# Functions 11

*Course Notes*

# Unit 3 – Quadratic Functions

***Functions to the Max (or Min…and sometimes Zero)***

*We will learn*

- *the meaning of a zero, and how to find them algebraically*
- *to determine the max or min value of a quadratic algebraically and graphically*
- *to sketch parabolas (using transformations, zeroes, the vertex and y-intercept)*
- *to solve real-world problems, including linear-quadratic systems*

Chapter 3 – Quadratic Functions Name_____________________.

**3.1 Properties of Quadratic Functions**

This lesson is a review of some of what we learned about quadratics in Grade 10. 😊

In Grade 10, we studied the three different forms of Quadratic Functions, and the information each gives.

1. Standard Form $f(x) = ax^2 + bx + c$

**Information:**

2. Zeros/Factored Form $f(x) = a(x - r)(x - s)$

**Information:**

3. Vertex Form $f(x) = a(x - h)^2 + k$

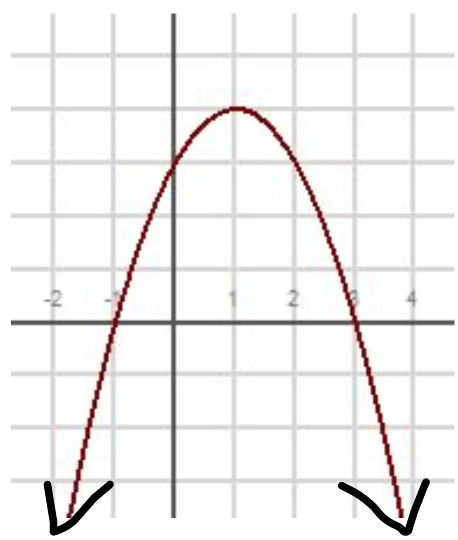
**Information:**

The three different forms are equivalent, meaning that they generate the exact same information or graph. Let's see if we remember how to convert between forms using simple algebra. We'll do converting Standard to Vertex Form in a later lesson since it involves a process called "Completing the Squares". Today, we'll test our learning and the skills acquired from the first unit and try the following two conversions:

Question 1: Convert $f(x) = 3x^2 + 18x - 48$ to the Factored Form.

Question 2: Covert $g(x) = -2(x-5)(x-3)$ to the Standard Form.

Do you recall the concept of the **<u>Axis of Symmetry?</u>**



1. The Axis of Symmetry (AoS) is

______________________________________________

______________________________________________

2. We can, therefore, write the coordinates of the

   vertex as ______________________

Determine the equation of the AoS and the Vertex for the given graph above.

## Now something NEW

The difference between consecutive values of y for constant increments of x are known as **finite differences**.

- If the first difference is the same, then a given table of values represents a linear relation.
- If the second difference is the same, then a given table of values represents a quadratic relation
- If the third difference is the same, then a given table of values represents a cubic relation
- If the fourth difference is the same, then a given table of values represents a quartic relation
- If the fifth difference is the same, then a given table of values represents a quintic relation
- And so on…

Determine if the function represented by the table of values is quadratic or not.

| x | f(x) | 1st Diff. | 2nd Diff. |
|---|---|---|---|
| 0 | 5 | | |
| 1 | 2 | | |
| 2 | -1 | | |
| 3 | -4 | | |
| 4 | -7 | | |

| x | f(x) | 1st Diff. | 2nd Diff. |
|---|---|---|---|
| -1 | 2 | | |
| 0 | 3 | | |
| 1 | 6 | | |
| 2 | 1 | | |

