

Functions 11

MCR3U

Course Notes

Unit 1 – Rational Expressions

FACTORED FORM SAVES THE UNIVERSE

We will learn

- how to simply polynomials through expanding and factoring
- how to simplify rational expressions involving the four basic operations of arithmetic (+, -, ×, ÷) being sure to state restrictions
- about equivalence between two given expressions (polynomial or rational)



2.1,2.2 Adding and Subtracting Polynomials; Multiplying(Expanding Polynomials)

Learning Goal:

Reviewing like terms and learning to add, subtract, expand and simplify polynomials.

a) Simplify: $(4x^2 - 5x^3 - 5) + (2x^3 - 4x + 2)$

$$= -3x^3 + 4x^2 - 4x - 3$$

b) Subtract $(4x^2 - 3x + 1)$ from $(6x^2 + x)$

$$= (6x^2 + x) - (4x^2 - 3x + 1)$$

$$= 6x^2 + x - 4x^2 + 3x - 1$$

$$= 2x^2 + 4x - 1$$

c) Expand:

a) $2x(3x - 8)$

$$= 6x^2 - 16x$$

DISTRIBUTIVE PROPERTY

b) $(2x - 3)(5x + 7)$

$$= 10x^2 + 14x - 15x - 21$$

$$= 10x^2 - x - 21$$

c) $-2(x - 5)(3x - 2)$

$$= -2[3x^2 - 2x - 15x + 10]$$

$$= -2[3x^2 - 17x + 10]$$

$$= -6x^2 + 34x - 20$$

d) $4n(n - 3) + (5n + 1)(3n + 2)$

$$= 4n^2 - 12n + 15n^2 + 10n + 3n + 2$$

$$= 19n^2 + n + 2$$

e) $(4x - 5)(4x + 5)$

$$= 16x^2 + 20x - 20x - 25$$

$$= 16x^2 - 25$$

f) $(4x - 5)^2$

$$= (4x - 5)(4x - 5)$$

$$= 16x^2 - 20x - 20x + 25$$

$$= 16x^2 - 40x + 25$$

Special Cases: There are two special cases which, when encountered in through expanding, can be done just by foiling (as we did with e and f above). However, if you understand their respective patterns, you will be able to understand how to factor them with ease.

Case 1: Difference of Squares – Notice what happened in *part e* above. After we expanded, the middle terms of $20x$ **cancelled out**. This happens in ALL Difference of Squares questions. All we need to do, then, is square the first term, and subtract the square of the second term. Let's try it out and see how quickly it goes!

$$\begin{aligned} \text{a) } & (3x+2)(3x-2) \\ & = (3x)^2 + (-6x) + (6x) - 2^2 \\ & = (3x)^2 - 2^2 \\ & = 9x^2 - 4 \end{aligned}$$

$$\begin{aligned} \text{b) } & (8x-5)(8x+5) \\ & = 64x^2 - 25 \end{aligned}$$

$$\begin{aligned} \text{c) } & (12x-9)(12x+9) \\ & = 144x^2 - 81 \end{aligned}$$

Case 2: Perfect Square – Again, notice what happened in f . The first and last terms are squared, but what about the middle terms? They were two $20x$'s, making a total of $40x$. Instead of cancelling out like in Difference of Squares, they double up. How can we use this to our benefit? Because this will happen with all perfect squares, the process is this: square the first term, multiply the two terms together and doubles them, then add the square of the second term. Let's try it out!

$$\begin{aligned} \text{a) } & (3x+2)^2 \\ & (3x+2)(3x+2) \\ & (3x)(3x) + 6x + 6x + (2)(2) \\ & (3x)^2 + 2(6x) + (2)^2 \\ & \quad \quad \quad 2(3x)(2) \\ \rightarrow & 9x^2 + 12x + 4 \end{aligned}$$

$$\begin{aligned} \text{b) } & (5x-8)^2 \\ & (5x-8)(5x-8) \\ & (5x)^2 - (5x)(8) - (5x)(8) + 8^2 \\ & 25x^2 - 80x + 64 \end{aligned}$$

$$\begin{aligned} \text{c) } & (9x+1)^2 \\ & = 81x^2 + 18x + 1 \end{aligned}$$

$$\begin{aligned} \text{d) } & (10x-3)^2 \\ & = 100x^2 - 60x + 9 \end{aligned}$$

Additional Note: Polynomial Expressions are considered **equivalent** if they contain **exactly the same terms**. So, you can tell if two expressions are equivalent just by looking.

