Math 10P, 11C, 11UC

Course Notes

Properties of Parabolas

quadratic

$a > 0$

$a < 0$

3.1 What are Parabolas → ax^2

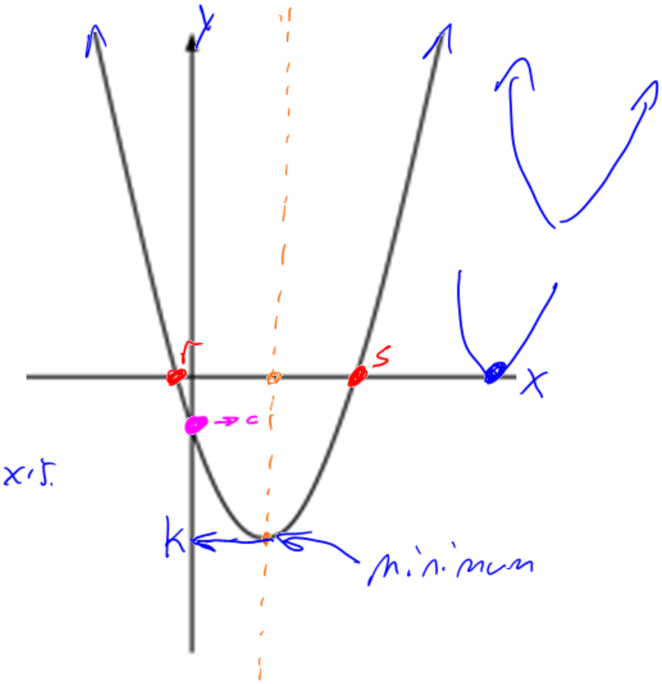
Learning Goal: We are learning the properties of Quadratics and how they relate to the parabola.

There are several important properties of quadratics that we will be learning:

- Direction of Opening

  opens up or down

  if $a > 0$, up   if $a < 0$, down



- Y-intercept

  - where the parabola touches the y-axis

  - $y = c$ $(0, c)$

- X-intercept(s), also called the zeroes

  - where the parabola touches the x-axis.

  → $x = r$, $x = s$

  $(r, 0)$ $(s, 0)$

- Axis of Symmetry (AoS)

  - a vertical line which represents a line of symmetry, meaning you could "fold" the parabola...

  $x = h$

- Maximum / Minimum Value

  - the top or bottom of the parabola

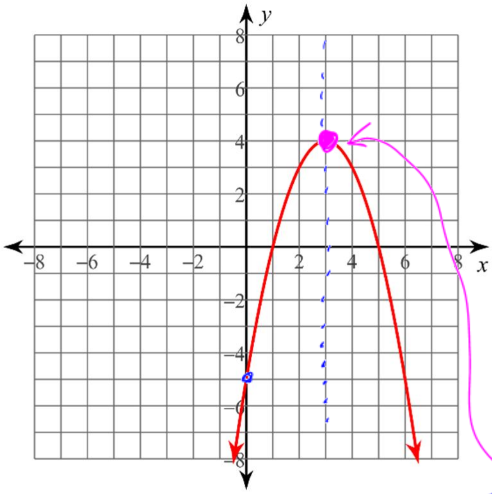
  $y = k$

- Vertex

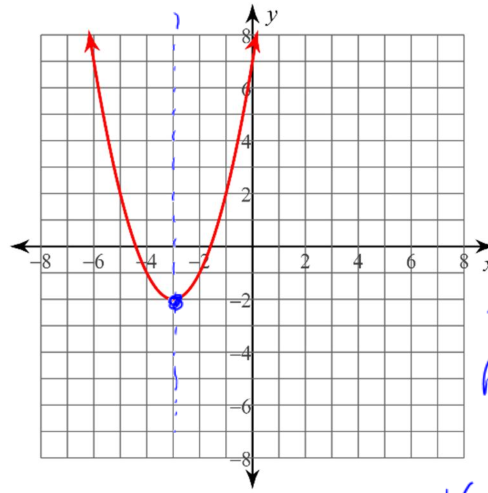
  - the highest or lowest point, it is the middle point, it is unique.

  (h, k)

Example 1: State the direction of opening, the y-intercept, the zeros (x-int), the AoS, the max/min value and the vertex.



