

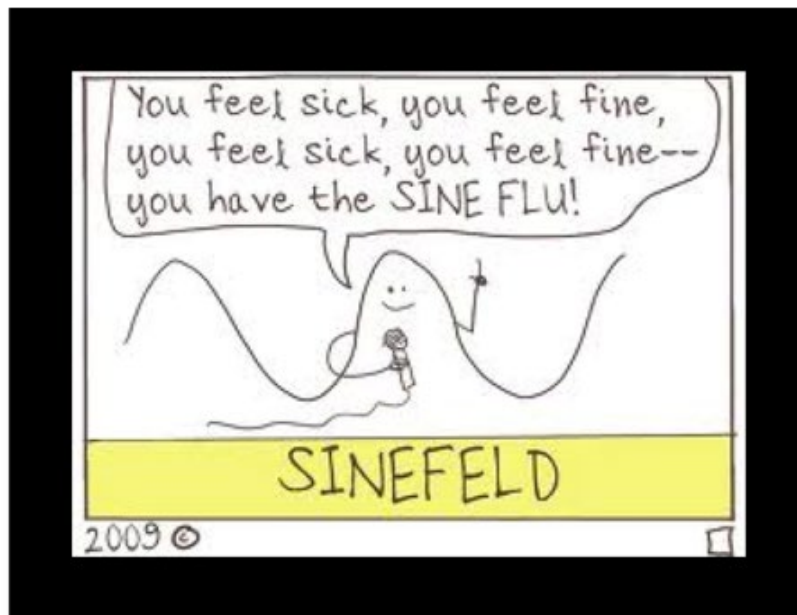
Functions 11

Course Notes

Unit 6 – Sinusoidal Functions

We will learn how to

- identify situations that can be modelled by sinusoidal or periodic functions
- interpret the graphs of sinusoidal or periodic functions
- graph sinusoidal functions with transformations
- determine the equations of sinusoidal functions from real-world situations



6.1 – Properties of Periodic Functions

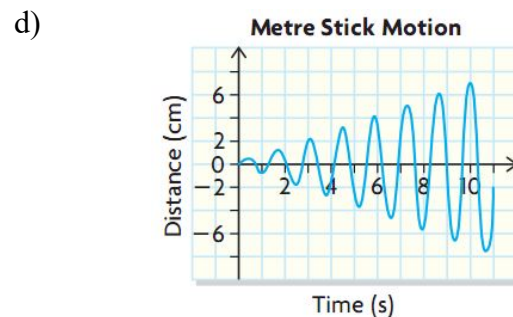
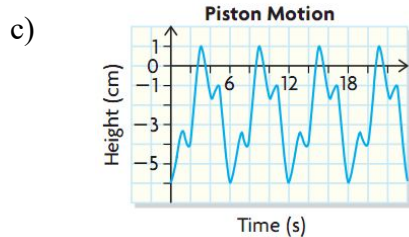
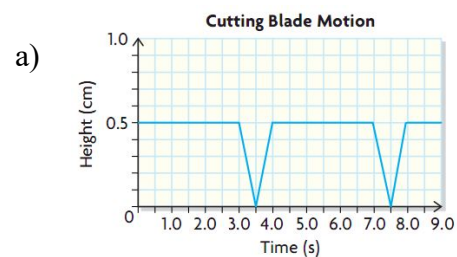
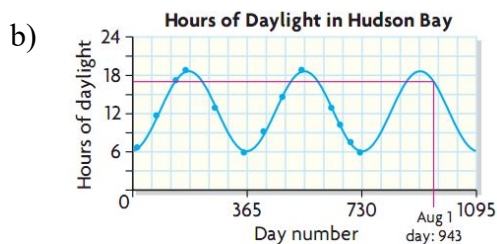
Learning Goal: We are learning to interpret and describe graphs that repeat at regular intervals.

Definition 6.1.1

A **PERIODIC FUNCTION** is one in which *the functional values repeat in a 'regular' way*.
i.e. a function whose graph repeats at regular intervals.

e.g. Consider the following pictures: Determine which are periodic.

Hint: Look for any repeating shape ☺



You might be wondering..... why we are talking about Periodic Functions when the Unit is **Sinusoidal Functions ?!!**.

