Name: ______________________________

PROPERTIES

of

QUADRATICS

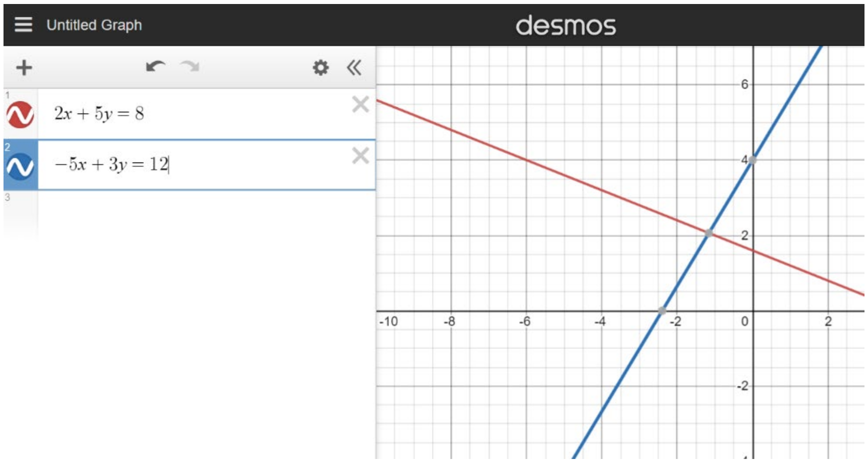
Unit Outline:

a. Properties of Parabolas
b. Zeros form
c. Vertex Form
d. Graphing from Vertex Form
e. Writing Quadratic Relations
f. Completing the Square

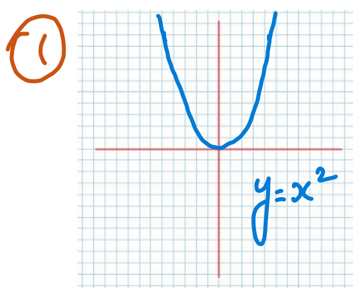
By the end of this unit:

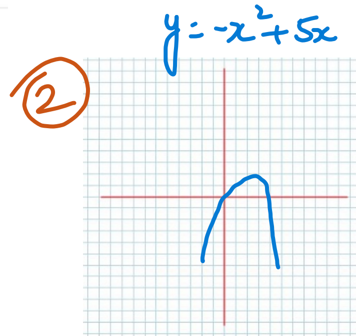
- I can describe what a quadratic relation looks like (as a graph and an equation);
- I can locate/describe the characteristics of a parabola (ie. vertex, axis of symmetry, etc.);
- I can describe the transformations of, and graph, quadratic relations in vertex form;
- I can convert quadratics in standard form to factored form and graph them
- I can convert quadratics in standard form to vertex form by completing the squares
- I can create equations of quadratic relations from graphs and word descriptions

Let's explore how quadratic expressions (*ALGEBRAIC MODEL)* appear when graphed in a cartesian plane (*GRAPHICAL MODEL of the same)*. Use the dynamic software Desmos for this. – www.desmos.com and draw a rough drawing of what you observed in the little graphs next to the quadratics they represent.