Key idea to remember-

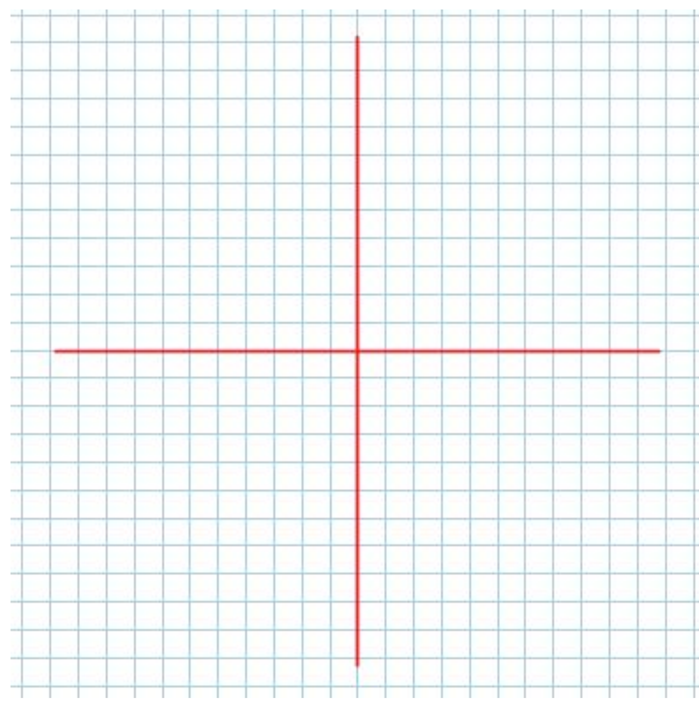
- Quadratic Functions have a constant non-zero second difference.
- If the 2nd Diff. > 0, then the leading coefficient a > 0. So, parabola opens UP.
- If the 2nd Diff. < 0, then the leading coefficient a < 0. So, parabola opens DOWN.

**Example 3.1.2**

Given the quadratic function $g(x) = -2(x+3)(x-1)$, state

a) The direction the parabola opens
b) The zeros of the quadratic
c) The equation of the axis of symmetry
d) The coordinates of the vertex
e) The function in vertex form

Sketch the graph of the function.

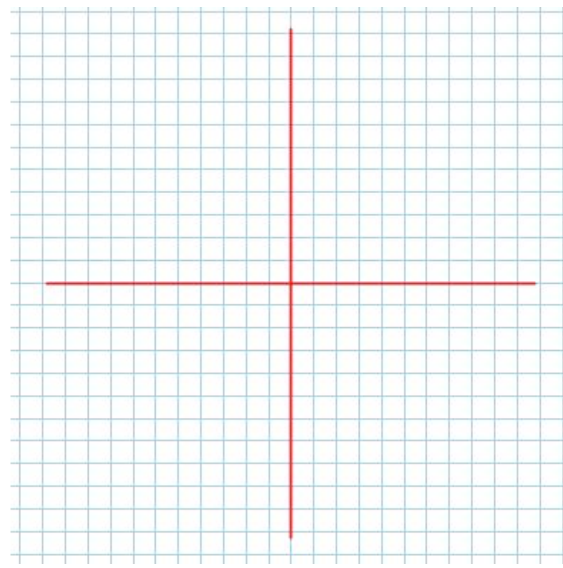


**Example 3.1.3**

Given the two points $(4,7)$, $(-5,7)$ which are on a parabola, determine the equation of the axis of symmetry.

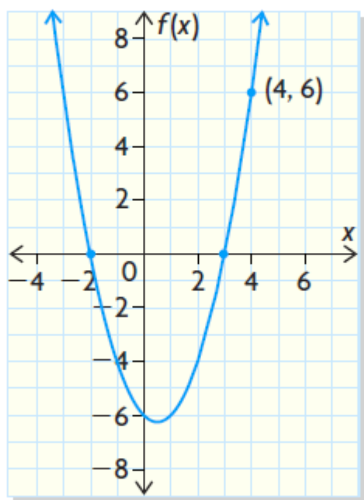
Complete the table, then graph the function.

| | Factored Form | Standard Form | Axis of Symmetry | Zeros | y-intercept | Vertex | Maximum or Minimum Value |
|---|---|---|---|---|---|---|---|
| a) | $R(x) = (40 - x)(10 + x)$ | $R(x) = -x^2 + 30x + 400$ | | | | | |



#13 from your text. Complete the table.

| | Zeros | Axis of Symmetry | Maximum or Minimum Value | Vertex | Function in Factored Form | Function in Standard Form |
|---|---|---|---|---|---|---|
| a) | 2 and 8 | | 6 | | | |




Find the equation of the parabola in both Factored form and Standard form.

The last thing I want to do is word problems. **DON'T OVERTHINK OR OVER COMPLICATE WORD PROBLEMS.** First of all, you will most likely be given a function to work with. Then, understand the **CONTEXT** of the problem.

The height of a football kicked from the ground is given by the function $h(t) = -5t^2 + 20t$, where $h(t)$ is the height in metres and $t$ is the time in seconds from its release.