Example 2.2.1:

Expand and Simplify:

$$\begin{aligned} \text{a) } & 3x(2x^2 - 3x + 2) \\ & = 6x^3 - 9x^2 + 6x \end{aligned}$$

$$\begin{aligned} \text{b) } & (2x - 5)(3x + 1) \\ & = 6x^2 + 2x - 15x - 5 \\ & = 6x^2 - 13x - 5 \end{aligned}$$

$$\begin{aligned} \text{c) } & (5 - 2x)^2 \\ & = 25 - 20x + 4x^2 \end{aligned}$$

$$\begin{aligned} \text{d) } & (x + 2)(3x - 1)(x + 5) \\ & = [3x^2 - x + 6x - 2](x + 5) \\ & = [3x^2 + 5x - 2](x + 5) \\ & = 3x^3 + 5x^2 - 2x \\ & \quad + 15x^2 + 25x - 10 \\ \hline & = 3x^3 + 20x^2 + 23x - 10 \end{aligned}$$

$$\begin{aligned} \text{e) } & (x - 2y + 3z)^2 \\ & (x - 2y + 3z)(x - 2y + 3z) \\ & = x^2 - 2xy + 3xz \\ & \quad - 2xy + 4y^2 - 6yz \\ & \quad + 3xz - 6yz + 9z^2 \\ \hline & = x^2 - 4xy + 6xz + 4y^2 - 12yz + 9z^2 \end{aligned}$$

HW - Section 2.1 and 2.2
Handout

Success Criteria:

- I can recognize when two functions are equivalent
- I can add and subtract polynomials by collecting like terms
- I can expand polynomials using the distributive property (FOIL)

2.3 Factoring Polynomials

Learning Goal: Review the Factoring Techniques

2.3: **FACTORING UNDOES THE DISTRIBUTIVE PROPERTY**

A. Factoring Polynomials: **Common Factors**

Goal: We are learning to factor polynomials by dividing out the Greatest Common Factor.

Okay, I feel rather silly teaching this concept. A common factor is something that can be divided into each term. You then divide every term by that common factor, leaving the common factor in front of the expression. Remember, you are creating equivalent expressions.

Example:

$$\begin{aligned} \text{a) } & \frac{4x^2}{4} + \frac{8x}{4} - \frac{20}{4} \\ & = 4(x^2 + 2x - 5) \end{aligned}$$

$$\begin{aligned} \text{b) } & \frac{15x^3y}{5xy} + \frac{40x^2y}{5xy} - \frac{35xy}{5xy} \\ & = 5xy(3x^2 + 8x - 7) \end{aligned}$$

Not too shabby. As we move into other factoring, it is *essential* that you *always* ask yourself if there is a common factor. If there is, your factoring will be much easier as you will be dealing with small numbers. So, while this is an easy concept, it is something that cannot be overlooked. Okay, now for something trickier.

$$\begin{aligned} \text{a) } & 7x(2x-5) + 4(2x-5) \\ & = (2x-5)(7x+4) \end{aligned}$$

$$\begin{aligned} \text{b) } & \frac{12n^3 - 16n^2}{4n^2} + \frac{15n - 20}{5} \\ & = 4n^2(3n-4) + 5(3n-4) \\ & = (3n-4)(4n^2+5) \end{aligned}$$

B. Factoring Polynomials: **Simple Quadratic Trinomials** $x^2 + bx + c$

Factoring is the opposite of expanding. When you expand $(x+4)(x+7)$ give us $x^2 + 11x + 28$.

Therefore, when we factor a trinomial of the form $x^2 + bx + c$, we should get $(x+r)(x+s)$, where r and s are integers (whole numbers, positive or negative).

