. While travelling in still air, the jet travels at constant speed.

? Which round trip takes less time?

EXAMPLE 1 Selecting a strategy for adding and subtracting rational expressions

Write expressions for the length of time required to fly from Toronto to Montreal in each situation. Determine which trip takes less time.

Basil's Solution

v is the jet's airspeed in still air.

$v + 80$ is the jet's airspeed from Toronto to Montreal.

$v - 80$ is the jet's airspeed from Montreal to Toronto.

I assigned a variable, v, to the jet's airspeed in still air, since its value is not given. So the speed with the wind from Toronto and the speed against the wind from Montreal are $v + 80$ and $v - 80$, respectively.

$\frac{540}{v}$ is the time elapsed when there is no wind.

$\frac{540}{v + 80}$ is the time elapsed from Toronto to Montreal.

$\frac{540}{v - 80}$ is the time elapsed from Montreal to Toronto.

Using the relation time $= \frac{\text{distance}}{\text{speed}}$, I determined expressions for the elapsed time for each way of the trip at each airspeed.

No wind

$$T_1 = \frac{540}{v} + \frac{540}{v}$$
$$= \frac{1080}{v}$$

I let T_1 represent the time on Monday, with no wind.

I let T_2 represent the time on Friday, with wind.

I found the round-trip times by adding the times for each way.

Wind

$$T_2 = \frac{540}{v + 80} + \frac{540}{v - 80}$$
$$= \frac{540(v - 80) + 540(v + 80)}{(v + 80)(v - 80)}$$
$$= \frac{1080v}{v^2 - 6400}$$

$$T_1 = \frac{1080}{v} \times \frac{v}{v}$$

I noticed that T_1 has the denominator v while T_2's denominator contains v^2. To compare T_1 with T_2, I need to have the same denominator, so I rewrote T_1 by multiplying its numerator and denominator by v.

Now the numerators are both the same.

$$= \frac{1080v}{v^2}$$

T_2 has a smaller denominator because 6400 is subtracted from v^2. Since I am dealing with division, the lesser of the two expressions is the one with the greater denominator, in this case T_1.

The trip without wind took less time.

Reflecting

A. Why were the expressions for time rational expressions?

B. How can you determine a common denominator of two rational functions?

C. How do the methods for adding and subtracting rational expressions compare with those for adding and subtracting rational numbers?

APPLY the Math

EXAMPLE 2 Using the lowest common denominator strategy to add rational expressions

Simplify and state any restrictions on the variables: $\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$.

Sheila's Solution

$\text{LCD} = 24x^3$

I found the **lowest common denominator** (LCD) by finding the least common multiple of $8x^2$, $4x$, and $6x^3$.

$$\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$$
$$= \frac{(3x)3}{(3x)8x^3} + \frac{(6x^2)1}{(6x^2)4x} - \frac{(4)5}{(4)6x^3}$$

I used the LCD to rewrite each term. For each term, I multiplied the denominator by the factor necessary to get the LCD. Then, I multiplied the numerator by the same factor.

$$= \frac{9x + 6x^2 - 20}{24x^3}; x \neq 0$$

I added and subtracted the numerators.

I determined the restrictions on the denominator by solving $24x^3 = 0$.

EXAMPLE 3 Using a factoring strategy to add expressions with binomial denominators

Simplify and state any restrictions on the variables: $\frac{3n}{2n + 1} + \frac{4}{n - 3}$.

Tom's Solution

$\text{LCD} = (2n + 1)(n - 3)$

I found the lowest common denominator by multiplying both denominators.

$$\frac{3n}{2n + 1} + \frac{4}{n - 3}$$
$$= \frac{(n - 3)3n}{(2n + 1)(n - 3)} + \frac{(2n + 1)4}{(2n + 1)(n - 3)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{(n-3)3n + (2n+1)4}{(2n+1)(n-3)}$$

$$= \frac{3n^2 - 9n + 8n + 4}{(2n+1)(n-3)}$$

I simplified by expanding the numerators.

$$= \frac{3n^2 - n + 4}{(2n+1)(n-3)}; \; x \neq -\frac{1}{2}, 3$$

I collected like terms and determined the restrictions by solving $(2n + 1)(n - 3) = 0$.

EXAMPLE 4 Using a factoring strategy to add expressions with quadratic denominators

Simplify and state any restrictions on the variables: $\frac{2t}{t^2 - 1} - \frac{t + 2}{t^2 + 3t - 4}$.