**a)** Write the function in factored form.

**b)** When will the football hit the ground?

**c)** When will the football reach its maximum height?

**d)** What is the maximum height the football reaches?

**e)** Graph the height of the football in terms of time without using a table of values.

Now your turn :)

Example 3.1.4 (From Pg. 147 in your text)

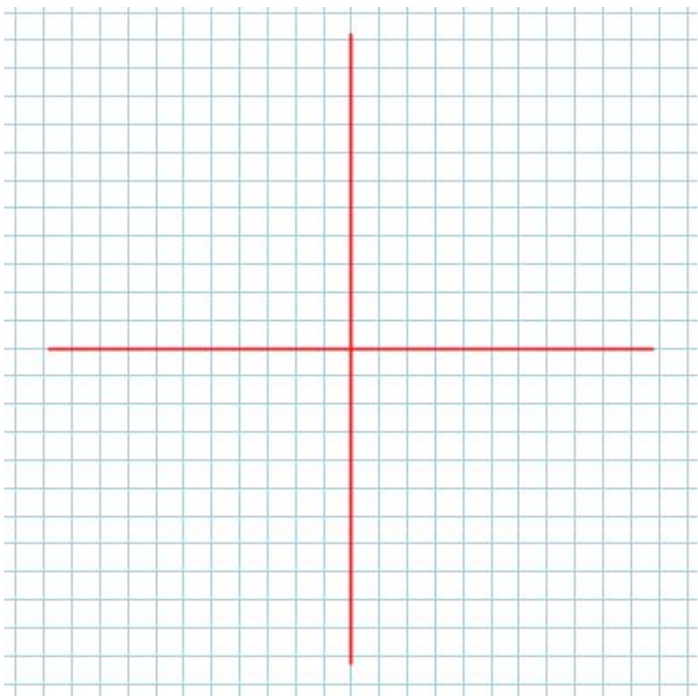
**11.** **A** The height of a rocket above the ground is modelled by the quadratic function $h(t) = -4t^2 + 32t$, where $h(t)$ is the height in metres $t$ seconds after the rocket was launched.

**a)** Graph the quadratic function.

**b)** How long will the rocket be in the air? How do you know?

**c)** How high will the rocket be after 3 s?

**d)** What is the maximum height that the rocket will reach?



# HW- Section 3.1

Pg. 145 – 147 #3, 4, 5bc, 6 (*expand*!), 7, 8, 9de, 12 (*tricky*!)

**Success Criteria:**

- I can recognize a quadratic function in standard, factored, and vertex form
- I can determine the zeros, direction of opening, axis of symmetry, vertex, domain and range from the graph or the algebraic form of a parabola
- I can determine the equation of quadratic function from its parabola (i.e. graph)

Chapter 3 – Quadratic Functions Name____________________.

**3.2 The Maximum or Minimum of Quadratic Functions**

**Learning Goal:** We are learning to determine the maximum/minimum value of a quadratic function.

One very important aspect of quadratic functions is that they all have either a maximum (if the associated parabola opens down) or a minimum (if the parabola opens up). **Max/Min's** have so many **applications** in the real world that it's **ridiculous**.

The **BIG QUESTION** we are faced with is this: **How do we find the Maximum or Minimum Value for some given Quadratic?**

But before that, let's do a quick review to reinforce our understanding of the vertex form.

Question 1: Convert $f(x) = 3(x+4)^2 - 18$ to the Standard Form.

Question 2: Given $g(x) = -4(x+5)^2 + 3$, state the vertex, axis of symmetry, direction of opening and range.

Question 3: Given the vertex (-3, -8) and the coordinate (-6, 37), determine the equation of the parabola.

Okay, Great!!! Now let's go back to the maximum and minimum.

In the Vertex Form, the *h* and *k* form together to give the coordinate of the vertex (h,k).

The maximum or minimum value of a quadratic function is the y-coordinate of the vertex.

So, clearly we do need to find the vertex. It's easy if the vertex form is given. If not, we need to find using algebraic techniques.

In order to find the vertex using algebra, we will consider the following techniques:

1) **USING THE ZEROS, TO FIND THE AXIS OF SYMMETRY**, and then the vertex (**this is the easiest technique**, ***assuming we can factor the quadratic***).
2) **COMPLETING THE SQUARE** to find the vertex (this is the toughest technique, but it's nice because you ***end up with the quadratic in vertex form***).
3) ☆ **USE PARTIAL FACTORING TO FIND THE AXIS OF SYMMETRY**, and then the vertex.

**Note: We can also use graphing calculators to find the max/min of a quadratic!**

**Example**

Determine the max or min value for the function $f(x) = -3x^2 - 12x + 15$ by finding **THE ZEROS** of the quadratic.