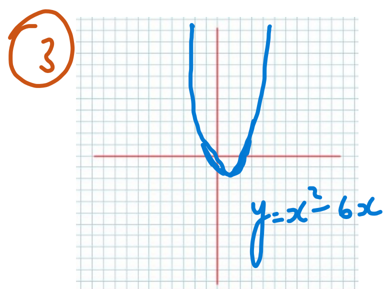
1. $y = 1x^2$

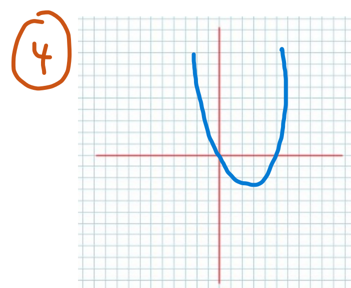




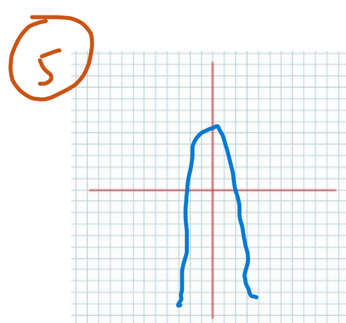
2. $y = -1x^2 + 5x$

3. $y = +1x^2 - 6x$





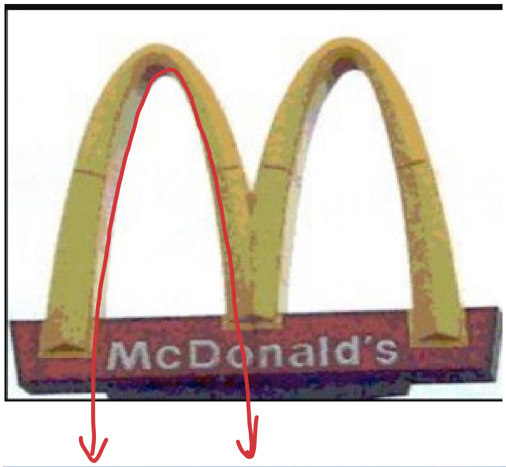
4. $y = +0.5x^2 - 3x$

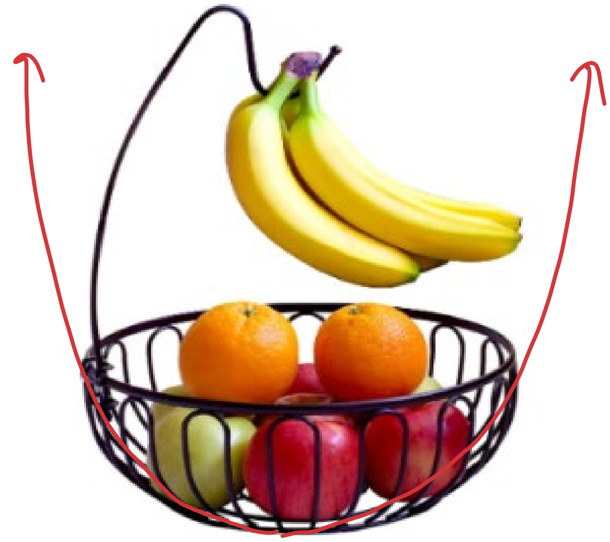


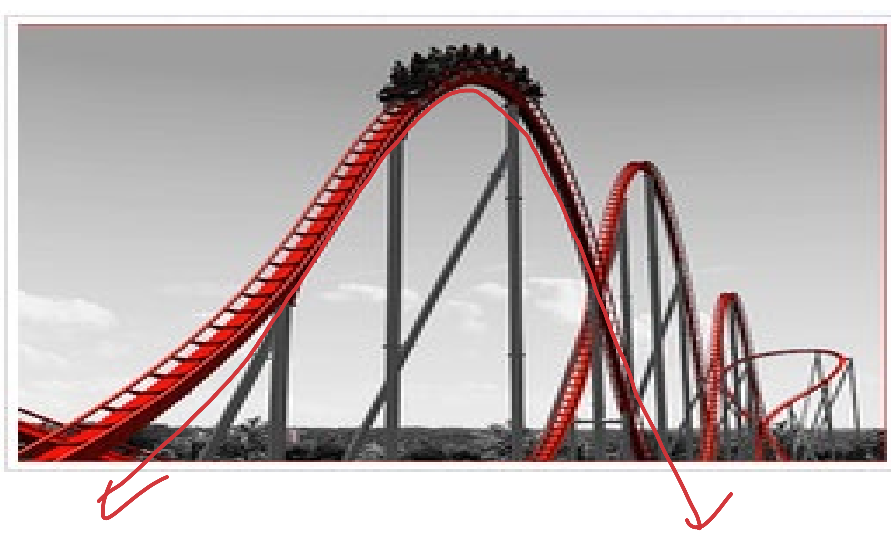
5. $y = -5x^2 + 35x + 5$

The shape we observed in all these graphs is called a PARABOLA

Look around you and you will be amazed at how commonly we see parabolas!!





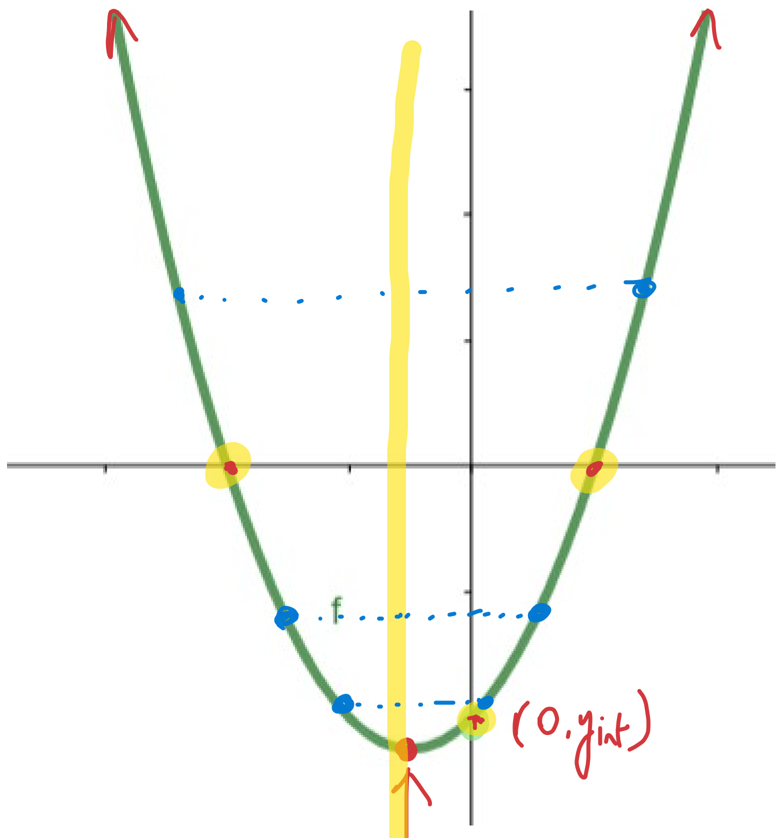


L1.Properties/Characteristics of Quadratics

Quadratic Expressions (Algebraically) can be represented in three ways/forms

Standard Form	Zeros Form	Vertex Form
$y=ax^2+bx+c$		

Vertex	Zeros	y-int	Vertical Stretch	Horizontal Shift	Vertical Shift	Axis of Symmetry AoS	Max/Min Value
(h,k)		c				$x=h$	k



1. **y-intercepts** *(where the graph intersects the y-axis)*

2. **x-intercepts** *(where the graph intersects the x-axis)*, Note: The x-intercepts are also called **the zeros** of the quadratic being graphed

3. **Vertex** *(the point at which the parabola changes direction)*

4. **Axis of Symmetry (AoS)** *(the line which cuts the parabola into identical two halves)*

5. **Max/Min Value** (*the max/min y-value i.e. k*)

VERTEX

$3x+2y=7$

Standard Form of a Quadratic Relation

$ax+by+c=0$

Review:

A linear relation is of the standard form $y=mx+b$ and as the name suggests has degree = 1.
(***Recall** that "**Degree**" of a polynomial is the highest exponent of the variable.)*

In this form **y= ax + b** of the linear relation, ***a*** represents the SLOPE and ***b*** represents the y-int.
of the line (graphical representation of the linear relation is a line).

So, we can learn some things about the linear graph from the algebraic model.

