

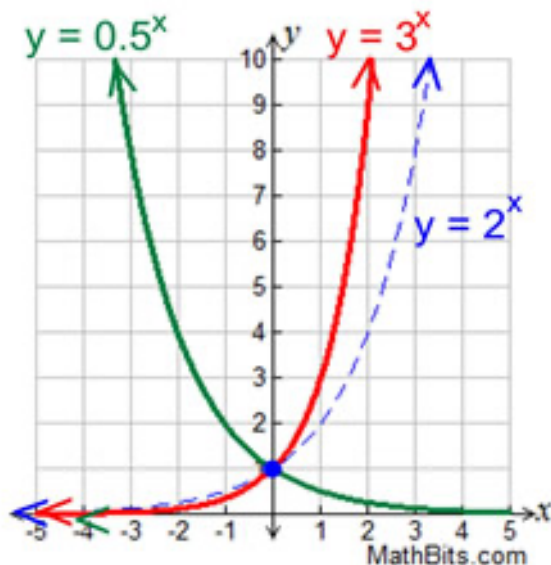
Functions 11

Course Notes

Unit 4 – Exponential Functions

We are learning to

- describe the characteristics of exponential functions and their graphs
- evaluate powers with integer and rational exponents and simplify expressions involving them
- explore the transformation of exponential functions
- use exponential functions to solve problems involving exponential growth and decay



4.2 Integer Exponents (Reviewing the known Power Laws and learning a few new ones 😊)

Learning Goal: We are learning to work with integer exponents.

Before beginning, we should quickly review: **THE POWER LAWS**

Consider a typical “power” a^n .

We call “ a ” the base. We call “ n ” the exponent and

the entire expression a^n is called a power.

Let’s recall **The Known Power Laws**:

Given the powers a^m and a^n , with exponents m and n , and the number $\frac{a}{b}$, and $a, b \neq 0$, then

$$1) 1^n = 1$$

$$2) a^1 = a$$

$$3) a^0 = 1$$

$$4) a^m \cdot a^n = a^{m+n}$$

$$a^3 \cdot a^2 = a \cdot a \cdot a \cdot a \cdot a = a^5$$

$$5) \frac{a^m}{a^n} = a^{m-n}$$

$$\frac{a^3}{a^2} = \frac{a \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a}} = a^1$$

$$6) (ab)^m = a^m b^m$$

$$a^0 = a^{3-3} = \frac{a^3}{\cancel{a^3}} = 1$$

$$7) \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$(a^5)^2 = a^{10}$$

$$8) (a^m)^n = a^{m \cdot n}$$

$$\left(\frac{2}{7}\right)^{-1} = \frac{7}{2} \quad 5^{-1} = \frac{1}{5}$$

Until now, for the most part, the exponents you've been working with have always been **NATURAL NUMBERS**. But, we now will examine some more **INTEGER EXPONENTS!!** i.e. we will also start dealing with Negative Exponents.

Let's explore Negative Exponents and create some additional Power Laws. 😊

Rule to remember for negative exponents: Negative Exponents reciprocate the Base

ADDITIONAL POWER LAWS:

$$9) a^{-n} = \frac{1}{a^n}$$

$$10) \left(\frac{a}{b}\right)^{-n} = \left(\frac{a^n}{b^n}\right)^{-1} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$$

$$11) \frac{a^{-m}}{a^{-n}} = \frac{a^n}{a^m} = a^{n-m}$$

$$\begin{aligned} &5^{-2} \\ &(5^2)^{-1} \\ &= \frac{1}{5^2} = \frac{1}{25} \end{aligned}$$

Example 4.2.1

Write each expression as a **single power with a positive exponent**:

a) $(4)^{-5} = \frac{1}{4^5} = \left(\frac{1}{4}\right)^5$

b) $\left(\frac{3}{2}\right)^{-4} = \left(\frac{2}{3}\right)^4$

c) $\frac{7^3}{7^9} = 7^{3-9} = 7^{-6} = \frac{1}{7^6} = \left(\frac{1}{7}\right)^6$