Example time!!

$$\begin{aligned} \text{a) } & x^2 + 12x + 32 \\ & M = 32 \\ & A = 12 \\ & = (x+4)(x+8) \end{aligned}$$

$$\begin{array}{l} 1 \times 32 \\ 2 \times 16 \\ 4 \times 8 \end{array}$$

$$\begin{aligned} \text{b) } & x^2 + 5x - 36 \\ & M = -36 \\ & A = 5 \\ & = (x-4)(x+9) \end{aligned}$$

$$\begin{array}{l} -1 \times 36 \\ -2 \times 18 \\ -3 \times 12 \\ -4 \times 9 \end{array}$$

$$\begin{aligned} \text{c) } & \frac{2x^2}{2} - \frac{20x}{2} + \frac{48}{2} \\ & = 2(x^2 - 10x + 24) \\ & M = 24 \\ & A = -10 \\ & = 2(x-4)(x-6) \end{aligned}$$

$$\begin{array}{l} -1 \times 24 \\ -2 \times -12 \\ -3 \times -8 \\ -4 \times -6 \end{array}$$

$$\begin{aligned} \text{d) } & x^2 - 12x - 1485 \\ & = (x+33)(x-45) \end{aligned}$$

C. Factoring Polynomials: Quadratic Trinomials $ax^2 + bx + c$ where $a \neq 1$.

When there is a number in front of the x^2 term that can't be factored out, we need to use the method of decomposition.

Here are the steps to set it up:

- 1) Multiply $a \times c$
- 2) List ALL the factors of that new number
- 3) Find a pair of factors that add to b . Keeping in mind that their signs must match ac .
- 4) Factor by Decomposition

It's easier to teach with examples, so let's just dive in:

$$\begin{aligned} \text{a) } & 5x^2 - 7x + 2 \\ & M = 10 \\ & A = -7 \\ & = \frac{5x^2 - 2x - 5x + 2}{x \quad -1} \\ & = x(5x-2) - 1(5x-2) \\ & = (5x-2)(x-1) \end{aligned}$$

$$\begin{array}{l} -1 \times -10 \\ -2 \times -5 \end{array}$$

$$\begin{aligned} \text{b) } & 6x^2 + 13x + 5 \\ & M = 30 \\ & A = 13 \\ & = \frac{6x^2 + 10x + 3x + 5}{2x \quad 1} \\ & = (2x+1)(3x+5) \end{aligned}$$

Let's try to cut out a step (you don't have to if you don't want to!!)

$$\begin{aligned} \text{c) } & 2x^2 - 7x - 30 \\ & M = -60 \\ & A = -7 \\ & = \frac{2x^2 + 5x - 12x - 30}{x \quad -6} \\ & = (x-6)(2x+5) \end{aligned}$$

$$\begin{array}{l} 1 \times -60 \\ 2 \times -30 \\ 3 \times -20 \\ 4 \times -15 \\ 5 \times -12 \end{array}$$

$$\begin{aligned} \text{d) } & 10x^2 + x - 24 \\ & M = -240 \\ & A = 1 \\ & = 10x^2 + 16x - 15x - 24 \\ & = (2x-3)(5x+8) \end{aligned}$$

$$\begin{aligned} \text{e) } & 35x^2 - 190x + 75 \\ & = 5(7x^2 - 38x + 15) \\ & = 5(x-5)(7x-3) \end{aligned}$$

$$\begin{aligned} \text{f) } & 60x^2 - 222x - 216 \\ & = 6(10x^2 - 37x - 36) \\ & = 6(5x+4)(2x-9) \end{aligned}$$

$$\begin{array}{r} -360 \\ -\cancel{6} \ 4 \\ \underline{} \ 2 \\ -458 \end{array}$$

D. Factoring Polynomials: **Special Cases**

Case 1: Difference of Squares

Remember expanding $(3x-5)(3x+5)$? We got:

Hence, when we factor difference of squares, recognize that it is a binomial, it has two square terms, and they are being subtracted. If it all fits, it is a difference of squares.