Frank's Solution

$$\frac{2t}{t^2 - 1} - \frac{t+2}{t^2 + 3t - 4}$$

I factored the denominators.

To find the LCD, I created a product by using the three unique factors:

$(t - 1)(t + 1)(t + 4)$

$$= \frac{2t}{(t-1)(t+1)} - \frac{t+2}{(t+4)(t-1)}$$

$$= \frac{(t+4)2t}{(t-1)(t+1)(t+4)} - \frac{(t+1)(t+2)}{(t+1)(t-1)(t+4)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{2t^2 + 8t - t^2 - 2t - t - 2}{(t-1)(t+1)(t+4)}$$

I simplified by expanding the numerators.

$$= \frac{t^2 + 5t - 2}{(t-1)(t+1)(t+4)}; \; t \neq 1, -1, -4$$

I collected like terms and determined the restrictions by solving $(t - 1)(t + 1)(t + 4) = 0$.

In Summary

Key Idea

- The procedures for adding or subtracting rational functions are the same as those for adding and subtracting rational numbers. When rational expressions are added or subtracted, they must have a common denominator.

Need to Know

- To add or subtract rational functions or expressions, determine the lowest common denominator (LCD). To do this, factor all the denominators. The LCD consists of the product of any common factors and all the unique factors.
- The LCD is not always the product of all the denominators.
- After finding the LCD, rewrite each term using the LCD as the denominator and then add or subtract numerators.
- Restrictions are found by finding the zeros of all denominators, that is, the zeros of the LCD.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{1}{3} + \frac{5}{4}$

b) $\frac{2x}{5} + \frac{6x}{2}$

c) $\frac{5}{4x^2} + \frac{1}{7x^3}$

d) $\frac{2}{x} + \frac{6}{x^2}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5}{9} - \frac{2}{3}$

b) $\frac{5y}{3} - \frac{y}{2}$

c) $\frac{5}{3x^2} - \frac{7}{5}$

d) $\frac{6}{3xy} - \frac{5}{y^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{3}{x - 3} - \frac{7}{5x - 1}$

b) $\frac{2}{x + 3} + \frac{7}{x^2 - 9}$

c) $\frac{5}{x^2 - 4x + 3} - \frac{9}{x^2 - 2x + 1}$

4. a) Evaluate $\frac{2}{(x^2 - 9)} + \frac{3}{(x - 3)}$ when $x = 5$.

b) Simplify the original expression by adding.

c) Evaluate the simplified expression when $x = 5$. What do you notice?

PRACTISING

5. Simplify. State any restrictions on the variables.

a) $\frac{2x}{3} + \frac{3x}{4} - \frac{x}{6}$

b) $\frac{3}{t^4} + \frac{1}{2t^2} - \frac{3}{5t}$

c) $\frac{2x}{3y} - \frac{x^2}{4y^3} + \frac{3}{5y^4}$

d) $\frac{n}{m} + \frac{m}{n} - m$

6. Simplify. State any restrictions on the variables.

a) $\frac{7}{a - 4} + \frac{2}{a}$

b) $\frac{4}{3x - 2} + 6$

c) $\frac{5}{x + 4} + \frac{7}{x + 3}$

d) $\frac{6}{2n - 3} - \frac{4}{n - 5}$

e) $\frac{7x}{x + 4} + \frac{3x}{x - 6}$

f) $\frac{7}{2x - 6} + \frac{4}{10x - 15}$

7. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{x+1} + \dfrac{4}{x^2 - 3x - 4}$

b) $\dfrac{2t}{t-4} - \dfrac{5t}{t^2 - 16}$

c) $\dfrac{3}{t^2 + t - 6} + \dfrac{5}{(t+3)^2}$

d) $\dfrac{4x}{x^2 + 6x + 8} - \dfrac{3x}{x^2 - 3x - 10}$

e) $\dfrac{x-1}{x^2 - 9} + \dfrac{x+7}{x^2 - 5x + 6}$

f) $\dfrac{2t+1}{2t^2 - 14t + 24} + \dfrac{5t}{4t^2 - 8t - 12}$

8. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{4x^2 + 7x + 3} - \dfrac{5}{16x^2 + 24x + 9}$