Example 4.2.2

Simplify, then **evaluate** leaving final answers in rational form:

a) $3^5 \cdot (3^{-2})$
 $= 3^{5+(-2)}$
 $= 3^3$
 $= 27$

b) $(2^{-3} \cdot 2^4)^{-5}$
 $= (2^{-3+4})^{-5}$
 $= (2^1)^{-5}$
 $= 2^{-5}$
 $= \frac{1}{2^5}$
 $= \frac{1}{32}$

c) $\frac{5^{-3}}{(5^2)^{-2}}$
 $= \frac{5^{-3}}{5^{-4}} = \frac{5^4}{5^3}$
 $= 5^{-3-(-4)} = 5^{4-3}$
 $= 5^1$
 $= 5$

d) $(xy^3z^7)^0$
 $= 1$

Negative Bases:

Investigate what happens when the base is negative...

Example: Expand, then evaluate the following:

a. $(-1)^2 = (-1)(-1) = 1$

b. $(-1)^3 = (-1)(-1)(-1) = -1$

c. $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

d. $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{(-2)(-2)(-2)} = \frac{1}{-8}$

e. $-1^4 = -(1)(1)(1)(1) = -1$

f. $-2^{-4} = -\frac{1}{2^4} = -\frac{1}{16}$

Negative Bases: $(-x)^a$

Write a rule for powers with a negative base:

If the exponent is even, the power is positive.

If the exponent is odd, the power is negative.

Practice:

1. Simply the following. Express answers in rational form (as a fraction):

a. $(-3)^2 = 9$

b. $(-3)^3 = -27$

c. $-4^2 = -16$

d. $-5^{-2} = -\frac{1}{25}$

e. $(-10)^{-3} = -\frac{1}{1000}$

f. $(-3)^3(-3)^{-3} = \frac{-27}{-27} = 1$
Handwritten note: $3 + (-3) = 0$, so $(-3)^0 = 1$

Example 4.2.3

Evaluate and express in rational form:

$$\begin{aligned} \text{a) } 3^2(6^{-3}) &= \frac{3^2}{6^3} \\ &= \frac{3 \times 3}{6 \times 6 \times 6} \\ &= \frac{1}{24} \end{aligned}$$

$$\begin{aligned} \text{b) } 2^{-3} + 10^{-3} - 3(5^{-3}) &= \frac{1}{2^3} + \frac{1}{10^3} - \frac{3}{5^3} \\ &= \frac{1}{8} + \frac{1}{1000} - \frac{3 \times 8}{125 \times 8} \\ &= \frac{125 + 1 - 24}{1000} = \frac{102}{1000} = \frac{51}{500} \end{aligned}$$

$$\begin{aligned} \text{c) } 13^{-5} \times \left(\frac{13^2}{13^8}\right)^{-1} &= 13^{-5} \cdot \frac{13^8}{13^2} \\ &= 13^{-5+8-2} \\ &= 13^1 = 13 \end{aligned}$$

Example 4.2.4

Evaluate using the laws of exponents (the power rules):

$$\begin{aligned} \text{a) } 3^2 \times 9^{-3} \div 3^{-7} &= 3^2 \times (3^2)^{-3} \div 3^{-7} \\ &= 3^2 \times 3^{-6} \div 3^{-7} \\ &= 3^{2+(-6)-(-7)} \\ &= 3^3 \\ &= 27 \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{4^{-2} + 3^{-1}}{5^{-1} + 2^{-2}} &= \frac{\frac{1}{4^2} + \frac{1}{3}}{\frac{1}{5} + \frac{1}{2^2}} \\ &= \frac{\frac{1}{16} + \frac{1}{3}}{\frac{1}{5} + \frac{1}{4}} \\ &= \frac{\frac{3+16}{48}}{\frac{4+5}{20}} \\ &= \frac{19}{48} \div \frac{9}{20} \\ &= \frac{19}{48} \times \frac{20}{9} = \frac{95}{108} \end{aligned}$$