Factor:

$$\begin{aligned} \text{a) } & 16x^2 - 49 \\ & = (4x+7)(4x-7) \end{aligned}$$

$$\begin{aligned} \text{b) } & 64x^2 - 81 \\ & = (8x+9)(8x-9) \end{aligned}$$

$$\begin{aligned} \text{c) } & \frac{5x^2}{5} - \frac{45}{5} \\ & = 5(x^2 - 9) \\ & = 5(x+3)(x-3) \end{aligned}$$

$$\text{d) } 25x^2 + 36$$

Not factorable

Case 2: Perfect Squares

Similarly, what happens when we expand $(3x+5)^2$?

When we come up a trinomial that looks like a perfect square, we just need to do a quick mental check:

1. Is the first term a perfect square? 2. Is the last term a perfect square? 3. Take the square roots of each term, multiply them and double them. If you get the middle term, you have a perfect square!

Factor:

$$\begin{aligned} \text{a) } & \frac{9x^2}{(3x)^2} - 2(3x)(4) + \frac{16}{4^2} \\ & = (3x+4)^2 \end{aligned}$$

$$\begin{aligned} \text{b) } & 49x^2 + 84x + 36 \\ & = (7x)^2 + 2(7x)(6) + 6^2 \\ & = (7x+6)^2 \end{aligned}$$

$$\text{c) } 200x^2 + 360x + 162$$

HW- Section 2.3 Handout

Success Criteria:

- I can factor polynomials using a variety of methods (GCF, a=1, decomposition, difference of squares, perfect squares)

2.4 Simplifying Rational Expressions

Learning Goal: We are learning to define rational functions, and explore methods of simplifying the related rational expression.

Definition: A rational expression is of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$

i.e. A **RATIONAL EXPRESSION** is constructed by “**DIVIDING**” one **POLYNOMIAL EXPRESSION** by another.

e.g. $\frac{3x^2-4x+1}{x^2-1}$ is a rational expression

Simplification of rational expressions involves the following steps:

1. Factor the Numerator and Denominator completely
2. State any restrictions on the variables in the Denominator (like the denominator can never be zero)
3. Reduce!! In other words, cancel out all the common factors. 😊
4. Write the expression in the simplified form.

Note that stating restrictions **MUST BE DONE BEFORE CANCELLING!!!!** If you cancel before stating the restrictions, **YOU RUN THE RISK OF EXPLODING THE UNIVERSE**. Don't do it. **FOR THE SAKE OF ALL HUMANITY, PLEASE DON'T DO CANCEL BEFORE STATING RESTRICTIONS!**

RESTRICTIONS ON A RATIONAL EXPRESSION

Consider the rational expression $\frac{2x-5}{x+2}$. Because x is a variable, we can substitute different (varying) values for it and calculate different values for the rational expression.

Denominator $\neq 0$

$$x + 2 \neq 0$$

$$x \neq -2$$

However, there is one value which we cannot substitute:

$$x \neq -2$$

$$\frac{x+5}{3x+2}$$

$D \neq 0$

$$3x + 2 \neq 0$$

$$3x \neq -2$$

$$x \neq -\frac{2}{3}$$

2.6 Multiplying and Dividing Rational Expressions

Learning Goal: We are learning to multiply and divide rational expressions.

This concept extends what we learned in section 2.4: **Simplifying Rational Expressions.**, BUT we are adding a small (but **ridiculously fun**) twist.

Now we will consider more than one rational expression at a time, and MULTIPLY OR DIVIDE them. Since Rational Expressions are analogous to fractions, we need to remind ourselves of the rules for multiplying and dividing fractions.

Rules:

1. Express as a product
2. Factor the Numerators and Denominators completely
3. State restrictions on the variables
4. Reduce! In other words, cancel out all the common factors. 😊
5. Write the expression in the simplified form.