But, can we do this in quadratics too?

Absolutely!!! ☺

Recall,
A quadratic relation has degree = 2.

We observed earlier, the graph of any quadratic relation is a **parabola.**

So, what can we learn about the parabola from the standard algebraic form of a quadratic relation?

$$\boxed{y=ax^2+bx+c} \leftarrow \text{STANDARD FORM}$$

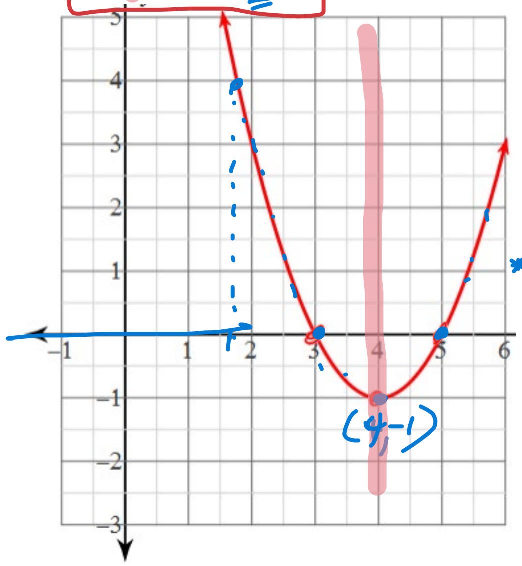
1. ***a*** reveals the direction of opening of the parabola.

   When a is POSITIVE $(a>0)$, the parabola opens **up.**
   When a is NEGATIVE $(a<0)$, the parabola opens **down.**

2. ***c*** is a constant value representing the y-intercept of the parabola.

Example:
On the next page is a quadratic relation and a picture of the graph it represents. Use the standard quadratic equation and its graph to state the different pieces of information that you can gather about it.

$y = x^2 - 8x + 15$



Important points on a Parabola:

Direction of opening OPENS UP

y-Intercept (0,15)

Zeros (3,0) and (5,0)

* x-intercepts

Vertex (4,-1)

V(h,k)

Axis of Symmetry $x=4$

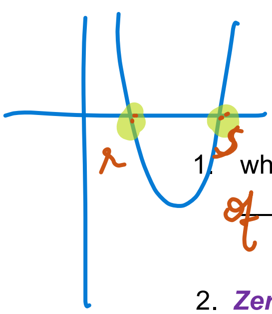
AoS: $x=h$

Max/min value Min = -1

max/min value = k

L2. Zeros Form or Factored Form of a Quadratic Relation

Factored form of a Quadratic Equation is

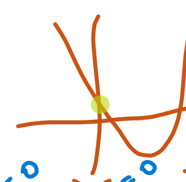


$$y = a(x-r)(x-s)$$

1. where ***r*** and ***s*** are the x-intercepts or zeros / roots and ***a*** can tell us about the direction of opening just like in the standard form.

2. ***Zeros*** **are the points where the parabola intersects the x-axis**
i.e. they are the x-intercepts and occur when y = 0.

y-intercept occurs when x = 0.



$y = 3(x-2)(x+1)$

$y_{int} = 3(0-2)(0+1) = 3(-2)(1) = -6$

3. The ***axis of symmetry*** divides the parabola into two identical halves and so can be found using the following formula:

$$X = \frac{r+s}{2} = h$$

4. The ***vertex*** ***(maximum/minimum)*** can be found by plugging in the ***x*** **value** of the **axis of symmetry** and **solving for y** $= k$

$x = h$

Let's understand this better in the topic that follows with the help of examples.

Graphing the Quadratic Equation:
Converting from Standard Form to Factored Form

1. For the following quadratic relation: $\boxed{y = x^2+10x+16}$
   i) Determine the **zeros** (values of x where y=0)
   ii) Determine the **y-intercept** (value of y where x = 0)
   iii) Determine the equation of the **axis of symmetry** (x=h)
   iv) Determine the coordinates of the **vertex** (V(h,k))
   v) Sketch the graph

$y = 2x^2 + 24x + 70$

$2(\quad x \quad)$

(i) $y = x^2 + 10x + 16$ (8, 2)

$y = (x+8)(x+2)$

$1(x-(-8))(x-(-2))$

$y = a(x-r)(x-s)$

$y = (x+8)(x+2)$

$\therefore$ zeros: $(-8,0)$ and $(-2,0)$

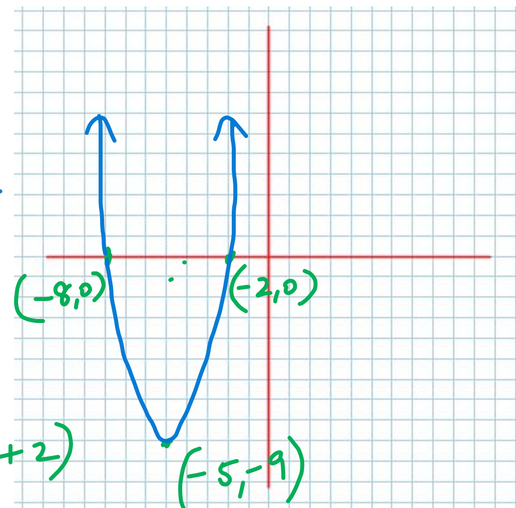
(ii) $\boxed{y_{int} = c = 16}$

(iii) AoS:

$x = \frac{r+s}{2}$

$x = \frac{-8+-2}{2}$

$\boxed{x = -5}$ h



$y = (x+8)(x+2)$

$k = (-5+8)(-5+2)$

$= (3)(-3) = -9$

(iv) $V(h,k)$

$\boxed{y = x^2 + 10x + 16}$

$k = y_{x=-5=h} = (-5)^2 + 10(-5) + 16$

$= 25 - 50 + 16 = -9$ $\therefore V(h,k) = (-5,-9)$

2. A parabola has zeros at -3 and 1. There is a y-intercept of -2. What is the equation of the parabola? (hint: obviously we have to use Algebra)