**Example 3.2.3**

COMPLETE THE SQUARE to find the vertex of the quadratic and state **where** the max (min) is and **what** the max (min) is.

$$g(x) = 2x^2 + 8x - 5$$

**Example 3.2.4**

Using PARTIAL FACTORING determine the axis of symmetry. Then find the vertex and state the min or max value.

$$h(x) = 5x^2 + 15x - 3$$

**Example:**

The height above the ground of a bungee jumper is modelled by the quadratic function $h(t) = -5(t-0.3)^2 + 110$ , where height, *h(t)*, is in metres and time, *t*, is in seconds.

a) When does the bungee jumper reach a maximum height? Why is it a maximum?

b) What is the maximum height reached by the jumper?

c) Determine the height of the platform from which the bungee jumper jumps.

## HW- Section 3.2

Pg. 153 #1, 3, 4abc (*one method is fine*), 6 (*Desmos*), 7bc, 8, 9 (*try Partial Factoring*), 11 (*ask for help on c if you feel the need!*)

**Success Criteria:**

- I can recognize when a function has a maximum or minimum value (based on "*a*")
- I can find the max/min (vertex) value using various methods
- I can find the max/min (vertex) value using various methods (partial factoring)

Chapter 3 – Quadratic Functions                                        Name_____________________.

**3.4 Operations with Radicals**

**Learning Goal:** We are learning to simplify and perform operations on radicals.

First we need to understand that **RADICALS** (*square roots, cube roots, etc*) **ARE NUMBERS**, and working with them should not induce any kind of fear in your spirit. So, **FEAR NOT!**

A COUPLE OF THINGS TO REMEMBER:

1) The square root of a square number/perfect square is a nice integer.

   e.g. $\sqrt{25} =$

   $\sqrt{49} =$

2) The cube root of a cubed number/perfect cube is a nice integer

   e.g. $\sqrt[3]{27} =$

   $\sqrt[3]{125} =$

Now, if we don't have a radical with a perfect square (or cube as the case may be) we could use a calculator to find the root.

e.g. $\sqrt{24} = 4.898979485566356196394568114941 18...$

**BUT** the "**DECIMAL EXPANSION**" is **unending** and **doesn't repeat** and so we can only **APPROXIMATE THE VALUE** of $\sqrt{24}$ because of the need to **ROUND-OFF**.

However,

"**EXACT NUMBERS**" like $\sqrt{24}$ are sometimes preferred in mathematical solutions and so **we do need to know how to work with these radical NUMBERS.** Working with radical numbers means we'll be:

- adding/subtracting
- multiplying/dividing them.

Before beginning, there is one thing to keep in mind: We will not be dealing with negative square roots as this will require the use of Complex numbers which you may not have learnt about yet.

## COEFFICIENTS WITH COEFFICIENTS, RADICALS WITH RADICALS

**e.g.** The number $2\sqrt{5}$ has a coefficient part of and a radical part of

Such a number (with both a coefficient and a radical part) is called a Mixed Radical.

***Rules:**

**1.) When multiplying radicals $\sqrt{a}$ X $\sqrt{b}$ = $\sqrt{ab}$ ; a ≥ 0, b ≥ 0**

**2.) When dividing radicals $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ ; b ≠ 0**

**3.) We can add or subtract only like radicals. For example: $3\sqrt{5}$ and $-7\sqrt{5}$ are like radicals but $7\sqrt{5}$ and $7\sqrt{3}$ are unlike radicals.**

**Example 3.4.1**

Multiply the following:

a) $\sqrt{5}\times\sqrt{3}$

b) $-2\sqrt{7}\times 3\sqrt{6}$

c) $5\sqrt{10}\times\sqrt{5}$

d) $\sqrt{2}\times\sqrt{2}$

**Example 3.4.2**

Simplify the following:

a) $\sqrt{50}$

b) $-3\sqrt{27}$

c) $2\sqrt{50}\times\left(-3\sqrt{24}\right)$

**Example 3.4.3**

Add the following:

a) $3\sqrt{2}+7\sqrt{2}$

b) $5\sqrt{7}-3\sqrt{5}-7\sqrt{7}$

c) $2\sqrt{5}-3\sqrt{20}$

d) $-3\sqrt{300}+\sqrt{243}$

**Note**: **We can only ADD OR SUBTRACT "LIKE" RADICALS.**
**e.g.** $2\sqrt{3}$ and $-5\sqrt{3}$ **ARE LIKE, but** $2\sqrt{5}$ and $3\sqrt{20}$ **ARE NOT (*or aren't they?.....*)**