Let's start by reminding ourselves the idea of multiplying and dividing simple rational numbers. We will then extend the idea to rational expressions. Remember to state the restrictions on the variables!

$$1. \frac{15}{10} \times \frac{12}{18} = 1$$

$$2. \frac{15}{10} \div \frac{12}{18} = \frac{15}{10} \times \frac{18}{12} = \frac{9}{4}$$

$$3. \frac{14xy}{21y} \times \frac{x^2}{3x^2} \div \frac{6x^2}{7x}$$

$y \neq 0, x \neq 0$

$$= \frac{14\cancel{y}}{3\cancel{y}} \times \frac{\cancel{x}^2}{3\cancel{x}^2} \times \frac{7\cancel{x}}{6\cancel{x}} = \frac{7}{27}$$

$$4. \frac{3(x-2)}{x} \times \frac{6}{x-2}$$

$x \neq 0, 2$

$$= \frac{9}{x}$$

Why have I coloured the denominators?

$$x \neq \frac{2}{3}, -1, \frac{3}{2}, -5$$

5. $\frac{x^2 - 25}{3x^2 + x - 2} \times \frac{6x^2 - 13x + 6}{2x^2 + 7x - 15}$

$$\frac{(x+5)(x-5)}{(3x-2)(x+1)} \times \frac{(3x-2)(2x-3)}{(2x-3)(x+5)}$$

$N_1 = x^2 - 25 = (x+5)(x-5)$
 $N_2 = 6x^2 - 13x + 6$
 $M = 3, A = -13 < -9$
 $= \frac{6x^2 - 9x - 4x + 6}{3x} = \frac{(3x-2)(2x-3)}{3x}$

$D_1 = 3x^2 + x - 2$
 $M = -6, A = 1$
 $= 3x^2 + 3x - 2x - 2$
 $= (3x-2)(x+1)$

$$= \frac{(x-5)}{(x+1)}$$

$D_2 = 2x^2 + 7x - 15$
 $M = -30 < -3, A = 7$
 $= 2x^2 + 10x - 3x - 15$
 $= (2x-3)(x+5)$

6. $\frac{(2x+1)^2}{25x^2 - 1} \div \frac{-4x-2}{20x+4}$

$$= \frac{(2x+1)(2x+1)}{(5x+1)(5x-1)} \times \frac{4(5x+1)}{-2(2x+1)}$$

$$x \neq -\frac{1}{5}, \frac{1}{5}, -\frac{1}{2}$$

$$= \frac{-2(2x+1)}{(5x-1)} = \frac{-4x-2}{5x-1}$$

7. $\frac{9y^2 - 4}{4y - 12} \div \frac{9y^2 + 12y + 4}{18 - 6y}$

$$y \neq 3, -\frac{2}{3}$$

$$= \frac{(3y+2)(3y-2)}{4(y-3)} \times \frac{6(3-y)}{(3y+2)(3y+2)}$$

$$= \frac{-3(3y-2)}{2(3y+2)} = \frac{-9y+6}{6y+4}$$

HW - Section 2.6

Pg. 121 - 123 #1ab, 3, 5, 6abc, 7ab, 8, **Try 13 for a fun challenge**

Success Criteria:

- I can multiply rational expressions by following the appropriate steps
- I can divide rational expressions by multiplying by the reciprocal of the divisor, then following the multiplication steps
- I can determine the restrictions from the factored form of the rational expression

2.6 Adding and Subtracting Rational Expressions

Learning Goal: We are learning to add and subtract rational expressions.

This is it. The pinnacle of Rational Expressions. **The most difficult thing you can do with Rational Expressions is to add or subtract them.** That's right! Adding and subtracting is the most difficult thing. Thankfully, you all can handle it!!

COMMON DENOMINATOR

Getting a Common Denominator is the key to the whole scene. A “**COMMON**” denominator must contain enough **FACTORS** (*or better: THE CORRECT FACTORS*) to “**KEEP EVERYONE HAPPY**”, so to speak. Keep in mind that we **STILL MUST STATE RESTRICTIONS** when faced with factors containing variables!!!

Rules:

1. Factor completely the Numerators and Denominators of each rational expression stating restrictions on the variables and reducing if possible.
2. Find the Least Common Denominator (LCD)
3. Rewrite each rational expression with the LCD as the denominator for each.
4. Add or Subtract the numerators (*which may require some expanding so that you can collect like terms!*)
5. Reduce if possible! In other words, cancel out all the common factors from the new numerator and new denominator. 😊
6. Write the final expression (the sum or difference found) in the simplified form.