Well, this is because **Sinusoidal Functions are a subset of Periodic Functions** i.e. Sinusoidal Functions are a group of some special Periodic Functions whose graph look like smooth symmetrical waves.

But why are we interested in Sinusoidal Functions? Well, that is because graphs of Sinusoidal Functions can be created by transforming the graph of the trigonometric functions $f(\theta) = \text{Sin}(\theta)$ and $f(\theta) = \text{Cos}(\theta)$.

The cool thing to note here is that the Sine and Cosine functions can be used as models to solve problems that involve many types of repetitive motions and trends and that's the reason we are interested in studying them!! ☺

Definition 6.1.2

The **Period** of a periodic function is the amount of the **domain values (x-values)** where **one cycle** takes place.

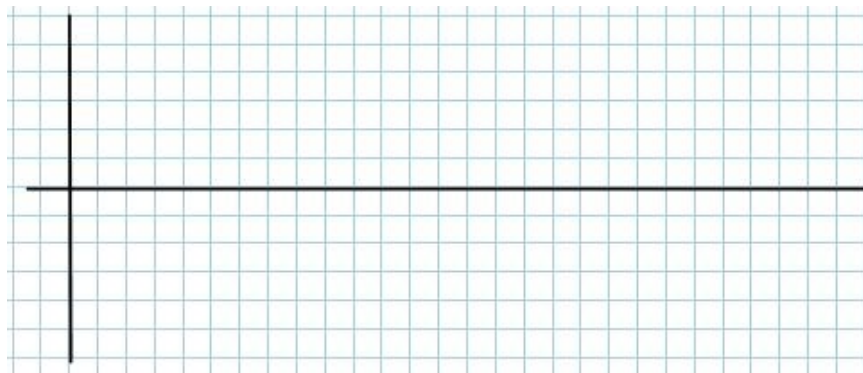
(Note that a **cycle is the portion of the periodic function graph that repeats.**)

Example 6.1.1

Determine the periods of the graphs of periodic functions on the previous page:

Definition 6.1.3

- The **Peak** of a periodic function is the maximum point on the graph.
- The **Trough** of a periodic function is the minimum point on the graph.



Definition 6.1.4

- The **Central Axis or the Equation of the Axis** is half way between the peak (maximum value) and the trough (minimum value).

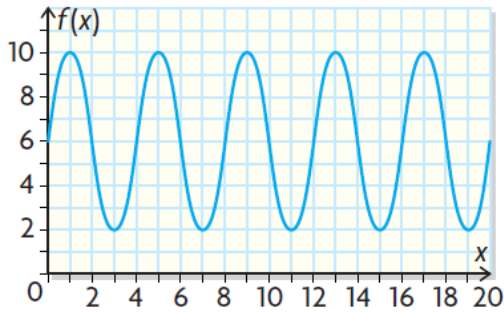
The equation of The Central Axis is given by: $y = \frac{\text{max} + \text{min}}{2}$.

- The **Amplitude** of a periodic function is the **distance from the peak or trough to the central axis**. So, it is half of the distance between a peak (maximum value) and a trough (minimum value). Amplitude being distance is always POSITIVE! ☺

$$\text{Amplitude} = \frac{\text{max} - \text{min}}{2}$$

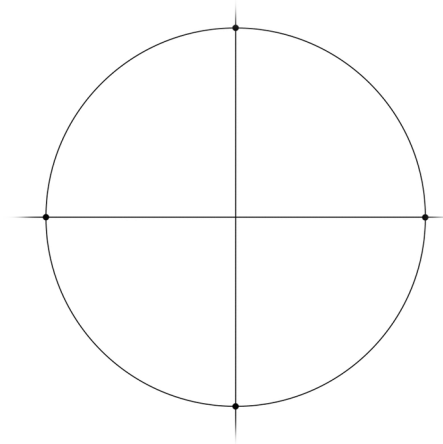
Example 6.1.1

Determine the range, period, equation of the axis, and amplitude of the function shown.