- opens down
- y-int = -5 c
- zeros: x = 1 r
  x = 5 s
- AoS: x = 3 h
- Max of 4 K
- Vertex: (3,4)
  h K



- up
- y-int = 7

zeros:
x = -1.5
x = -4.5

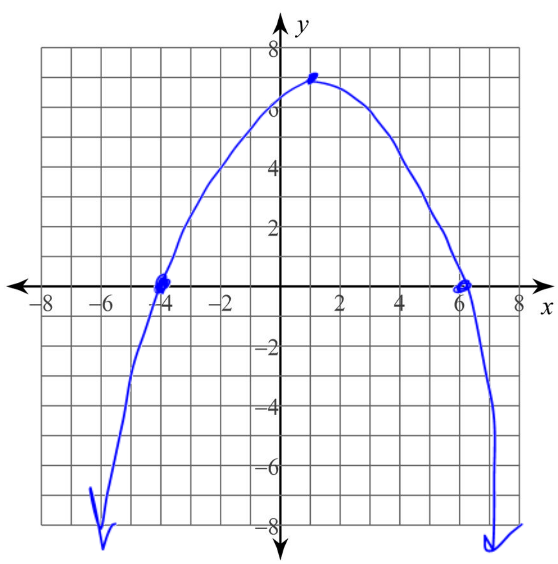
AoS: x = -3

Min of -2

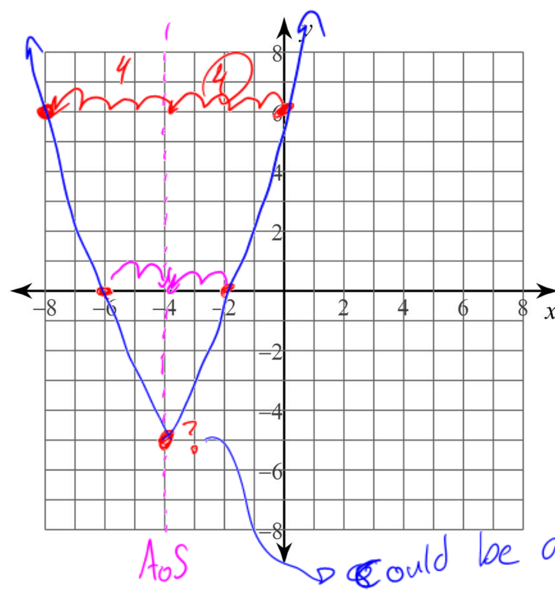
Vertex: (-3,-2)

Example 2: Sketch parabolas based on the following properties. Be reasonably accurate.

a) Zeros at $x = -4$ and $x = 6$. Vertex at (1,7).

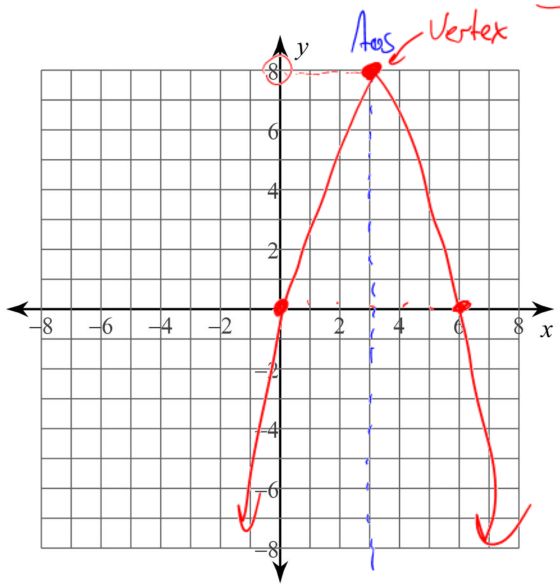


b) Zeros at $x = -2$ and $x = -6$. Y-int at (0,6).

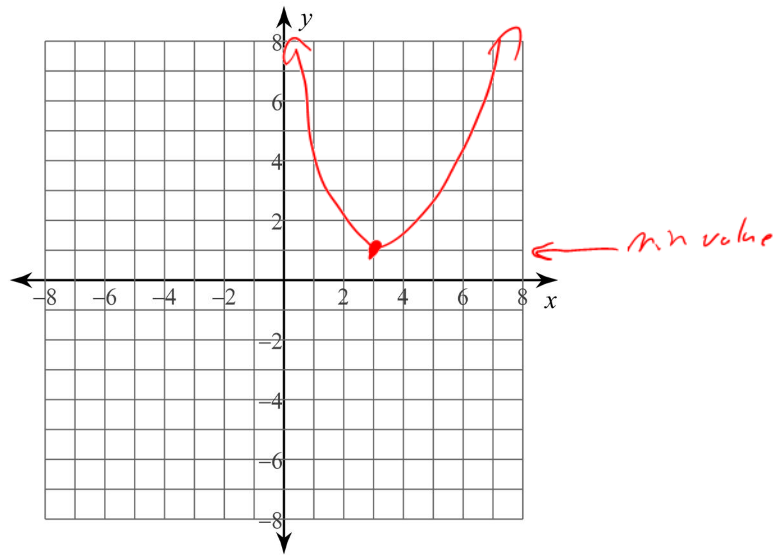


could be anywhere.

c) AoS at $x = 3$, Max Value at $y = 8$, and y-int at 0.



d) No zeros, min value at $y = 1$.