HW- Section 4.2 Pg. 222 #5ace, 6ace, 7abf, 8cef, 13adefgh

Success Criteria:

- I can apply the exponent laws
- I can recognize that a negative exponent represents a reciprocal expression

4.3 Rational Exponents (Learning a few more new power laws 😊)

Learning Goal: We are learning to work with powers involving rational (fractional) exponents and to evaluate expressions containing them.

A **RATIONAL EXPONENT** can be a **FRACTION**.

$$3\left(\frac{1}{4}\right)$$

For example, we can consider the number $(16)^{\frac{3}{4}}$. Of course, the question we need to ask is:

What the rip is that thing??



Let's put our calculators to some good use. Use a calculator to complete the following:

i. $9^{\frac{1}{2}}$ = 3 . and $\sqrt{9}$ = 3 . So, $9^{\frac{1}{2}} = \sqrt{9}$.

ii. $64^{\frac{1}{3}}$ = 4 . and $\sqrt[3]{64}$ = 4 . So, $64^{\frac{1}{3}} = \sqrt[3]{64}$.

iii. $16^{\frac{1}{4}}$ = 2 . and $\sqrt[4]{16}$ = 2 . So, $16^{\frac{1}{4}} = \sqrt[4]{16}$.

Can you generalize the rule from the above examples? Note that each of the powers in the examples were **unit fractions**.

General Rule:

$$x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

Yes, that's right!! But what if the power is not a unit fraction like $(16)^{\frac{3}{4}}$?

As you know, a fraction has two parts: a numerator, and a denominator. When a fraction is used as an exponent, the two parts of the fraction carry two related (but different) meanings in terms of “powers”.

Definition 4.3.1

Given a power with a “rational” (fractional) exponent $a^{\frac{m}{n}}$, the **numerator** of the exponent is a “**power**” in the usual sense, and the **denominator** represents a “**root**” or “**radical**”.

So, with this knowledge let's come back to our example in the beginning.

e.g. For the number $16^{\frac{3}{4}}$ $= \sqrt[4]{16^3} = (\sqrt[4]{16})^3$

$4^2 = 16$
 $4 = \sqrt{16}$

$2^4 = 16$
 $2 = \sqrt[4]{16}$

$10^5 = 100000$
 $10 = \sqrt[5]{100000}$

Ans: $(2)^3 = 8$

So, **General Rule:** $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Also, remember that **the exponent laws that apply to powers are the same for rational exponents too.**

Example 4.3.1

From your text: Pg. 229 #2.

Write in exponent form, and then evaluate:

a) $\sqrt[7]{512}$
 $= (512)^{\frac{1}{7}}$
 $= 2$

c) $\sqrt[3]{27^2}$
 $= 27^{\frac{2}{3}}$
 $= 9$

f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$
 $= \left(\frac{16}{81}\right)^{-\frac{1}{4}}$
 $= \left(\frac{81}{16}\right)^{\frac{1}{4}}$
 $= \frac{3}{2}$

Handwritten diagram for f): A tree diagram showing 81 as 9*9, and 9 as 3*3*3*3. Arrows point from 81 to 3^4 and from 16 to 2^4.

b) $\sqrt[3]{-27}$
 $= (-27)^{\frac{1}{3}}$
 $= -3$

Note: We **CANNOT** take an **Even Root** of a **negative radicand**.
 We **CAN** take an **Odd Root** of a negative radicand, however.
Why?