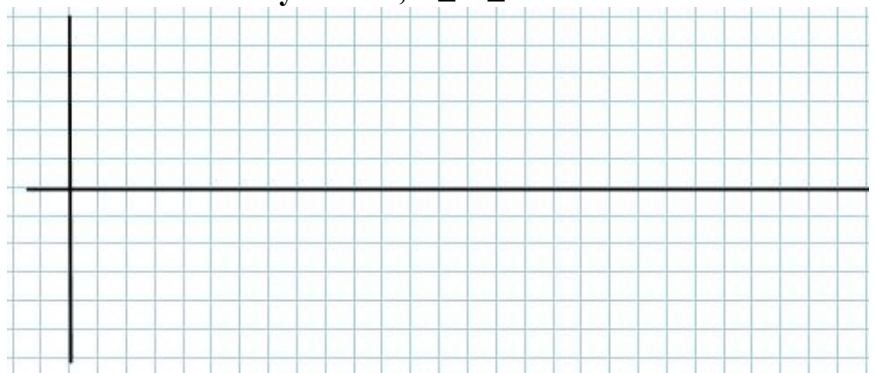
So, What about the so-called Sinusoidal Functions? What do the graphs of functions $f(\theta) = \sin(\theta)$ and $f(\theta) = \cos(\theta)$ look like?

Let's try and sketch thembut first consider our dear friend the unit circle which may hopefully make sketching $f(\theta) = \sin(\theta)$ and $f(\theta) = \cos(\theta)$ easy! 😊



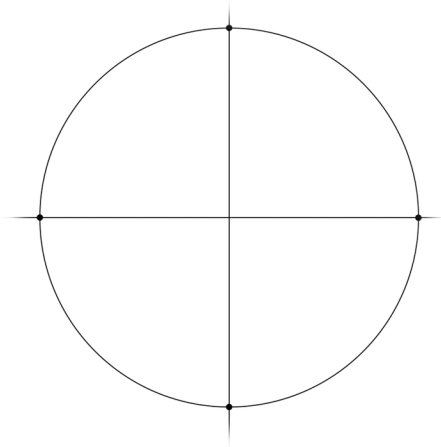
x	y=Sin x

$y = \sin x, 0 \leq x \leq 720$



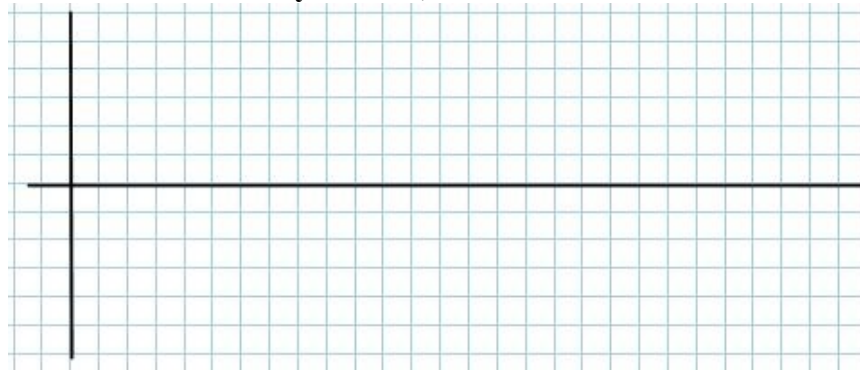
Notice the Axis Angles $0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$ in the unit circle above to make the table of values.

Do you think considering these special axis angles could make sketching the Sine and Cosine graphs easier?



x	y=Cos x

$$y = \text{Cos } x, 0 \leq x \leq 720$$



So,

1. The graph of $f(\theta) = \text{Sin}(\theta)$ has these characteristics:

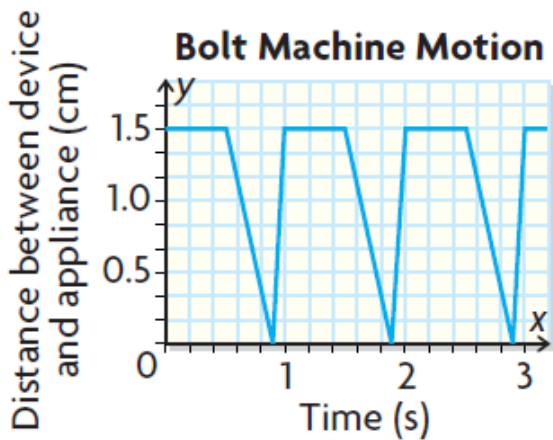
- The period is _____.
- The amplitude is _____, the max value is _____, and the min value is _____.
- The domain is _____, and the range is _____.
- The zeros are located at _____.