b) $\dfrac{a-1}{a^2 - 8a + 15} - \dfrac{a-2}{2a^2 - 9a - 5}$

c) $\dfrac{3x+2}{4x^2 - 1} + \dfrac{2x-5}{4x^2 + 4x + 1}$

9. Simplify. State any restrictions on the variables. Remember the order of operations.

a) $\dfrac{2x^3}{3y^2} \times \dfrac{9y}{10x} - \dfrac{2y}{3x}$

b) $\dfrac{x+1}{2x-6} \div \dfrac{2(x+1)^2}{2-x} + \dfrac{11}{x-2}$

c) $\dfrac{p+1}{p^2 + 2p - 35} + \dfrac{p^2 + p - 12}{p^2 - 2p - 24} \times \dfrac{p^2 - 4p - 12}{p^2 + 2p - 15}$

d) $\dfrac{5m-n}{2m+n} - \dfrac{4m^2 - 4mn + n^2}{4m^2 - n^2} \div \dfrac{6m^2 - mn - n^2}{3m + 15n}$

10. K Simplify. State any restrictions on the variables.

a) $\dfrac{3m+2}{2} + \dfrac{4m+5}{5}$

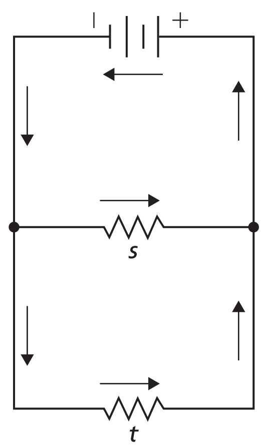
b) $\dfrac{5}{x^2} - \dfrac{3}{4x^3}$

c) $\dfrac{2}{y+1} - \dfrac{3}{y-2}$

d) $\dfrac{2x}{x^2 + x - 6} + \dfrac{5}{x^2 + 2x - 8}$

11. T When two resistors, s and t, are connected in parallel, their combined resistance, R, is given by $\dfrac{1}{R} = \dfrac{1}{s} + \dfrac{1}{t}$.

If s is increased by 1 unit and t is decreased by 1 unit, what is the change in R?



12. **A** Fred drove his car a distance of $2x$ km in 3 h. Later, he drove a distance of $x + 100$ km in 2 h. Use the equation speed $= \dfrac{\text{distance}}{\text{time}}$.

a) Write a simplified expression for the difference between the first speed and the second speed.

b) Determine the values of x for which the speed was greater for the second trip.

13. Matthew is attending a very loud concert by The Discarded. To avoid permanent ear damage, he decides to move farther from the stage. Sound intensity is given by the formula $I = \dfrac{k}{d^2}$, where k is a constant and d is the distance in metres from the listener to the source of the sound.

Determine an expression for the decrease in sound intensity if Matthew moves x metres farther from the stage.

14. **C** **a)** For two rational numbers in simplified form, the lowest common denominator is always one of the following:

i) one of the denominators

ii) the product of the denominators

iii) none of the above

Give an example of each of these.

b) Explain how you would determine the LCD of two simplified rational functions with different quadratic denominators. Illustrate with examples.

Extending

15. In question 11, you encountered an equation of the form $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$, which can be written as $\dfrac{1}{x} = \dfrac{1}{z} - \dfrac{1}{y}$. Suppose you want to determine natural-number solutions of this equation; for example, $\dfrac{1}{6} = \dfrac{1}{2} - \dfrac{1}{3}$ and $\dfrac{1}{20} = \dfrac{1}{4} - \dfrac{1}{5}$.

a) Show that the difference between reciprocals of consecutive positive integers is the reciprocal of their product,

$$\frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1}$$

b) State two more solutions of the equation $\dfrac{1}{x} = \dfrac{1}{z} - \dfrac{1}{y}$.

16. A Pythagorean triple is a triple of natural numbers satisfying the equation $a^2 + b^2 = c^2$. One way to produce a Pythagorean triple is to add the reciprocals of any two consecutive even or odd numbers. For example, $\dfrac{1}{5} + \dfrac{1}{7} = \dfrac{12}{35}$.

Now, $12^2 + 35^2 = 1369$. This is a triple if 1369 is a square, which it is: $1369 = 37^2$. So 12, 35, 37 is a triple.

a) Show that this method always produces a triple.

b) Determine a triple using the method.

2 Chapter Review

FREQUENTLY ASKED Questions

Q: What is a rational function and how do you determine its simplified form?

A: A rational function is a function that can be expressed as a quotient of two polynomials.

The domain of a rational function is the set of all real numbers, except the zeros of the denominator.

To simplify, divide out common factors of the numerator and denominator.