**Example 3.4.4**

Simplify:

a) $2\sqrt{3}\ \left(3\sqrt{2} - 5\sqrt{6}\right)$

b) $\left(3\sqrt{12} - 5\sqrt{2}\right)\left(2\sqrt{3} + 3\sqrt{2}\right)$

c) $\left(5 - 2\sqrt{2}\right)^2$

HW- **Section 3.4**

Pg. 167 – 168 #3 – 5abc, 6 – 7acef, 8 - 13

**Success Criteria:**

- I can recognize "like" radicals. Totally awesome dude!
- I can write a radical in simplest form
- I can simplify radicals by adding, subtracting, multiplying, and dividing
- I can appreciate that a radical is an EXACT answer and therefore SUPERIOR to decimals

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**3.5 Solving Quadratic Equations**

**Learning Goal:** We are learning to solve quadratic functions in different ways.

1. **Solving by Factoring**

## Steps

1) Make one side of your equation equal to zero (stuff = zero)
2) **Factor**. The solutions will be the x values when each factor is set equal to zero

Example 1:
Solve $x^2 - 9x + 12 = -8$

Example 2:
Solve $16x^2 - 25 = 0$

Example 3:
Solve $2(x+3)^2 = 5(x+3)$

Example 4:

The profit of a skateboard company can be modelled by the function $P(x) = -63 + 133x - 14x^2$, where *P(x)* is the profit in thousands of dollars and *x* is the number of skateboards sold, also in thousands of dollars. When will the company break even, and when will it be profitable?

Example 5:

The path a dolphin travels when it rises above the ocean's surface can be modelled by the function $h(d) = -0.2d^2 + 2d$, where $h(d)$ is the height of the dolphin above the water's surface and *d* is the horizontal distance from the point where the dolphin broke the water's surface, both in feet. When will the dolphin reach a height of 1.8 feet?

2. **Solving by using Quadratic Formula** (especially useful when not factorable)

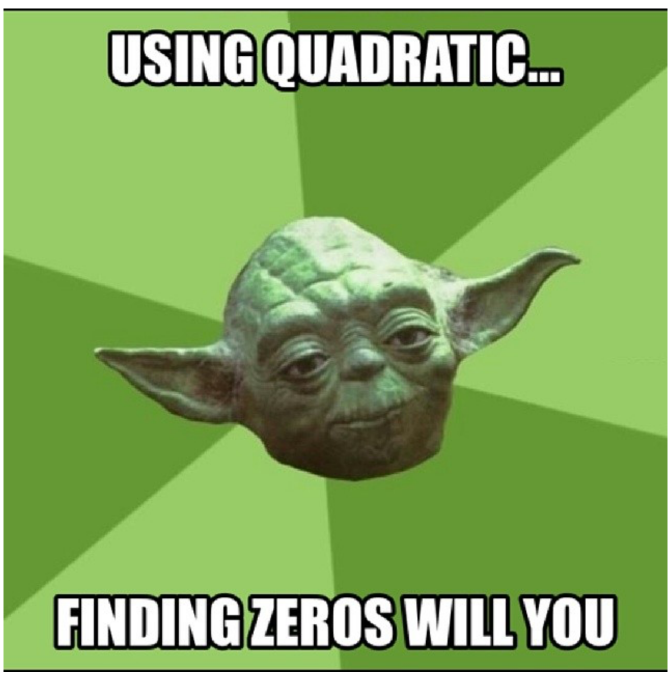
The quadratic formula is the one formula to solve them all. With great power, comes great responsibility. It is a tricky formula, but once you learn how to properly use it, you will be that much happier. The key is to write and communicate your math carefully.

Given $ax^2 + bx + c = 0$ , then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**Two notes:**

1. Inside the square root, you start with $b^2$. No matter what you plug in, you get a positive number. If $b = 9, \quad b^2 = 81$. If $b = -5, \quad b^2 = 25$. Why do I make note of this? I have seen many people do it wrong, so don't be one of those people.

2. Since we are calculating a square root, we have three options. If the number inside the square root is positive, there are two solutions. If the number is zero, there is only one solution. If the number is negative, there are no solutions since you cannot square root a negative. This is a fact. Don't try to square root a negative as it absolutely 100% cannot be done. Please don't try to do it. It doesn't work. No solution is an answer, so do not fret. If you don't like the negative, double check your work.