2. The graph of $f(\theta) = \text{Cos}(\theta)$ has these characteristics:

- The period is _____.
- The amplitude is _____, the max value is _____, and the min value is _____.
- The domain is _____, and the range is _____.
- The zeros are located at _____.

Example 6.1.2

3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.
- What is the period of one complete cycle?
 - What is the maximum distance between the device and the appliance?
 - What is the range of this function?
 - If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.
 - Determine the equation of the axis.
 - Determine the amplitude.
 - There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of “attaching the bolt.”



HW - Section 6.1

Pg. 352 – 355 #4, 5, 7 – 10

Success Criteria:

- I can find the range, period, central axis, and amplitude of a periodic function
- I can determine IF a function is periodic

6.5 – Sketching Sinusoidal Functions

Learning Goal: We are learning to sketch the graphs of sinusoidal functions using transformations.

By this point in your illustrious High School careers, you have a solid understanding of Transformations of Functions in general. In terms of the sinusoidal functions Sine and Cosine in particular, the concepts are as you expect

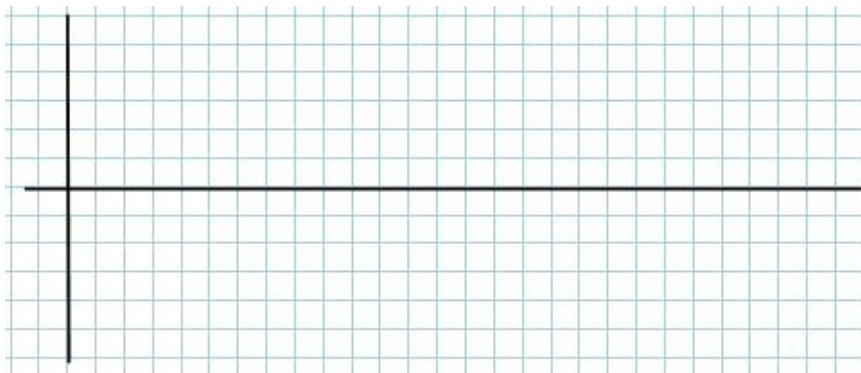
i.e. the graphs of the functions $g(x) = a \sin(k(x - d)) + c$ and $g(x) = a \cos(k(x - d)) + c$ are obviously periodic in the same way that the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ are, and the differences are only in the placement of the graph (ie the shifts) and how stretched it is.

However, what is new is that **the transformations have specific meanings relating to the nature of the sinusoidal “wave”**.

But first,

Using the special angles of 0° , 90° , 180° , 270° , and 360° , let's graph the following functions again:

$$f(\theta) = \sin(\theta) \quad 0^\circ \leq \theta \leq 720^\circ$$

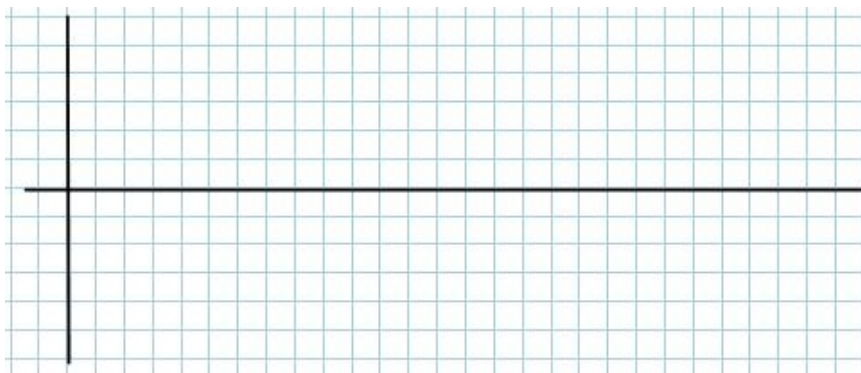


So,

- Key points for $f(x) = \sin x$
 $(0^\circ, 0)$, $(90^\circ, 1)$, $(180^\circ, 0)$, $(270^\circ, -1)$, $(360^\circ, 0)$

**Note that from 0 to 720, we see 2 periods.*

$$g(\theta) = \cos(\theta) \quad 0^\circ \leq \theta \leq 720^\circ$$



So,

- Key points for $f(x) = \cos x$
 $(0^\circ, 1)$, $(90^\circ, 0)$, $(180^\circ, -1)$, $(270^\circ, 0)$, $(360^\circ, 1)$

**Note that from 0 to 720, we see 2 periods in the base Cosine graph too!*

It's finally time to transform the above graphs!! 😊

NOTE that to graph the transformed function $g(x)$, you need to apply the transformations to only the key points of $f(x) = \sin x$ and $f(x) = \cos x$ and NOT to every point on $f(x)$.