Three Forms of a Quadratic Equation

Standard Form

$y = ax^2 + bx + c$

open up/down

y-int

ex: $y = \underbrace{-3}x^2 + 5x \underbrace{-8}$

opens down

y-int

Factored Form Zeros Form

$y = a(x - r)(x - s)$

open up/down

x-intercepts

ex: $y = 5(x - \boxed{8})(x + 2)$

$x - (-2)$

x-ints are:

$x = 8$

$x = -2$

Vertex Form

$y = a(x - \boxed{h})^2 + k$

open up/down

(h,k) the vertex

ex: $y = -10(x+3)^2 - 5$

Success Criteria

- I can identify the properties of a parabola from a graph
- I can recognize the three different forms of a quadratic

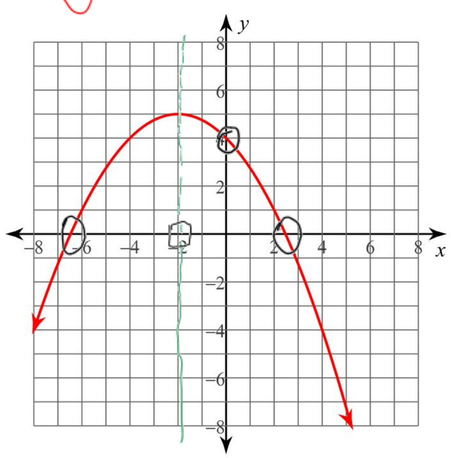
3.2 Factored Form $y = a(x - r)(x - s)$

Learning Goal: We are learning to work with the factored form of a quadratic, and to convert from the standard form to the factored form and vice-versa.

Working with the Factored Form

Let's review from yesterday. From the parabola below, tell us as many properties as you can.

$y = -\frac{1}{4}x^2 - x + 4$



① opens down $a = -\frac{1}{4}$
② y-intercept $y = 4$
③ zeros $x = -6.5$
$x = 2.5$
④ AoS $x = -2$
⑤ Max of 5
Min
⑥ Vertex $(-2, 5)$
(AoS, max/min)

Now, what happens if we take the average of the two zeros? What number do we get?

$\frac{(-6.5)+(2.5)}{2} = \frac{-4}{2} = -2$ we get the AoS!! :)

Factored form is immensely useful, because it gives us a lot of the properties of a parabola. It tells us the direction of opening, and the zeros. But we can THEN use the zeros to determine the Axis of Symmetry! And then we use the AoS to determine the Max/Min value....which is therefore the coordinates of the vertex!

Determine the AoS, Max/Min value, and the coordinates of the vertex.

a) The parabola $y = -x^2 + 8x - 12$ has zeros at $x = 2$ and $x = 6$.

① AoS = average of the zeros
$= \frac{2+6}{2}$
$= \frac{8}{2}$
$= 4$

② Maximum because a is negative
$y = -(4)^2 + 8(4) - 12$
$= -16 + 32 - 12$
$y = 4$

$\therefore$ vertex is $(4, 4)$

a positive

b) The parabola $y = 2x^2 - 4x - 6$ has zeros at $x = -1$ and $x = 3$.

① AoS
$= \frac{-1+3}{2}$
$= \frac{2}{2}$
$= 1$

② Minimum
$y = 2(1)^2 - 4(1) - 6$
$= 2 - 4 - 6$
$y = -8$

∴ vertex
$(1, -8)$

Converting between Factored and Standard Forms

Sadly, you will not always be given a quadratic in factored form ☹. So you must be able to convert from the standard form into factored form by factoring. Furthermore, if you are given the factored form, you can convert to standard form by expanding, and this will help you find the y-intercept.