Example 1:

$3x^2 - 24x + 45 = 0$

Example 2:

$3x^2 + 2x + 15 = 0$

Example 3:

$4x^2 - 8x + 10 = 2x + 7$

Example 4: The profit on a school drama production is modelled by the quadratic equation $P(x) = -60x^2 + 790x - 1000$, where *P(x)* is the profit in dollars and *x* is the price of the ticket, also in dollars.

a) Use the quadratic formula to determine the break-even price for the tickets.

b) At what price should the drama department set the tickets to maximize their profit?

Another Example:

**8.** The population of a region can be modelled by the function $P(t) = 0.4t^2 + 10t + 50$, where $P(t)$ is the population in thousands and $t$ is the time in years since the year 1995.

**a)** What was the population in 1995?

**b)** What will be the population in 2010?

**c)** In what year will the population be at least 450 000? Explain your answer.

## HW Section 3.5

Pg 177 – 178 #1bc, 2bcd, 4abef, 6cd, 7 (*Hint: what is the height of the ball when it is on the ground?*), 9, 11 (#9 and 11 are tricky – ask for help!), 14

**Success Criteria:**

- I can solve quadratic functions by factoring, then setting each factor equal to zero
- I can solve quadratic functions by using the quadratic formula

Chapter 3 – Quadratic Functions Name___________________.

**3.6 Zeros of Quadratic Functions**

**Learning Goal:** We are learning to determine the number of zeros of a quadratic function.

Before beginning we should look at the difference between a Quadratic **FUNCTION** and a Quadratic **EQUATION**. A function such as $f(x)=3x^2-5x+1$ has a graph with **infinitely** many points. On the other hand, a quadratic equation (in standard form) looks like:

$$3x^2-5x+1=0$$

(*What is the difference between the function and the equation?)*

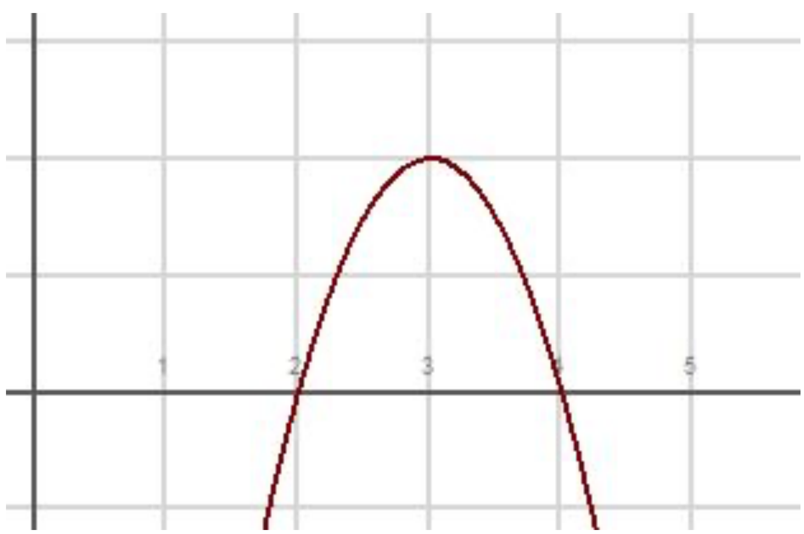
As it turns out, solving a quadratic equation is **Exactly the Same as finding the zeros of quadratic functions.**

Quadratic functions, therefore can have _____, ______, or _______ **ZEROS**.

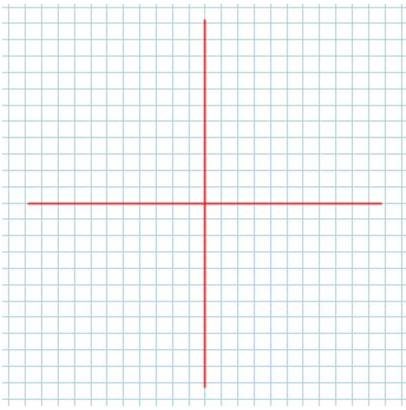
Remember – **FUNCTIONS CAN BE DESCRIBED AS A SET OF ORDERED PAIRS**, where the "ordered pair" is a pair of numbers: a **domain value** and a **range value** which can look like $(x, f(x))$. We have talked about the vertex of a parabola. Consider a parabola opening down (which means it will have a maximum value.