EXAMPLE

$$f(x) = \frac{4\cancel{(x+1)}^{1}(x+2)}{2\cancel{(x+1)}_{1}(x+3)} = \frac{2(x+2)}{(x+3)};\ x \neq -1, -3$$

Study Aid

- See Lesson 2.4, Examples 1 to 5.
- Try Chapter Review Questions 9, 10, and 11.

Q: How do we add, subtract, multiply, and divide rational expressions?

A: Rules for adding, subtracting, multiplying, and dividing rational expressions are the same as those for rational numbers.

EXAMPLE

$$\frac{2x^2}{(x-1)^2} \div \frac{4x}{x^2-1} + \frac{7}{2x-2}$$

$$= \frac{2x^2}{(x-1)(x-1)} \div \frac{4x}{(x+1)(x-1)} + \frac{7}{2(x-1)}$$

$$= \frac{2x^2}{(x-1)(x-1)} \times \frac{(x-1)(x+1)}{4x} + \frac{7}{2(x-1)}$$

$$= \frac{x(x+1)}{2(x-1)} + \frac{7}{2(x-1)}$$

$$= \frac{x^2+x+7}{2(x-1)};\ x \neq 1, -1, 0$$

Study Aid

- See Lesson 2.6, Examples 1 to 4 for multiplication and division.
- See Lesson 2.7, Examples 1 to 4 for addition and subtraction.
- Try Chapter Review Questions 12 to 17.

Q: Why are there sometimes restrictions on the variables in a rational expression, and how do you determine these restrictions?

A: The restrictions occur because division by zero is undefined. To determine restrictions, set all denominators equal to zero before simplifying and solve, usually by factoring.

In the preceding example, set

$(x-1)^2 = 0,\quad x^2 - 1 = 0,\quad 2x - 2 = 0,\quad$ and $\quad 4x = 0$

Solve by factoring:

$(x-1)^2 = 0,\quad (x-1)(x+1) = 0,\quad 2(x-1) = 0,\quad$ and $\quad 4x = 0$

Solving gives the restrictions $x \neq 1, -1, 0$.

Study Aid

- See Lessons 2.4, 2.6, and 2.7, all Examples.
- Try Chapter Review Questions 9 to 17.

PRACTICE Questions

Lesson 2.1

1. Simplify.

a) $(7x^2 - 2x + 1) + (9x^2 - 4x + 5) - (4x^2 + 6x - 7)$

b) $(7a^2 - 4ab + 9b^2) - (-a^2 + 2ab + 6b^2)$

2. Determine two non-equivalent polynomials $f(x)$ and $g(x)$, such that $f(0) = g(0)$ and $f(1) = g(1)$.

3. Ms. Flanagan has three daughters: Astrid, Beatrice, and Cassandra. Today, January 1, their ages are, respectively,

$$A(n) = -(n + 30) + (2n + 5)$$

$$B(n) = (7 - n) - (32 - 2n)$$

$$C(n) = (n - 26) - (n + 4) + (n - 3)$$

All ages are expressed in years, and n represents Ms. Flanagan's age.

a) Are the daughters triplets? Explain.

b) Are any of them twins? Explain.

c) How old was Ms. Flanagan when Cassandra was born?

Lesson 2.2

4. Expand and simplify.

a) $-3(7x - 5)(4x - 7)$

b) $-(y^2 - 4y + 7)(3y^2 - 5y - 3)$

c) $2(a + b)^3$

d) $3(x^2 - 2)^2(2x - 3)^2$

5. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Determine the volume of the cone in simplified form if the radius is increased by x and the height is increased by $2x$.