ZEROS FORM: $y = a(x-r)(x-s)$

$y_{int} = -2$

$P(0,-2)$

$y = a(x+3)(x-1)$

$-2 = a(0+3)(0-1)$

$-2 = a(3)(-1)$

$\frac{-2}{-3} = \frac{-3a}{-3}$

$\boxed{\frac{2}{3} = a}$

$\therefore$ EQUATION:

$y = \frac{2}{3}(x+3)(x-1)$

L3. Vertex Form of a Quadratic Relation

VERTEX: (h,k)

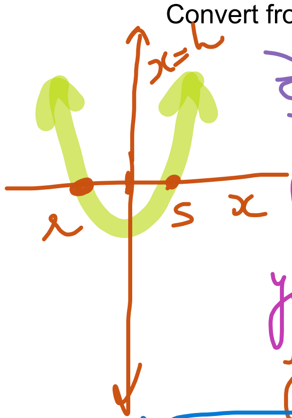
AoS : $x=h$

Direction of Opening : $a>0$ ↑, $a<0$ ↓

max/min. value k.

Vertex Form : $\boxed{y=a(x-h)^2+k}$

Convert from Standard Form to Vertex Form by finding the zeros, AoS, and the vertex.



$y = 5x^2 + 60x + 175$

$y = 5(x^2+12x+35)$

$y = 5(x+5)(x+7)$

$y = a(x-r)(x-s)$

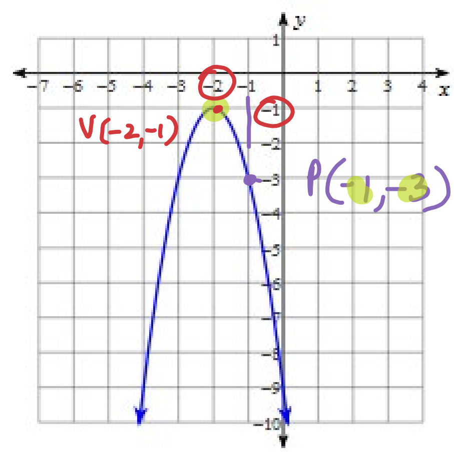
$h = -6$

$k = y_{x=h} = 5(-6)^2+60(-6)+175$

$= 180 - 360 + 175 = -5$

ZEROS	$(r,0),(s,0)$	$(-5,0)$ and $(-7,0)$
AoS	$x=h=\frac{r+s}{2}$	$x = \frac{-5+-7}{2} = \frac{-12}{2} = -6$
VERTEX	(h,k)	$(-6,-5)$
VERTEX FORM	$y=a(x-h)^2+k$	$y=5(x+6)^2-5$

Find the Equation of the following parabola.



$y=a(x-h)^2+k$

$y=a(x+2)^2-1$

$-3=a(-1+2)^2-1$

$-3=a-1$

$-3+1=a$

$\boxed{-2=a}$

$\boxed{\text{EQUATION: } y=-2(x+2)^2-1}$

L4. Transformations of Quadratic Relations (Parabolas)

Learning Goal: We are learning to apply combinations of transformations in a systematic order to sketch graphs of Quadratic Relations.

To **TRANSFORM** something is to CHANGE THE FORM.

of the BASE QUADRATIC / PARENT $y=x^2$

In previous classes of this course, you may remember having heard repeatedly that the **Vertex Form $y = a(x-h)^2 + k$** is a very powerful form of the Quadratic Relation. Through the lesson today, you will discover that this form is indeed powerful due to the information it contains regarding the transformations with respect to the parent or base quadratic.

There are **THREE BASIC TRANSFORMATIONS**

1) Flips (*Reflections "across" an axis*)
2) Stretches (*Dilations*)
3) Shifts (*Translations*)

So, we can have **Horizontal** flips, stretches and/or shifts, and **Vertical** flips, stretches and/or shifts. Now let's look at how transformations can be applied to functions with the help of a simulation. ☺

$y=x^2 \qquad y=a(x-h)^2+k$

$(x, y) \longrightarrow (x+h, ay+k)$

Observations:

1. The most basic quadratic relation is $y=x^2$ Hence, it is called a Parent or a Base Relation. Here, a=1, b=0, c=0, h=0, k=0, r=0, and s=0. So, basically, all other quadratic relations are just transformations (flips, stretches, and shifts) of this parent graph.

2. The **a** value transforms the basic quadratic by __multiplying__ to the y-values.

   We call this a __VERTICAL STRETCH__ by a factor of __a__.

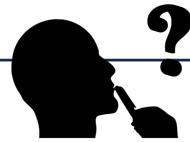
   (Remember: The sign of **a** also indicates the direction of opening. So, the sign of **a** indicates whether the parent parabola will have to FLIP or not to get the new transformed daughter parabola.)

3. The **k** value transforms the basic quadratic by __adding__ to the y-values.

   We call this a __VERTICAL SHIFT__ by __k__ units UP/DOWN

4. The **h** value transforms the basic quadratic by __adding__ to the x-values.

   We call this a __HORIZONTAL SHIFT__ by __h__ units LEFT/RIGHT



In case you were wondering…
A quadratic relation does not have a horizontal stretch because a horizontal stretch (or compression) would involve multiplying the entire x by a constant number, which would change the shape and position of the parabola. Instead, quadratics undergo vertical stretches or compressions, which change the vertical "height" of the parabola.

Putting it all together

Every point (x, y) of the parent quadratic will have a corresponding image point $(x+h,\ ay+k)$ on the transformed quadratic with transformations a, h, k

Create a Table of Values to sketch a graph of the following Transformed Quadratic.