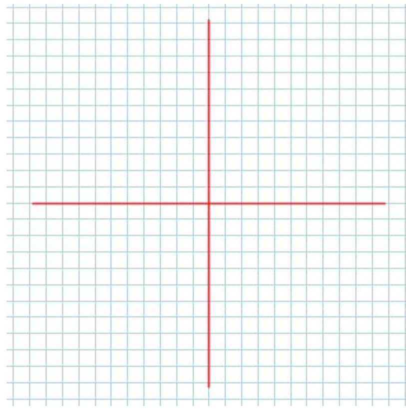
The vertex of that parabola is NOT the maximum. Instead, the vertex is a POINT which is made up of two special numbers. The domain value is WHERE the max occurs and the functional value (the "y" value) is the maximum.

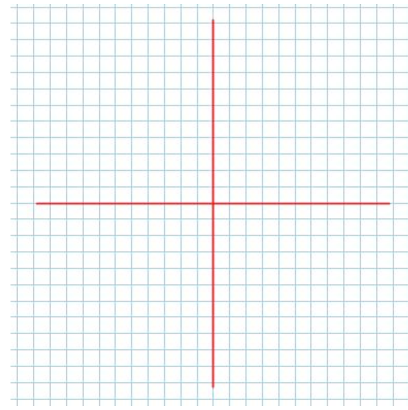
When we talk about the ZEROS of a quadratic we need to understand what we mean by that. Consider the sketch of the graph of the quadratic function $f(x)=-2(x-3)^2+2$



Q. Do all quadratics have 2 zeros? NO!!!!!!!







## Finding the Zeros of a Quadratic

We can find those pesky zeros in a number of ways:

1) Writing the quadratic in zeros form (by factoring)
2) Using the quadratic formula (but the quadratic ***MUST BE IN STANDARD FORM*** - $f(x) = ax^2 + bx + c$)
3) Using graphing technology (lame, but legit)

**Example 3.6.1**

Determine the zeros by factoring:

a) $f(x) = x^2 - 3x - 4$

b) $g(x) = 2x^2 + x - 1$

**Example 3.6.4**

Using the quadratic formula, determine the zeros of the quadratic:

In case you've forgotten, the quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

a) $f(x) = 2x^2 + 3x - 7$

b) $g(x) = 3x^2 - 2x + 4$

# The Discriminant

The Discriminant of the quadratic formula is called the **DISCRIMINANT** because

The Discriminant is $b^2 - 4ac$

1) If $b^2 - 4ac > 0$, then the quadratic has ____________________zeros

2) If $b^2 - 4ac = 0$, then the quadratic has ____________________zeros

3) If $b^2 - 4ac < 0$, then the quadratic has ____________________zeros

Example 3.6.5

Determine the number of zeros using the discriminant:

a) $f(x) = 2x^2 + 3x - 2$

b) $g(x) = -x^2 + 4x - 4$

c) $h(x) = 3x^2 + 5x + 6$

**HW Section 3.6**

Pg 185 – 186 #1 – 3abc, 4, 6 – 9 (*these are a bit tricky…ask for help!*), 15

**Success Criteria:**

- I can recognize that a quadratic function may have 0, 1, or 2 zeros
- I can use the discriminant of the quadratic formula to determine the number of zeros

Chapter 3 – Quadratic Functions                                        Name___________________.

**3.7 Families of Quadratic Functions**

**Learning Goal:** We are learning the properties of families of quadratic functions.

**A group of parabolas that all share a common characteristic lie in the same family.**

Consider the two quadratic functions:

$f(x)=2(x-3)^2+1$, and $g(x)=-3(x-3)^2+1$          **What's Different?**

Clearly $f(x)$ and $g(x)$ are different functions, but they do share the same vertex, and the same axis of symmetry. These quadratics are said to be in the same "family"

(some might say they are in the same vertex family)

**Family 1: If the value of a is varied in a quadratic function expressed in vertex form, $f(x) = a(x-h)^2 + k$, a family of parabolas with the same vertex and axis of symmetry is created.**

Next,

consider $h(x)=3(x+2)(x-4)$, and $f(x)=\frac{2}{3}(x+2)(x-4)$.          **What's Different?**

We see another kind of family here because $h(x)$ and $f(x)$ share the same zeros, and the same axis of symmetry.

(some might say these quadratics are in the same zeroes family)

**Family 2: If the value of a is varied in a quadratic function expressed in factored form, $f(x) = a(x-r)(x-s)$, a family of parabolas with the same x-intercepts and axis of symmetry is created.**

Finally consider the third form of a quadratic. Consider

$f(x) = -3x^2 - 2x + 7$, and $g(x) = 2x^2 + 8x + 7$

**Family 3: If the values of a and b are varied in a quadratic function expressed in standard form, $f(x) = ax^2 + bx + c$, a family of parabolas with the same y-intercept is created.**

 Name___________________.