Lesson 2.3

6. Simplify.

a) $(2x^4 - 3x^2 - 6) + (6x^4 - x^3 + 4x^2 + 5)$

b) $(x^2 - 4)(2x^2 + 5x - 2)$

c) $-7x(x^2 + x - 1) - 3x(2x^2 - 5x + 6)$

d) $-2x^2(3x^3 - 7x + 2) - x^3(5x^3 + 2x - 8)$

e) $-2x[5x - (2x - 7)] + 6x[3x - (1 + 2x)]$

f) $(x + 2)^2(x - 1)^2 - (x - 4)^2(x + 4)^2$

g) $(x^2 + 5x - 3)^2$

7. Factor.

a) $12m^2n^3 + 18m^3n^2$

b) $x^2 - 9x + 20$

c) $3x^2 + 24x + 45$

d) $50x^2 - 72$

e) $9x^2 - 6x + 1$

f) $10a^2 + a - 3$

8. Factor.

a) $2x^2y^4 - 6x^5y^3 + 8x^3y$

b) $2x(x + 4) + 3(x + 4)$

c) $x^2 - 3x - 10$

d) $15x^2 - 53x + 42$

e) $a^4 - 16$

f) $(m - n)^2 - (2m + 3n)^2$

Lesson 2.4

9. Simplify. State any restrictions on the variables.

a) $\dfrac{10a^2b + 15bc^2}{-5b}$

b) $\dfrac{30x^2y^3 - 20x^2z^2 + 50x^2}{10x^2}$

c) $\dfrac{xy - xyz}{xy}$

d) $\dfrac{16mnr - 24mnp + 40kmn}{8mn}$

10. Simplify. State any restrictions on the variables.

a) $8xy^2 + 12x^2y - \dfrac{6x^3}{2xy}$

b) $\dfrac{7a - 14b}{2(a - 2b)}$

c) $\dfrac{m + 3}{m^2 + 10m + 21}$

d) $\dfrac{4x^2 - 4x - 3}{4x^2 - 9}$

e) $\dfrac{3x^2 - 21x}{7x^2 - 28x + 21}$

f) $\dfrac{3x^2 - 2xy - y^2}{3x^2 + 4xy + y^2}$

11. If two rational functions have the same restrictions, are they equivalent? Explain and illustrate with an example.

Lessons 2.6 and 2.7

12. Simplify. State any restrictions on the variables.

a) $\frac{6x}{8y} \times \frac{2y^2}{3x}$

b) $\frac{10m^2}{3n} \times \frac{6mn}{20m^2}$

c) $\frac{2ab}{5bc} \div \frac{6ac}{10b}$

d) $\frac{5p}{8pq} \div \frac{3p}{12q}$

13. Simplify. State any restrictions on the variables.

a) $\frac{x^2}{2xy} \times \frac{x}{2y^2} \div \frac{(3x)^2}{xy^2}$

b) $\frac{x^2 - 5x + 6}{x^2 - 1} \times \frac{x^2 - 4x - 5}{x^2 - 4} \div \frac{x - 5}{x^2 + 3x + 2}$

c) $\frac{1 - x^2}{1 + y} \times \frac{1 - y^2}{x + x^2} \div \frac{y^3 - y}{x^2}$

d) $\frac{x^2 - y^2}{4x^2 - y^2} \times \frac{4x^2 + 8xy + 3y^2}{x + y} \div \frac{2x + 3y}{2x - y}$

14. Simplify. State any restrictions on the variables.

a) $\frac{4}{5x} - \frac{2}{3x}$

b) $\frac{5}{x + 1} - \frac{2}{x - 1}$

c) $\frac{1}{x^2 + 3x - 4} + \frac{1}{x^2 + x - 12}$

d) $\frac{1}{x^2 - 5x + 6} - \frac{1}{x^2 - 9}$

15. Simplify and state any restrictions on the variables.

a) $\frac{1}{2x} - \frac{7}{3x^2} + \frac{4}{x^3}$

b) $\frac{3x}{x + 2} + \frac{4x}{x - 6}$

c) $\frac{6x}{x^2 - 5x + 6} - \frac{3x}{x^2 + x - 12}$

d) $\frac{2(x - 2)^2}{x^2 + 6x + 5} \times \frac{3x + 15}{(2 - x)^2}$

e) $\frac{(x - 2y)^2}{x^2 - y^2} \div \frac{(x - 2y)(x + 3y)}{(x + y)^2}$

f) $\frac{2b - 5}{b^2 - 2b - 15} + \frac{3b}{b^2 + b - 30} \times \frac{b^2 + 8b + 12}{b + 3}$

16. Fred's final mark in an online course was determined entirely by two exams. The mid-term exam was out of x marks and was worth 25% of his final mark. The final exam was out of $2x$ marks and was worth 75% of his final mark. Fred scored 40 marks on the first exam and 60 marks on the second exam. Determine the value of x if Fred earned a final mark of 50% in the course.

17. Sam plays a game in which he selects three different numbers from 1 to n $(n > 3)$. After he selects his numbers, four different winning numbers from 1 to n are chosen, one at a time. Sam wins if all three of his numbers are among the four winning numbers.

The first number chosen is one of Sam's! His probability of winning is now given by

$$P(n) = \frac{24}{n^3 - 3n^2 + 2n} \div \frac{3}{n}$$

a) Simplify $P(n)$ and state the restrictions on n.

b) What would Sam's probability of winning be if

i) $n = 5$?

ii) $n = 4$?