Parent Table $x \to \boxed{x^2} \to x^2$

$\boxed{y = x^2}$

VERTEX

x	y
-2	4
-1	1
0	0
1	1
2	4

Image Table 1

$\boxed{y = -2(x-1)^2 + 4}$
$a(x-h)^2 + k$

$x+1$	$-2y+4$
$-2+1=-1$	$-2(4)+4=-4$
$-1+1=0$	$-2(1)+4=2$
$0+1=1$	$-2(0)+4=4$
$1+1=2$	$-2(1)+4=2$
$2+1=3$	$-2(4)+4=-4$

Image Table 2

$\boxed{y = 3(x+4)^2 - 2}$

$x-4$	$3y-2$
-6	10
-5	1
-4	-2
-3	1
-2	10

Let's summarize:

The general form of a "transformed" quadratic is:

$$\boxed{y = a(x-h)^2 + k}$$

This is known as the Vertex Form of a Quadratic/Parabola

where

- a tells us the V. STRETCH / V. FLIP.
- h tells us the H. SHIFT ↔.
- k tells us the V. SHIFT ↕..

with respect to the basic/parent/base quadratic $y=x^2$

$y=1x^2$

$y=a(x-h)^2+k$

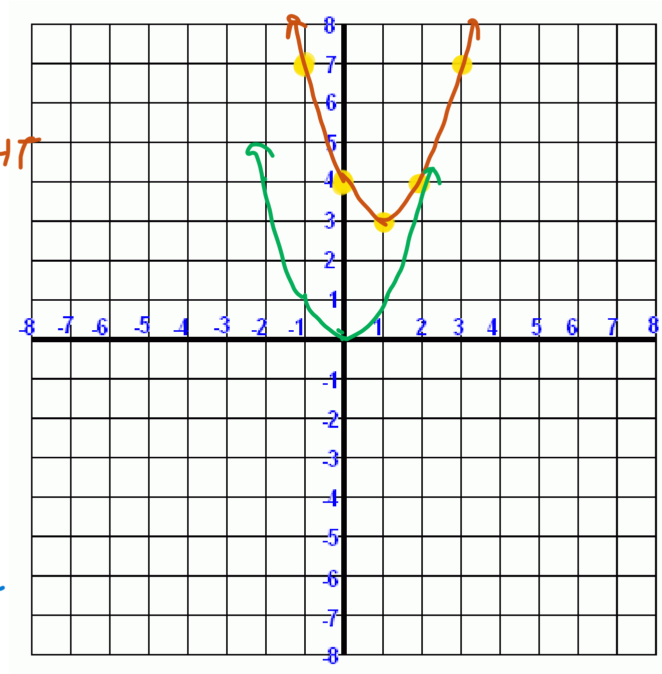
Describe the transformations for $\boxed{y=1(x-1)^2+3}$. Then make a sketch of the relation.

$a=1$ No V. STRETCH / V. FLIP

$h=1$ H. SHIFT by 1 unit RIGHT

$k=3$ V. SHIFT by 3 units UP.

x	y	x+1	y+3
-2	4	-1	7
-1	1	0	4
0	0	1	3
1	1	2	4
2	4	3	7



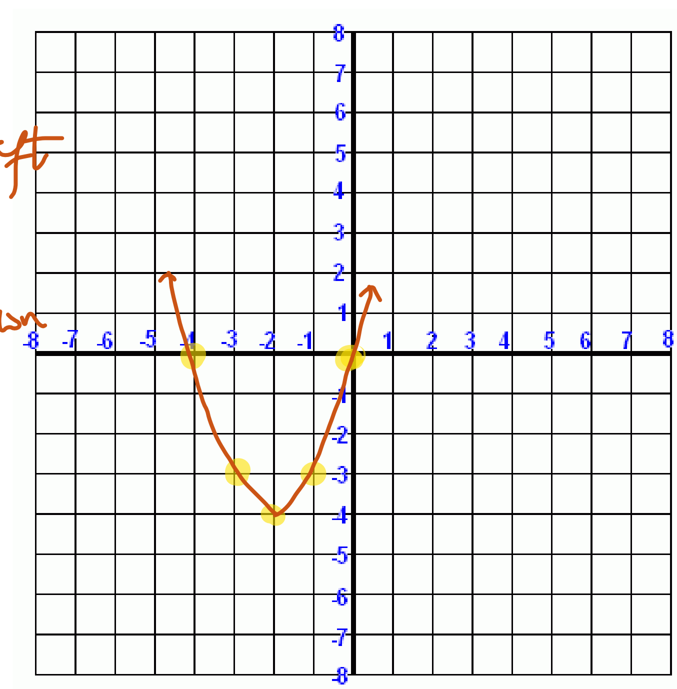
Describe the transformations for $\boxed{y=1(x+2)^2-4}$. Then make a sketch of the relation.

$a=1$ No V. STRETCH / V. FLIP

$h=-2$ H. SHIFT by 2 units Left

$k=-4$ V. SHIFT by 4 units Down

x	y	x-2	y-4
-2	4	-4	0
-1	1	-3	-3
0	0	-2	-4
1	1	-1	-3
2	4	0	0



Putting it All Together: Vertex Form of a Quadratic Relation

The general form for a quadratic in Vertex Form is:

$$\boxed{y = a(x-h)^2 + k}$$

where a is the stretch factor and reflection(flip)

and the point (h,k) is the vertex of the parabola.