Convert from Factored Form to Standard Form, then state the y-intercept.

a) $y = -3(x-5)(x+8)$ FOIL first

$= -3(x^2 + 8x - 5x - 40)$
$= -3(x^2 + 3x - 40)$
$= -3x^2 - 9x \boxed{+120}$

y-int: (0,120)

b) $y = \frac{1}{2}(x+2)(x+10)$

$= \frac{1}{2}(x^2 + 10x + 2x + 20)$
$= \frac{1}{2}(x^2 + 12x + 20)$
$= \frac{1}{2}x^2 + 6x \boxed{+10}$

y-int = (0,10)

Convert from Standard Form to Factored Form

a) $y = x^2 + 10x \boxed{+24}$ y-int

1 24
2 12
$\boxed{+4 \; +6}$

$y = 1(x+4)(x+6)$

$(x-(-4))$

$x = -4$ and $x = -6$ ← zeros

b) $y = -2x^2 - 6x + 108$

*Always factor a first!

$y = -2(x^2 + 3x - 54)$

1 54
2 27
3 18
$\boxed{-6 \; +9}$

$y = -2(x-6)(x+9)$

$x = 6$ and $x = -9$ ← zeros

Now that we are in factored form, what are the zeros, AoS, max/min Value, and vertex?

AoS $= \frac{-4 + -6}{2} = \frac{-10}{2} = \boxed{-5}$
$x = -5$

a +, minimum $y = (-5)^2 + 10(-5) + 24$
$= 25 - 50 + 24$
$= -1$

So vertex is $(-5, -1)$

AoS $= \frac{6 + (-9)}{2} = \frac{-3}{2} = -1.5$

a −, maximum $y = -2(-1.5)^2 - 6(-1.5) + 108$
$= -4.5 + 9 + 108$
$= 112.5$

vertex $(-1.5, 112.5)$

Success Criteria

- I can find the zeros, AoS, max/min value, and vertex of a parabola if its equation is in factored form
- I can convert from factored form to standard form by expanding the brackets
- I can convert from standard to factored form by factoring by decomposition

3.3 Vertex Form $y = a(x - h)^2 + k$

Learning Goal: We are learning to work with the vertex form of a quadratic.

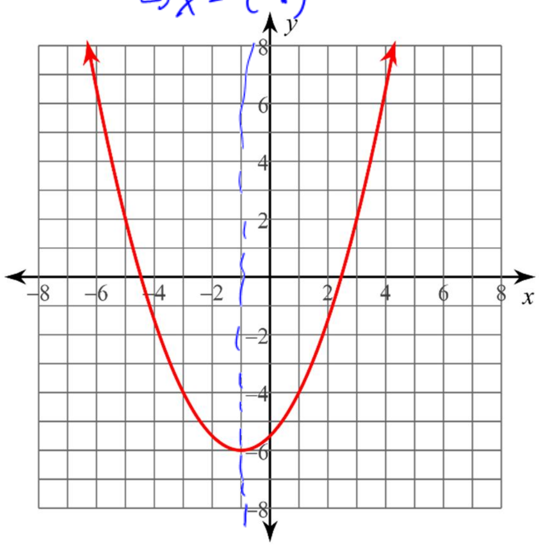
What is Vertex Form? What does it tell us?

Take a look at the graph below. Identify the properties of the parabola.

(direction of opening, y-int, zeros, Aos, max/min value, vertex)

$y = \frac{1}{2}(x + 1)^2 - 6$

$\hookrightarrow x - (-1)$



- opens up
- y-int is -5.5
- zeros: $x = 2.5$ and $x = -4.5$
- Aos: $x = -1$
- Min of $y = -6$
- Vertex $(-1, -6)$

Now, compare the equation to the vertex you identified. What do you notice?

This is the power of the Vertex Form! It tells us the...wait for it... vertex !!!! But be careful, the x value of the vertex is the opposite of what you see in the equation (similar to zeros in factored form). Also, notice that the "a" value of ½ is positive, and our graph also opens up.

Today, we are going to do two things

1) Convert the Vertex Form back into Standard Form
2) Learn how to graph a parabola from Vertex Form