The key points are the points corresponding to the special angles.

General Form of the Sine and Cosine Functions

$$f(\theta) = a \sin(k(\theta - d)) + c$$

$$g(\theta) = a \cos(k(\theta - d)) + c$$

Transformation	Properties
$a =$ Pattern of Sin: CA-max-CA-min-CA If $a < 0$, pattern: Pattern of Cos: max-CA-min-CA-max If $a < 0$, pattern:	$a = \frac{\max - \min}{2}$ $ a $ is the AMPLITUDE of the graph <i>Remember that Amplitude is the distance from the peak (or trough) to the central axis.</i>

$k =$

$$\text{Period Factor} = \frac{360^\circ}{|k|}$$

Remember that Period is the set of all the x-values in one cycle of the repeating graph

$d =$

Note: To determine d you **MUST** first FACTOR k from the θ

*The Horizontal translation(shift) of a Sinusoidal Function is called a **PHASE SHIFT***

$c =$

$$c = \frac{\text{max} + \text{min}}{2}$$

Note that x-axis is the central axis of our base Sine and Cosine function graphs and the vertical shift of the base graph is the same as the vertical shift of the central axis.

So,
Equation of Central Axis :

Example 6.5.1

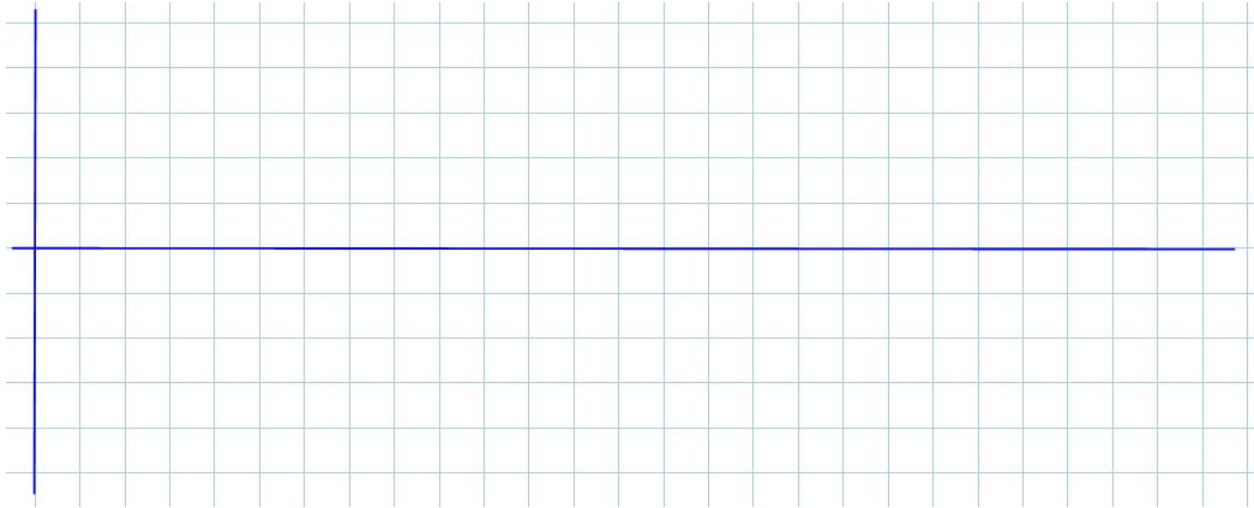
Determine the amplitude, period, phase shift and the equation of the central axis for:

a) $f(\theta) = 2 \sin(\theta + 60^\circ) + 1$

b) $g(\theta) = 3 \cos(2\theta - 90^\circ)$

Example 6.5.2

Sketch $f(x) = \sin(x)$ and $g(x) = \sin(2x)$ for $0^\circ \leq x \leq 360^\circ$ on the same set of axes.