Fill in the chart

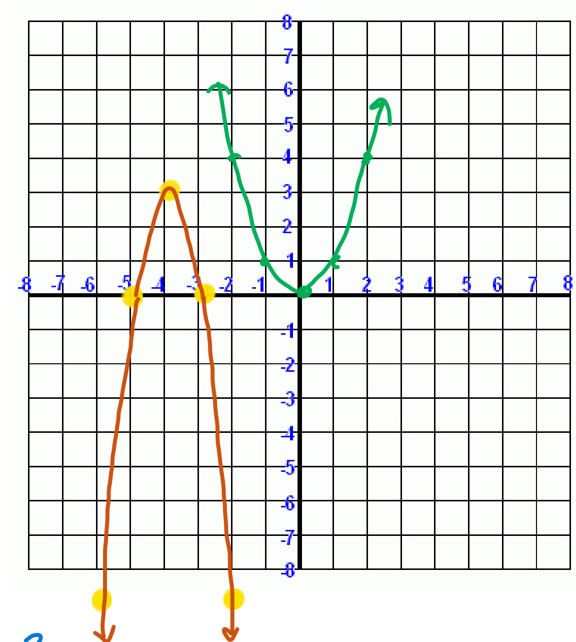
	$y=-2(x-1)^2+4$	$y=(x-3)^2+0$	$y=3(x+4)^2+2$
stretch factor a	-2	1	3
Open up or down $a>0$ $a<0$	OPENS DOWN	OPENS UP	OPENS UP
Vertical shift k	4 UP.	0	2 UP
Horizontal shift h	1 RIGHT	3 RIGHT	4 LEFT
Vertex (h,k)	$(1,4)$	$(3,0)$	$(-4,2)$
Equation of the Axis of symmetry $x=h$	$x=1$	$x=3$	$x=-4$
y-intercept $y(x=0)$	$(0,2)$	$(0,9)$	$(0,50)$
Max/min value k	4	0	2
Max/min point $V(h,k)$	$(1,4)$	$(3,0)$	$(-4,2)$

*When graphing/sketching a Quadratic, **you must <u>plot the vertex first</u>**, then **<u>use *a* to step the parabola</u>**.

Graphing using the Sketch of the Graph

To sketch $\boxed{y=-3(x+4)^2+3}$, start by drawing a sketch of $y=x^2$

Describe the transformations and graph the equation



$a=-3$ V. STRETCH by 3, V. FLIP.

$h=-4$ H. SHIFT 4 units LEFT

$k=3$ V. SHIFT 3 units UP.

x	y	$x-4$	$-3y+3$
-2	4	-6	-9
-1	1	-5	0
0	0	-4	3
1	1	-3	0
2	4	-2	-9

L5. Completing the Squares

A quadratic in the standard form $y=ax^2+bx+c$ can be written in the vertex form $y=a(x-h)^2+k$ by creating a **perfect square (x-h)^2** . The method adopted for this is often called **Completing the Square.**

The motivation for this method lies in the algebraic identity we were introduced to earlier where a trinomial $a^2 - 2ab + b^2$ was represented as a perfect square binomial $(a - b)^2$

$$\boxed{(a - b)^2 = a^2 - 2ab + b^2}$$

Our aim here is something very similar. We are aiming to get (x – h)^2. So, how do we do this?

Let's figure it out using some examples and asking ourselves some intelligent questions on the way!

1. Consider: $8^2 = 8 \times 8 = 64$
   So, is 64 a perfect square? Why?

   ____.

2. Consider: $(x+2)^2 = \boxed{x^2+4x+4} = (x+2)(x+2) = (x+2)^2$
   So, using the same reasoning can we say **x^2 + 4x + 4** is a perfect square? Why?

   ____.

3. Now, Consider: $\boxed{x^2 + 4x} +4 -4$
   Is it a perfect square? If not, how could we make it a perfect square (*may be we can do some algebraic manipulation)*? What could that manipulation be?

   ____.

Time to put this understanding to convert a quadratic in standard form into its vertex form.

Steps to Completing the Square

1. **Factor** the a from the x^2 and x terms
2. Use the coefficient of the **2nd term,** divide it by 2 and square it. **Rewrite** the equation by adding and subtracting this square term in the brackets
3. **Move** the subtracted square outside the brackets (Remember to multiply it by the a as moved out of the bracket)
4. **Simplify** - collect like terms outside the bracket
5. **Apply** the perfect square identity to convert a(**x^2 - 2xh + h^2**) to a(**x – h**)^2