$y = ax^2 + bx + c$

Converting Vertex to Standard Form

a) $y = 2(x-5)^2 + 8$ FOIL

$y = 2(x-5)(x-5) + 8$

$y = 2(x^2 - 5x - 5x + 25) + 8$

$y = 2(x^2 - 10x + 25) + 8$

$y = 2x^2 - 20x + 50 + 8$

$y = 2x^2 - 20x + \underbrace{58}_{y\text{-int}}$

b) $y = -(x+11)^2 - 22$

$y = -(x+11)(x+11) - 22$

$y = -(x^2 + 11x + 11x + 121) - 22$

$y = -1(x^2 + 22x + 121) - 22$

$y = -x^2 - 22x - 121 - 22$

$y = -x^2 - 22x - 143$

c) $y = \frac{2}{3}(x+6)^2 + 7$

$y = \frac{2}{3}(x+6)(x+6) + 7$

$y = \frac{2}{3}(x^2 + 6x + 6x + 36) + 7$

$y = \frac{2}{3}(1x^2 + 12x + 36) + 7$

$y = \frac{2}{3}x^2 + 8x + 24 + 7$

$y = \frac{2}{3}x^2 + 8x + 31$

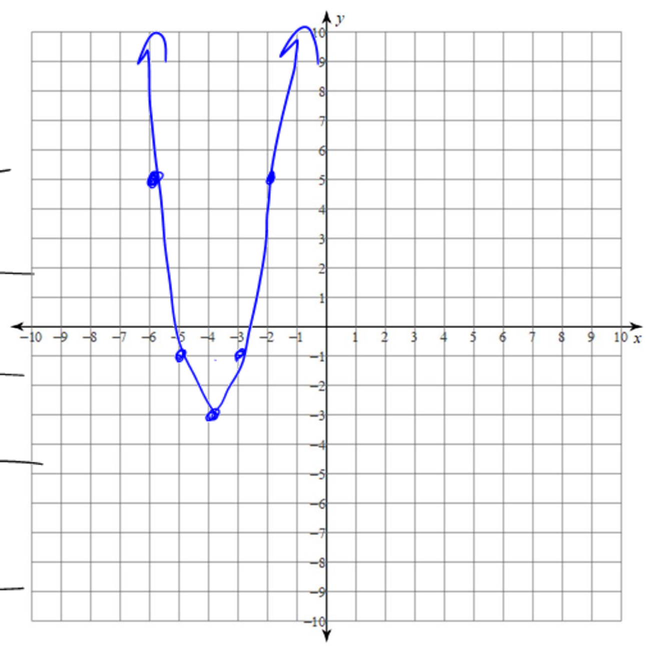
$\frac{2}{3} \times \frac{12}{1} = \frac{24}{3} = 8$

Graphing from Vertex Form

The vertex is the most important part of a parabola. In fact, it is considered the “optimal” point on the parabola. So, if we know the vertex (which we do in vertex form) we can easily find other points around the vertex using a table of values.

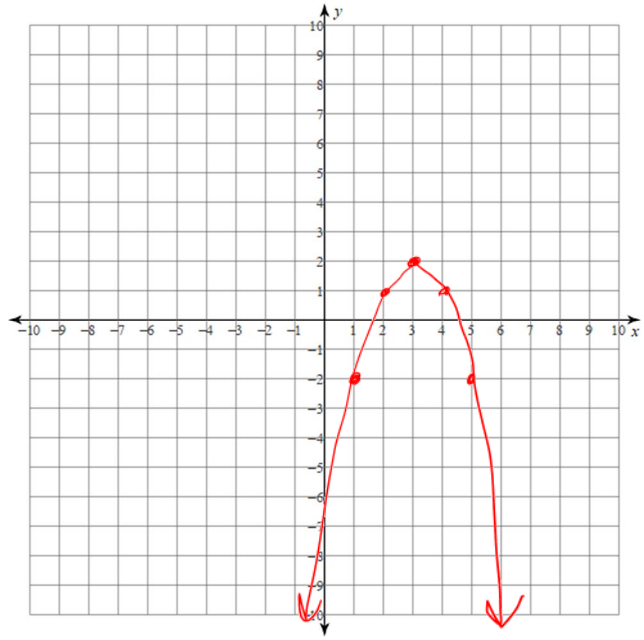
a) $y = 2(x+4)^2 - 3$ opposite vertex $(-4, -3)$

x	y	(x, y)
-6	$y = 2(-6+4)^2 - 3 = 5$	$(-6, 5)$
-5	$y = 2(-5+4)^2 - 3 = -1$	$(-5, -1)$
-4	-3 vertex = middle	$(-4, -3)$
-3	-1	$(-3, -1)$
-2	5	$(-2, 5)$