Analyzing $g(x)$:

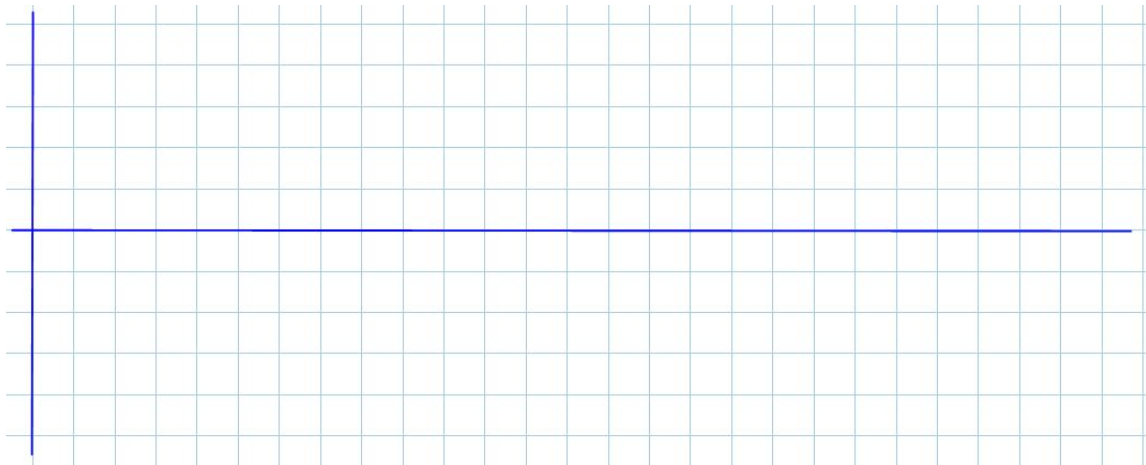
Now...what would $\sin\left(\frac{1}{2}x\right)$ look like?

Notes about Domain and Range: Consider the function $f(x) = -2\cos(3x + 90^\circ) + 3$.

Determine all the transformations for this function. Without graphing, determine the range of the function. Determine the domain of the function for: 1 cycle; 2 cycles; 3 cycles.

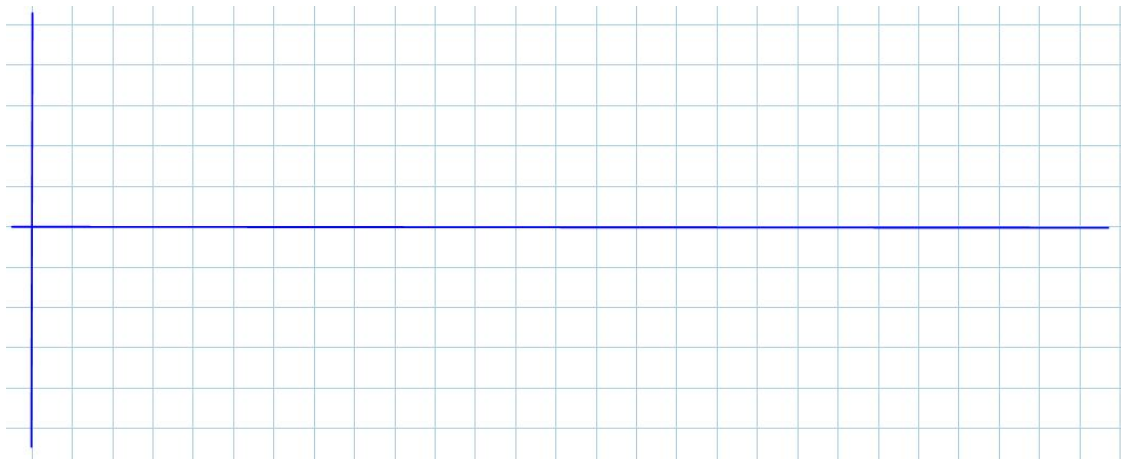
Example 6.5.3

Sketch $f(\theta) = -2\cos(\theta - 60^\circ) + 1$ on $0^\circ \leq \theta \leq 360^\circ$. State transformations, create tables, and state domain and range of the function.