STANDARD FORM $\longrightarrow$ VERTEX FORM

$y = a(x-h)^2 + k$

Examples – Complete the Square and State the Vertex

1. $y = \boxed{x^2 + 8x} + 15$

STEP 1: $y = 1(x^2 + 8x \quad) + 15$

$\frac{8}{2} = 4$

$\pm 4^2$

STEP 2: $y = 1(x^2 + 8x + 4^2 - 4^2) + 15$

STEP 3: $y = 1((x+4)^2 - 16) + 15$

STEP 4: $y = 1(x+4)^2 - 16 + 15 \Longrightarrow y = 1(x+4)^2 - 1$

$\therefore$ VERTEX $= (-4, -1)$

2. $y = \boxed{2x^2 + 12x} - 3$

2

$\boxed{y = a(x-h)^2 + k}$

①

② $y = 2(x^2 + 6x + 3^2 - 9) - 3$

$\frac{6}{2} = 3$

$(x+3)^2$

③ $y = 2((x+3)^2 - 9) - 3$

$(x+3)(x+3)$

$x^2 + 3x + 3x + 9$

$x^2 + 6x + 9$

④ $y = 2(x+3)^2 - 18 - 3 = 2(x+3)^2 - 21$ $\therefore$ VERTEX $= (-3, -21)$

3. $y = \boxed{-2x^2 + 12x} - 7$

-2

$y = -2(x^2 - 6x + 3^2 - 3^2) - 7$

$\frac{6}{2} = 3$

$(x-3)^2$

$y = -2((x-3)^2 - 9) - 7$

$y = -2(x-3)^2 + 18 - 7$

$y = -2(x-3)^2 + 11$ VERTEX $= (3, 11)$

4. $y=\frac{1}{2}x^2+6x+5$

$y=\boxed{0.5x^2+6x}+5$

$\frac{12}{2}=6$

$y=0.5(x^2+12x+6^2-6^2)+5$

$(x+6)^2$

$y=0.5((x+6)^2-36)+5$

$y=0.5(x+6)^2-18+5$

$$\boxed{y=0.5(x+6)^2-13}$$

VERTEX $(-6,-13)$

Conceptual Review: ***<u>Think and Answer!</u>***

1. What is the highest exponent of a polynomial in its standard form known as?

   DEGREE of a polynomial

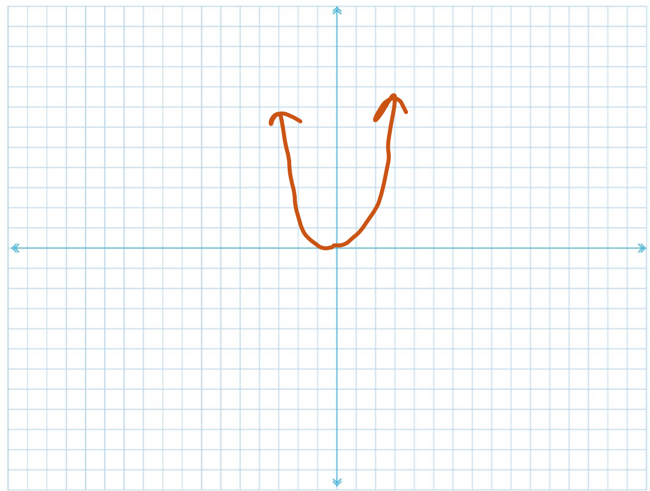
2. The algebraic model of a quadratic relation is a polynomial with <u>degree =</u> ___2___.

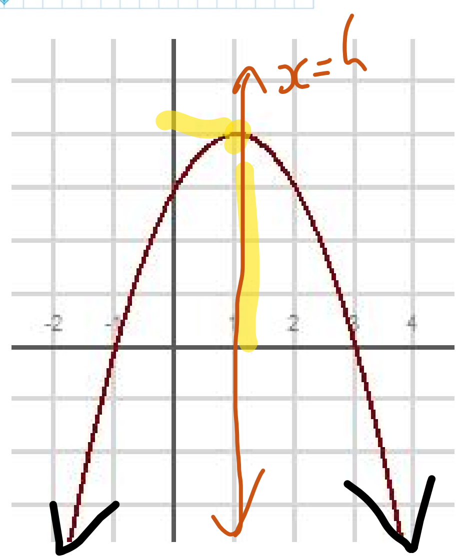
3. What does the graph of a quadratic relation look like?

   PARABOLA

4. Plot the graph of the parent quadratic function

x	$y = x^2$
-2	4
-1	1
0	0
1	1
2	4





5. i. What is the Vertex and the equation of the Axis of Symmetry of the given graph?

Vertex: (1,4) .

Axis of Symmetry: $x=1$.

ii. Can you find the maximum point or the minimum point on this graph? If yes, what is that point?

Maximum / Minimum? Max .

Coordinates of the Point? (1,4) .

Does this maximum/minimum point that you observed from the graph have a special name? If yes, what is it called? VERTEX .

6. You learnt about **THREE FORMS** of quadratic functions and the **information** they give. Do you recollect the general algebraic representations of the three forms?

1. Vertex Form: $y = a(x-h)^2+k$.

2. Factored Form: y = $a(x-r)(x-s)$.

3. Standard Form: y = ax^2+bx+c.

Now,

Compare the three algebraic forms of your parabola and recall the information that a gives you about the graph.

a POSITIVE

If $\boxed{a > 0,}$ will the graph open UP or DOWN? UP.

And what $\boxed{\text{if } a < 0,}$ will the graph open UP or DOWN? DOWN.

a NEGATIVE

Okay great!! 😊 Now think some more and answer:

If a > 0, will the vertex represent a maximum or a minimum? MINIMUM.

And what if a < 0 ? MAX.

Fantastic!! 😊 A little more thinking and answer the following:

Should there be a restriction on the leading coefficient **a** for the three different algebraic forms to be a quadratic?

If yes, what should the restriction be? $a \neq 0$.