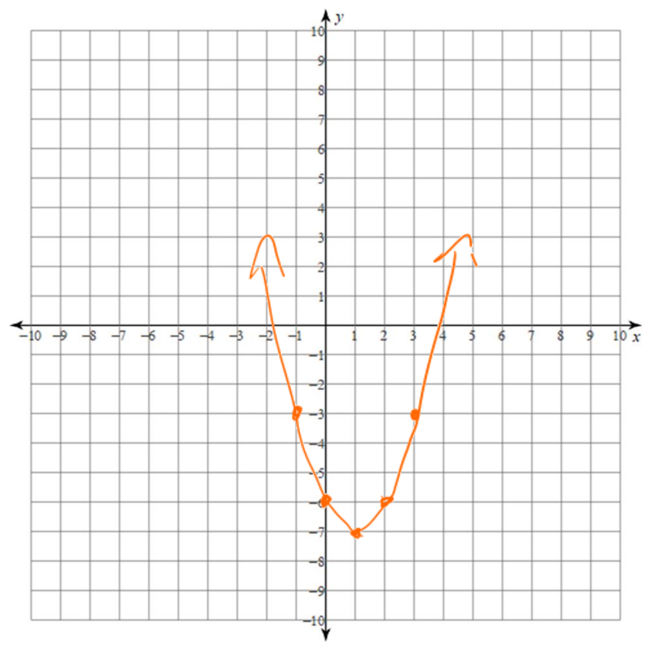
b) $y = -(x - \boxed{3})^2 + 2$ Vertex $(3, 2)$

x	y
1	-2
2	1
3	2
4	$y = (-)(4-3)^2 + 2 = 1$
5	$y = -(\underbrace{5-3}_{2})^2 + 2 = -2$



c) $y = (x - \boxed{1})^2 - 7$ vertex $(1, -7)$

x	y
-1	-3
0	-6
1	-7
2	$y = (2-1)^2 - 7 = -6$
3	$y = (3-1)^2 - 7 = -3$



Success Criteria

- I can identify the vertex, and direction of opening, if given a quadratic in vertex form
- I can convert from vertex form to standard form by expanding the brackets and collecting like terms
- I can graph a parabola if given a quadratic in vertex form

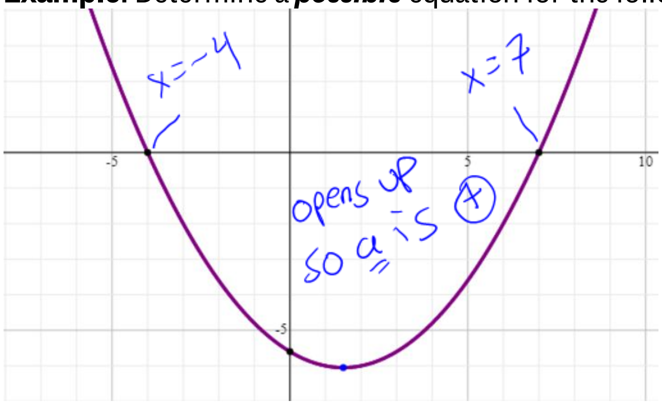
3.4 Determining an Equation

Learning Goal: We are learning to generate the equation of a quadratic in factored form based on its graph.

Finding an Equation using Factored Form

We can easily determine a ***possible*** equation of a quadratic in factored form by studying its zeros and the direction of opening. Since we know the zeroes, we should write our equation in factored form: $y = a(x - r)(x - s)$.

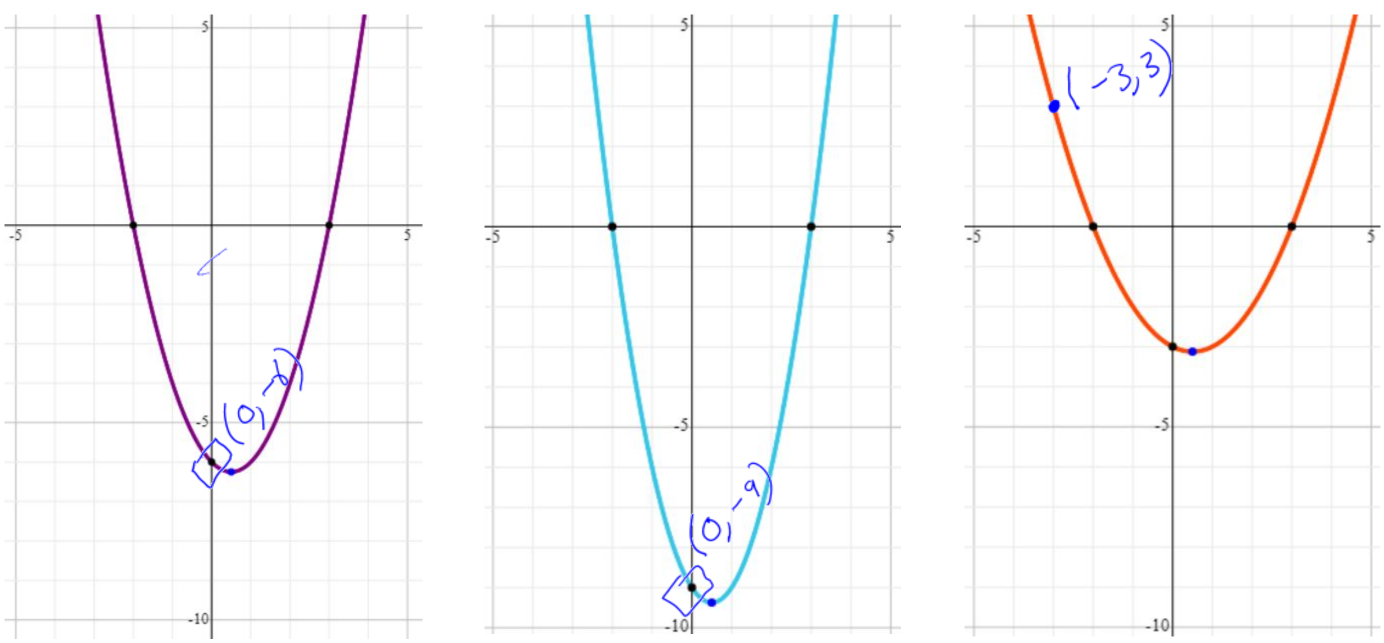
Example: Determine a ***possible*** equation for the following parabola.