**3.7 Families of Quadratic Functions**

**Example 3.7.1**

Determine the equation of the quadratic with zeros $x = 3,$ and $x = -1$ and that passes through the point $(5, 6)$.

**Example 3.7.2**

Determine the equation of the quadratic function $f(x)$ with a max value of $3$ and axis of symmetry with equation $x = -5$ if $f(2) = -18$.

**HW Section 3.7**

Page 192 #4 – 6, 8 – 10

**Success Criteria:**

- I can solve for "*a*" if given either the vertex or zeros
    1. Substituting the zeros into the linear equation to determine the corresponding y-values
- I can identify when solutions are inadmissible

 Name___________________.

**3.8 Linear Quadratic Systems**

**Learning Goal:** We are learning to solve problems involving the intersection of a linear and quadratic function.

Recall from Grade 10 that solving a **S**YSTEM **O**F **L**INEAR **E**QUATIONS could be interpreted to mean finding the point of intersection of the two lines. The solution to a **SoLE** is a point, $(x, y)$. From an algebraic point of view, we have two techniques for solving a SoLE:

1) Substitution
2) Elimination

**Example 3.8.1**

Solve the SoLE

$$2x + 3y = 7 \quad (1)$$

$$x - 2y = -7 \quad (2)$$

*Algebraically*

*Graphically*

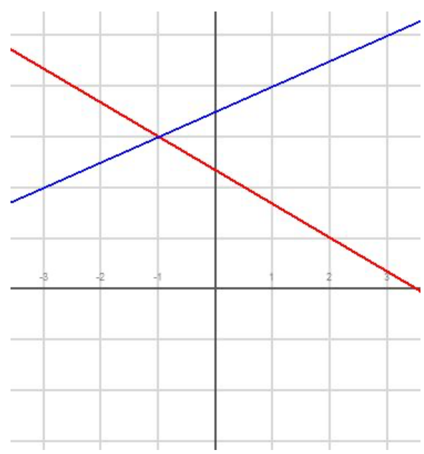



Figure 3.8.1

Solving a Linear-Quadratic System is more difficult, but we have the tools to succeed!
We will need to make use of (at least) one Property (or Rule) of Algebra:

### THE TRANSITIVE PROPERTY OF EQUALITY

***Rule***: Given three numbers (or more generally, three mathematical objects) $a$, $b$, and $c$, and **if** $c = a$ and $c = b$, **then** $a = b$.

***Example***: If $f(x) = -2x - 4$, and $g(x) = x^2 - 3x - 10$, and if $f(x) = g(x)$, then $x^2 - 3x - 10 = -2x - 4$

Note: **Solving a Linear-Quadratic System** is equivalent to **finding the solution(s) to a quadratic equation**. For L-QS`s we can therefore have 0, 1, or 2 solutions.

**We will apply the techniques for solving quadratic equations!**

**Example 3.8.2**

Solve the Linear-Quadratic System given directly above.

**Example 3.8.3 (#2c, on Page 198 from your text)**

Determine the point(s) of intersection of the two functions algebraically:

$f(x) = 3x^2 - 2x - 1, \; g(x) = -x - 6$

**Example 3.8.4**

Determine the number of points of intersection without solving the System:

$f(x) = x^2 + 2x + 14, \; g(x) = 8x + 5$ (Hint: To solve this problem you must be very discriminating)

**Example 3.8.5** (#9 on Page 199 in your text)

**9.** Determine the value(s) of $k$ such that the linear function $g(x) = 4x + k$ does not intersect the parabola $f(x) = -3x^2 - x + 4$.

**Example 3.8.6 (#10 in your text)**

**10.** A daredevil jumps off the CN Tower and falls freely for several seconds before releasing his parachute. His height, $h(t)$, in metres, $t$ seconds after jumping can be modelled by

$h_1(t) = -4.9t^2 + t + 360$ before he released his parachute; and
$h_2(t) = -4t + 142$ after he released his parachute.

How long after jumping did the daredevil release his parachute?

**HW Section 3.8**

Pg198 – 199 #1ab, 2ab, 3, 4bcd, 6, 8, 11 (*tangent means touching at one point!*), 12

**Success Criteria:**

- I can solve for the points of intersection by
  1. Making the functions equal to each other
  2. Solving for the zeros (x-coordinates) of the resulting quadratic function