Example 6.5.4

Sketch $f(\theta) = 3\sin(2\theta - 90) - 1$. State transformations, create tables, and state domain and range of the function.

**HW Section 6.5**

Pg. 383 – 3385 #1, 2, 4 – 7, 9

Success Criteria

- I can sketch the graph of a sinusoidal function by applying the transformations to the parent function.

6.6 – Models of Sinusoidal Functions

Learning Goal: We are learning to create a sinusoidal function from a graph or table of values.

In this section we will look at how to develop a sinusoidal function which can explain given information. In essence we will be writing sine or cosine functions based on given transformations.

Just as a reminder:

General Form of the Sine and Cosine Functions

$$f(\theta) = a \sin(k(\theta - d)) + c \qquad g(\theta) = a \cos(k(\theta - d)) + c$$

Another reminder (about the pattern of sinusoidal functions):

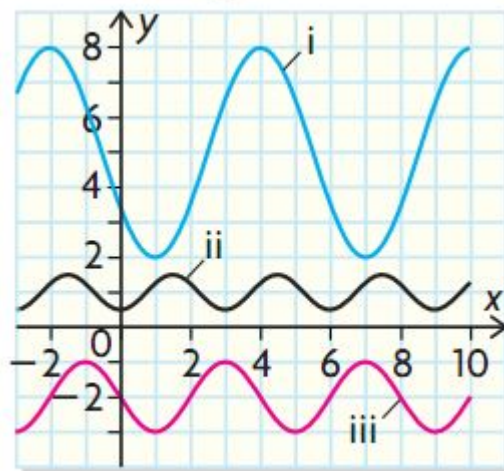
Sine functions “start” at the central axis and go up to a **max** if “ a ” is **positive**, or down to a **min** if “ a ” is **negative**.

Cosine functions “start” at a **max** if “ a ” is **positive**, or at a **min** if “ a ” is **negative**.

Example 6.6.1

From your text: Pg. 391 #4a

Determine a sinusoidal equation for each function:



Example 6.6.2

From your text: Pg. 392 #5a)

5. For each table of data, determine the equation of the function that is the simplest model.

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1

Example 6.6.3

From your text: Pg. 392 #6b)

6. Determine the equation of the cosine function whose graph has each of the following features.

	Amplitude	Period	Equation of the Axis	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	30°

Example 6.6.4

A sinusoidal function has an amplitude of 4 units, a period of 120° , and a maximum at $(0,9)$. Determine the equation of the function.

HW Section 6.6

Pg. 391 – 393 #4b, 5bcd, 6acd, 7, 11

Success Criteria:

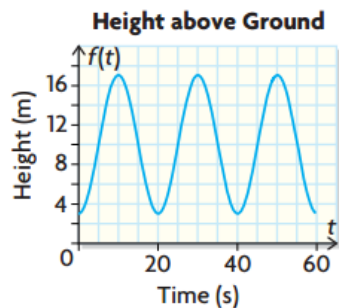
- I can create an sinusoidal function based on information from a graph or table
- I can recognize when it is best to use a sine or cosine function

6.7 – Problem Solving with Sinusoidal Functions

Learning Goal: We are learning to solve problems related to real-world applications of sinusoidal functions.

We can **use the sinusoidal properties** of **Period**, **Central Axis**, **Amplitude** and **Phase Shift** to describe and solve “real world” problems.

Example 6.7.1 (*From the text: Pg. 398 #2*)



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.
 - a) What is the equation of the axis of the function, and what does it represent in this situation?
 - b) What is the amplitude of the function, and what does it represent in this situation?
 - c) What is the period of the function, and what does it represent in this situation?
 - d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.
 - e) Determine the equation of the sinusoidal function.
 - f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?

Example 6.7.2

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches a maximum height of 11m at 10s and then reaches a minimum height of 1m at 55s. How high is John after 2 minutes?

Example 6.7.3 (Text pg. 396)

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right (+10 cm) and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At $t = 0$, the pole was momentarily at its resting position. Then it started moving to the right.

- a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.

- b) How does the situation affect the domain and range?

- c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

HW Section 6.7

Pg. 398 – 401 #4 – 6, 8, 10 (*a question of beauty*)

Success Criteria:

- I can create a sinusoidal function that represents information from a real-life scenario
- I can use the function to solve further problems