$y = +a(x-(-4))(x-7)$

$y = \boxed{a(x+4)(x-7)}$

Now, look at the three pictures below. They all have the same zeroes and direction of opening but are clearly different! So, what part of the equation $y = a(x - r)(x - s)$ tells them apart?



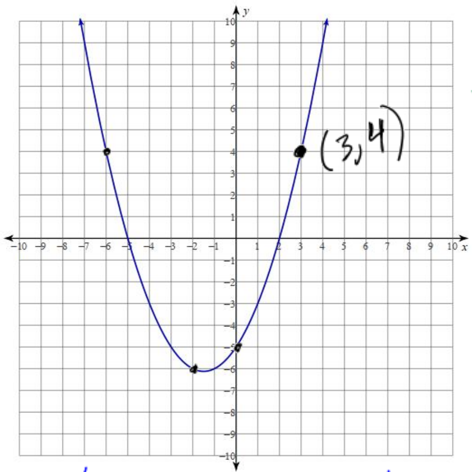
So if we want to find the ***exact*** equation for a particular quadratic, we need to find out what "*a*" is. We can do that by

1) Taking a point (x,y) that is on the parabola
2) Plugging it into the equation.
3) Then we can solve for "*a*".

In factored form, the only points that you CANNOT substitute in are the zeros themselves.

Example: Determine the equation of each parabola in factored form.

Example 1



Zeros

$x = -5$

$x = 2$

Start w/ Skeleton Equation

$y = a(x-(-5))(x-2)$

$y = a(x+5)(x-2)$

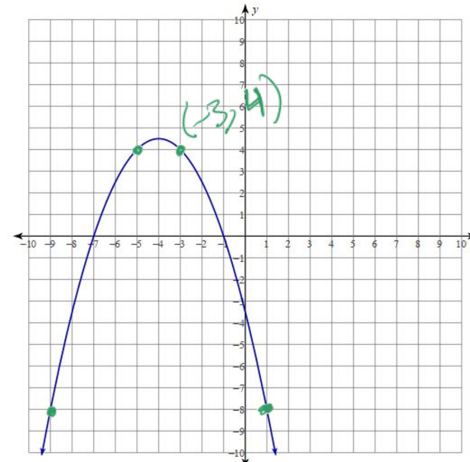
Substitute (3,4) into Equation + solve for a.

$4 = a(3+5)(3-2)$

$4 = a(8)(1)$

$\div 8 \quad 4 = 8a \div 8 \rightarrow \frac{1}{2} = a \rightarrow \boxed{\therefore y = \frac{1}{2}(x+5)(x-2)}$