2 Chapter Self-Test

1. Simplify.
   a) $(-x^2 + 2x + 7) + (2x^2 - 7x - 7)$
   b) $(2m^2 - mn + 4n^2) - (5m^2 - n^2) + (7m^2 - 2mn)$

2. Expand and simplify.
   a) $2(12a - 5)(3a - 7)$
   b) $(2x^2y - 3xy^2)(4xy^2 + 5x^2y)$
   c) $(4x - 1)(5x + 2)(x - 3)$
   d) $(3p^2 + p - 2)^2$

3. Is there a value of a such that $f(x) = 9x^2 + 4$ and $g(x) = (3x - a)^2$ are equivalent? Explain.

4. If Bonnie is away from Clyde for n consecutive days, then the amount of heartache Clyde feels is given by $h(n) = (2n + 1)^3$.
   a) If Bonnie is absent, by how much does Clyde's pain increase between day n and day $n + 1$?
   b) How much more pain will he feel on day 6 than on day 5?

5. Factor.
   a) $3m(m - 1) + 2m(1 - m)$
   b) $x^2 - 27x + 72$
   c) $15x^2 - 7xy - 2y^2$
   d) $(2x - y + 1)^2 - (x - y - 2)^2$
   e) $5xy - 10x - 3y + 6$
   f) $p^2 - m^2 + 6m - 9$

6. Use factoring to determine the x-intercepts of the curve $y = x^3 - 4x^2 - x + 4$.

7. Simplify. State any restrictions on the variables.
   a) $\dfrac{4a^2b}{5ab^3} \div \dfrac{6a^2b}{35ab}$
   b) $\dfrac{x - 2}{x^2 - x - 12} \times \dfrac{2x - 8}{x^2 - 4x + 4}$
   c) $\dfrac{5}{t^2 - 7t - 18} + \dfrac{6}{t + 2}$
   d) $\dfrac{4x}{6x^2 + 13x + 6} - \dfrac{3x}{4x^2 - 9}$

8. Mauro found that two rational functions each simplified to $f(x) = \dfrac{2}{x + 1}$.

   Are Mauro's two rational functions equivalent? Explain.

9. Roman thinks that he has found a simple method for determining the sum of the reciprocals of any three consecutive natural numbers. He writes, for example,

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}, \quad \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{74}{120}, \quad \text{or} \quad \frac{37}{60}$$

   Roman conjectures that before simplification, the numerator of the sum is three times the product of the first and third denominators, plus 2. Also, the denominator of the sum is the product of the three denominators. Is Roman's conjecture true?

The Algebraic Dominos Challenge

The game shown at the right consists of eight pairs of coloured squares called dominos.

1*R*	2*R*	2*P*	4*P*
1*P*	3*R*	5*R*	4*R*
6*P*	3*P*	5*P*	7*R*
6*R*	8*P*	8*R*	7*P*

Rules:

1. Write a polynomial in each square marked *P* and a rational function in each square marked *R*.
2. The expressions you write must satisfy each of these conditions:
   - Polynomials and numerators and denominators of each rational function must be quadratics without a constant common factor.
   - Restrictions on the variable of each rational function must be stated in its square.
   - When two polynomials are side by side, then one or both of the polynomials must be perfect squares.
   - When a polynomial and a *different-coloured* rational expression are side by side, their product must simplify.
   - When two rational expressions are side by side, their product must simplify.
   - When a polynomial is on top of another polynomial, their quotient must simplify.
   - When a polynomial is on top of a *different-coloured* rational expression (or vice versa), their quotient must simplify.
   - When a rational expression is on top of a rational expression, their quotient must simplify.
3. After you have completed the table, simplify the products and quotients wherever possible. You get one point for every *different* linear factor that remains in your table.
4. Count the linear factors and write your score next to your table.

? How can you maximize your score?

A. What form for the polynomials, including numerators and denominators, will make filling the table and counting your score as easy as possible?

B. Why should you avoid reusing a factor unless it is necessary?

C. Play the game by completing the table.

D. Tally your score.

E. Check your answers. What could you do to increase your score?

F. List some strategies you can use to maximize your score.

Task Checklist

- ✔ Does each square contain a polynomial and rational function of the right type?
- ✔ Are all of the rules satisfied?
- ✔ Did you check to see if you could make changes to improve your score?