Also,

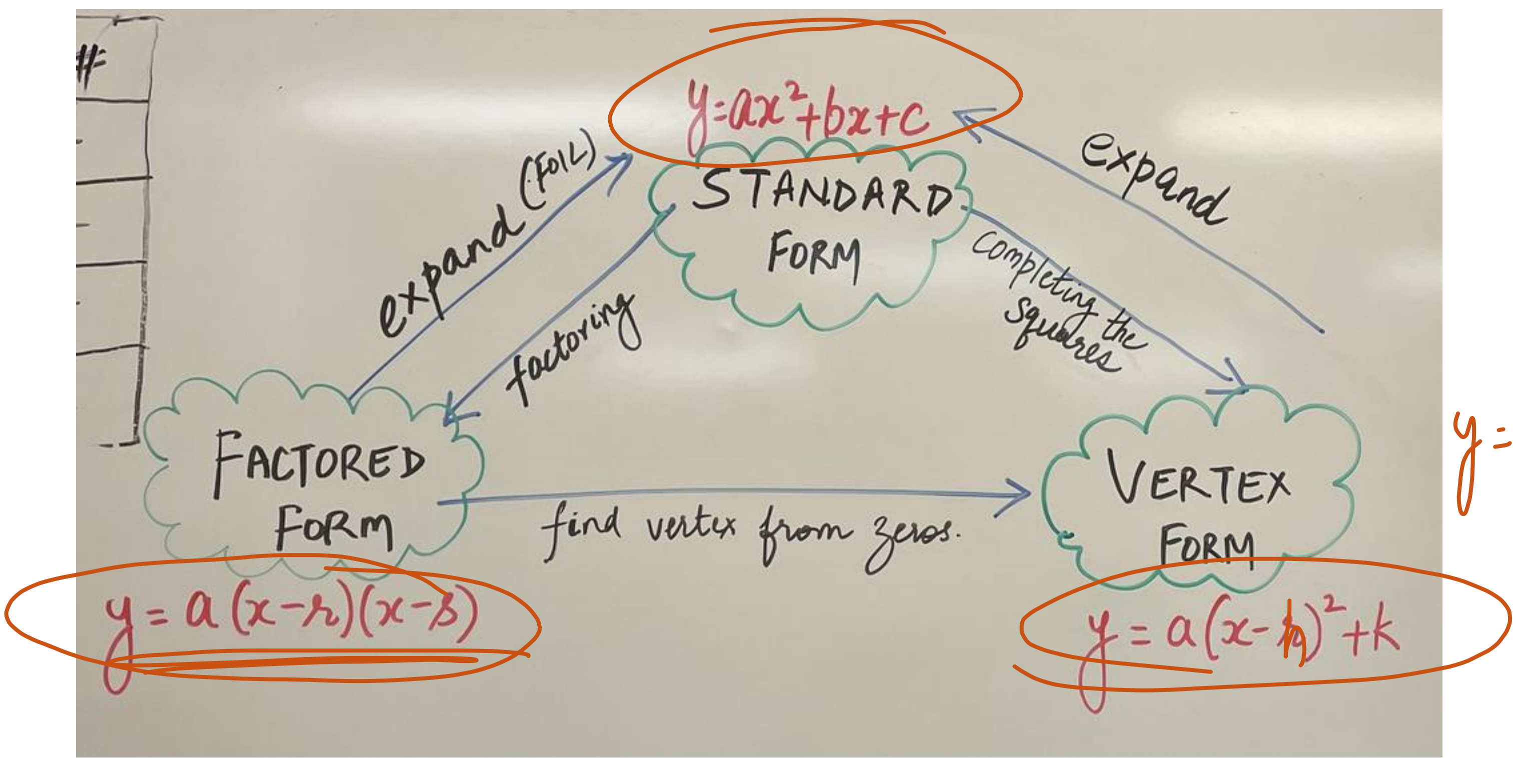
If $y=a(x-h)^2+k$ is the vertex form of any quadratic function, then the vertex is represented by (h,k) and the equation of the axis of symmetry is $x=h$.

Convert from one form to another.

$y = ax^2 + bx + c$ | y = 3x² – 12x + 9

$y = a(x-r)(x-s)$ | y = 3(x – 1)(x – 3)

$y = a(x-h)^2 + k$ | y = 3(x – 2)² – 3



$y = a(x-h)(x-h)+k$

Instructions: Fill in the chart! Complete your work below and behind, keep it organized. Do your conversion work on a separate paper.

	Vertex Form	Standard Form	Zeros Form	Opens up/down	Vertical Stretch	Horiz Shift	Vert. Shift	AoS	Max/Min Value	Vertex	Zeros	y-int	Any other Point
	$y=a(x-h)^2+k$	$y=ax^2+bx+c$	$y=a(x-r)(x-s)$	a	a	h	k	x=h	Max or min ~~of~~ k	(h,k)	r and s	c	(x,y)
1.	$y = 3(x-2)^2-3$	$y = 3x^2-12x+9$	$y = 3(x-1)(x-3)$	UP	3	2	−3	x=2	MIN = −3	(2,−3)	(1,0) (3,0)	(0,9)	(−1,20)
2.	$y=-7(x-1)^2+567$	$y=-7x^2+14x+560$	$y=-7(x-10)(x+8)$	DOWN	−7	1	567	x=1	MAX 567	(1,567)	(10,0) (−8,0)	(0,560)	(2,448)
3.	$y=2(x+4.25)^2-15.125$	$y=2x^2+17x+21$ $=\frac{2x^2+14x}{2x}+\frac{3x+21}{3}$ $=(2x+3)(x+7)$ 7 $\frac{3}{2}$	$y=(2x+3)(x+7)$ s=−7 $2x+3=0$ $2x=-3$ $x=-\frac{3}{2}=-1.5=r$	UP	2	$\frac{-7-1.5}{2}$ −4.25	$(2(-4.25)+3)(-4.25+7)$ $=(-5.5)(2.75)$ $=-15.125$	x=−4.25	MIN −15.125	(−4.25,−15.125)	(−1.5,0) (−7,0)	(0,21)	(1,40)
4.	$y=3(x+6)^2-12$	$y=3x^2+36x+96$	$y=3(x+4)(x+8)$	UP	3	−6	−12	−6	MIN −12	(−6,−12)	(−4,0) (−8,0)	(0,96)	(4,288)
	$y=\frac{1}{2}(x-4)^2-2$ $y=0.5(1-4)^2-2$ $=0.5(9)-2=2.5$	a $y=\frac{1}{2}x^2-4x+6$	$y=0.5(x^2-8x+12)$ $\boxed{y=0.5(x-6)(x-2)}$ y $k=0.5(4-6)(4-2)$ $=0.5(-2)(2)=-2$	UP	0.5	4	−2	x=4	min −2	(4,−2)	(6,0) (2,0)	6	(1,2.5)
	Challenge: Create the equation that has the following information. Then convert it to the other two quadratic forms. HINT: use (x, y) to solve for a.												
5.										(4,2)			(6,−6)

VERTEX FORM STANDARD FORM ZEROS FORM DoO V.stretch H.shift V.shift AoS max/min value Vertex Zeros y-int Any point