Example 2



Zeros

$x = -7$

$x = -1$

$y = a(x-(-7))(x-(-1))$

$y = a(x+7)(x+1)$

$4 = a(-3+7)(-3+1)$

$4 = a(4)(-2)$

$\frac{4}{-8} = \frac{-8a}{-8}$

$-\frac{1}{2} = a \qquad \therefore y = -\frac{1}{2}(x+7)(x+1)$

Determining an Equation using Vertex Form

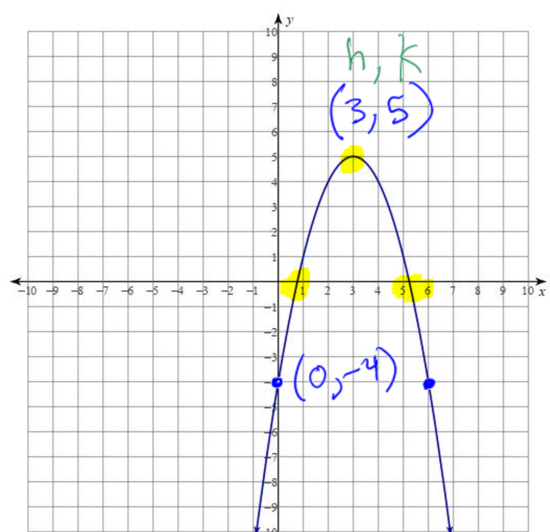
Take a look at the parabola to the right. Notice that the zeros do not line up nicely on axis, which means that factored form is not our friend.

But notice that the we can read the vertex nicely. It is at (3,5).

So that means a ***possible*** equation for the parabola to the right is:

$y = a(x-h)^2 + k$

$y = a(x-3)^2 + 5$



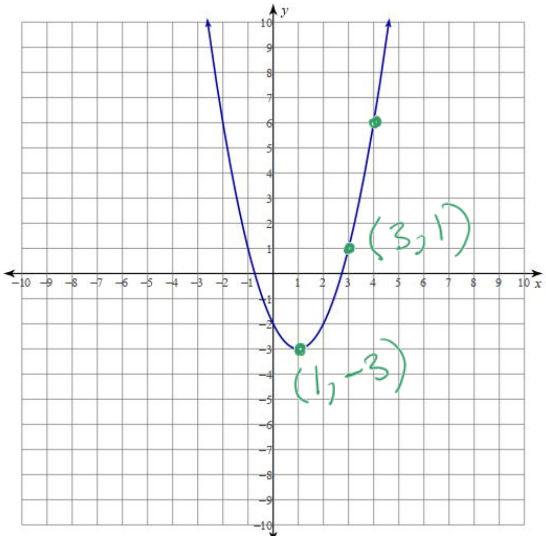
But if we want to find the ***exact*** equation the quadratic, we need to find out what "a" is. We can do that by

1) Taking a point *(x,y)* that is on the parabola
2) Plugging it into the equation
3) Solving for "*a*"

In vertex form, the only point that you cannot substitute into the equation is the vertex itself.

Example: Determine the equation of each parabola in vertex form.

Example 1



① Develop the Skeleton Equation

$y = a(x-1)^2 - 3$

② Sub in the point you chose.

(3,1)

$1 = a(3-1)^2 - 3$

$1 = a(4) - 3$

$1 = 4a - 3$

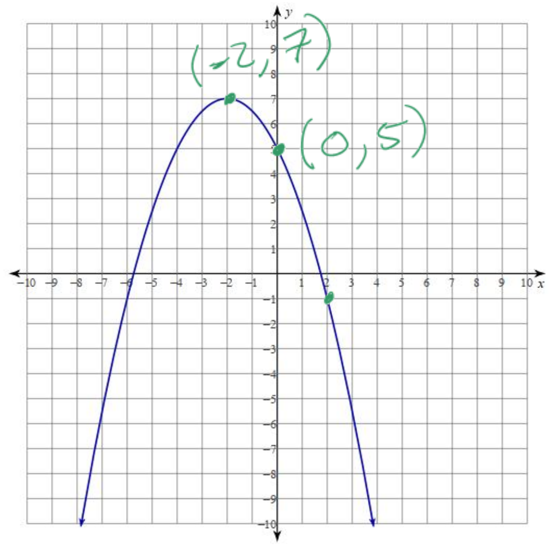
$+3 \qquad\qquad +3$

$\frac{4}{4} = \frac{4a}{4}$

$1 = a$

$\boxed{y = 1(x-1)^2 - 3}$

Example 2



$y = a(x-(-2))^2 + 7$

$y = a(x+2)^2 + 7$

Sub in (0,5)

$5 = a(0+2)^2 + 7$

$5 = a(4) + 7$

$5 = 4a + 7$

$-7 \qquad\qquad -7$

$\frac{-2}{4} = \frac{4a}{4}$

$-\frac{1}{2} = a$

$y = -\frac{1}{2}(x+2)^2 + 7$

Success Criteria

- I can determine the equation of a parabola in factored form
- I can determine the equation of a parabola in ~~factored~~ vertex form
- I can decide which form of the equation is the best to use (11UC)