Let's start by reminding ourselves the idea of adding and subtracting simple rational numbers. We will then extend the idea to rational expressions. Remember to state the restrictions on the variables!

$$1. \frac{1}{10} - \frac{2}{25} + \frac{3}{20} - \frac{1}{4}$$

$\begin{matrix} 10 \times & & 4 \times & & & & 5 \times & & & & 25 \times \\ 10 \times & 10 & 4 \times & 25 & & & 5 \times & 20 & & & 4 \times 25 \\ & & & & & & & & & & & \end{matrix}$

$$= \frac{10 - 8 + 15 - 25}{100}$$

Clearly LCD=100!!!! But what next?

$$= \frac{-8}{100} = \frac{-2}{25}$$

$$2. \frac{10x^3}{10x^4} + \frac{5t^2x}{5t^2t^2} - \frac{3 \times 2t^3}{5t \times 2t^3}$$

$LCD = 10t^4$

$$= \frac{30 + 5t^2 - 6t^3}{10t^4} ; t \neq 0$$

$$3. \frac{4x}{x^2 + 6x + 8} - \frac{3x}{x^2 - 3x - 10}$$

$$= \frac{4x(x-5)}{(x-5)(x+2)(x+4)} - \frac{3x(x+4)}{(x-5)(x+2)(x+4)}$$

$LCD = (x+2)(x+4)(x-5)$

$$= \frac{4x^2 - 20x - 3x^2 - 12x}{(x+2)(x+4)(x-5)} = \frac{x^2 - 32x}{(x+2)(x+4)(x-5)} ; x \neq -2, -4, 5$$

$$4. \frac{(p+1)}{p^2 + 2p - 35} + \frac{p^2 + p - 12}{p^2 - 2p - 24} \times \frac{p^2 - 4p - 12}{p^2 + 2p - 15}$$

BEDMAS, folks, BEDMAS!!!!!!!!!!!! (Seriously...bedmas)

$$= \frac{(p+1)}{(p+7)(p-5)} + \left(\frac{(p+4)(p-3)}{(p-6)(p+4)} \times \frac{(p-6)(p+2)}{(p+5)(p-3)} \right)$$

$p \neq -7, 5, 6, -4, -5, 3$

$$= \frac{(p+1)(p+5)}{(p+7)(p-5)} + \frac{(p+2)(p+7)(p-5)}{(p+5)(p+7)(p-5)} = \frac{p^2 + 6p + 5 + p^3 + 4p^2 - 31p - 70}{(p+5)(p+7)(p-5)}$$

$$\begin{aligned} & (p+2)(p+7)(p-5) \\ &= (p^2 + 9p + 14)(p-5) = p^3 + 9p^2 + 14p - 5p^2 - 45p - 70 \\ &= p^3 + 4p^2 - 31p - 70 \end{aligned}$$

$$= \frac{p^3 + 5p^2 - 25p - 65}{(p+5)(p+7)(p-5)}$$

HW- Section 2.7

Pg 128 - 130 #1bc, 3b, 5c, 6ad, 7acef, 8c, 9ac

Success Criteria:

- I can add and subtract expressions through determining the LCD
- I can recognize that the LCD is not always the product of all the denominators
- I can identify restrictions from the factored form of the LCD

Unit 1 (Chapter 2) – Polynomial and Rational Expressions

Chapter 2 Review

In this chapter we have not worked with functions. Instead, we flexed our “algebraic muscles” through simplification of expressions.

Polynomial Expressions

The main thing here is that we add/subtract **LIKE TERMS**

Keep in mind the distinction between **Term** and **Factor**!

Rational Expressions

Restrictions...Restrictions...holy cow RESTRICTIONS

Cancelling...**Factors** are the only things which can be

cancelled...FACTORS (NOT TERMS!!!!)

Get your **Restrictions BEFORE YOU CANCEL the factors**

Don't forget the small twist in finding restrictions for division problems!

Review- Chapter Review Section

Pages 132 – 133 #7abcd, 8ace, 10cd, 12ac, 13b, 14, 15bcf

* Chapter Review Practice Handout.