Example 4.3.2

From your text: Pg. 229 #3

Write as a single power:

$$\begin{aligned} \text{a) } & \left(8^{\frac{2}{3}}\right)\left(8^{\frac{1}{3}}\right) && a^m \cdot a^n = a^{m+n} \\ & = 8^{\frac{2}{3} + \frac{1}{3}} \\ & = 8^{\frac{3}{3}} = 8^1 = \boxed{8} \end{aligned}$$

$$\begin{aligned} \text{b) } & 8^{\frac{2}{3}} \div 8^{\frac{1}{3}} && a^m \div a^n = a^{m-n} \\ & = 8^{\frac{2}{3} - \frac{1}{3}} \\ & = 8^{\frac{1}{3}} \\ & = \sqrt[3]{8} \\ & = \boxed{2} \end{aligned}$$

Example 4.3.3

From your text: Pg. 229 #4

Write as a single power, then evaluate. Express answers in **rational form**.

$$\begin{aligned} \text{a) } & \sqrt{5} \cdot \sqrt{5} \\ & = (\sqrt{5})^2 \\ & = 5^{\frac{2}{2}} \\ & = 5 \end{aligned}$$

$\sqrt{25} = 5^1$

$$\begin{aligned} \text{b) } & \frac{\sqrt[3]{-16}}{\sqrt[3]{2}} = \frac{(-16)^{\frac{1}{3}}}{(2)^{\frac{1}{3}}} \\ & = \left(\frac{-16}{2}\right)^{\frac{1}{3}} \\ & = -2 \end{aligned}$$

HW- Section 4.3

Pg 229 #2de, 3cdef, 5, 6, 8, 10

Success Criteria:

- I can understand that the numerator of a fractional exponent is the power, while the denominator is the root.

4.4 Simplifying Expressions Involving Exponents

Learning Goal: We are learning to simplify algebraic expressions involving powers and radicals.

Keep the **EXPONENT RULES** in your mind at all times.

One of the Keys of the exponent rules is “**SAMENESS**”.

- When you have the **SAME BASE**, (but possibly different exponents) you can combine powers.

e.g. $\frac{x^3 \times x^4}{x^7} = x^{3+4-7} = x^0 = 1$

- When you have the **SAME EXPONENT** (but possibly different bases) you can “combine the bases under the same exponent”.

e.g. $\frac{\sqrt[3]{12} \times \sqrt[3]{36}}{\sqrt[3]{16}} = \frac{(12)^{\frac{1}{3}} \cdot (36)^{\frac{1}{3}}}{(16)^{\frac{1}{3}}} = \left(\frac{12 \cdot 36}{16}\right)^{\frac{1}{3}} = (27)^{\frac{1}{3}} = 3$

↓

$$= \sqrt[3]{\frac{12 \times 36}{16}}$$

$$= \sqrt[3]{27}$$

$$= 3$$

Now we turn to problems involving both numbers and variables being exponentized (not a word, but it should be because of how awesome it sounds).

Example 4.4.1

Simplify, leaving your answer with only positive exponents:

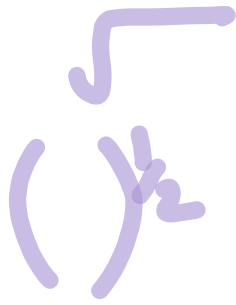
$$\begin{aligned} \text{a) } & (x^3)^2 (x^{-8}) \\ &= x^{6+(-8)} \\ &= x^{-2} \\ &= \frac{1}{x^2} \end{aligned}$$

$$\begin{aligned} \text{b) } & \frac{z^4 (z^{-2})}{z^{-3}} \\ &= z^{4+(-2)-(-3)} \\ &= z^{4-2+3} \\ &= z^5 \end{aligned}$$

$$\begin{aligned} \text{c) } & \sqrt{36x^{-6}} \\ &= \sqrt{36} x^{-\frac{6}{2}} \\ &= 6x^{-3} \\ &= \frac{6}{x^3} \end{aligned}$$

$$\begin{aligned} \text{d) } & \left(\frac{(3x^2y)^{-1} (x^3y^{-4})}{(x^3y^2)^{-2}} \right)^{-2} \\ &= \left(\frac{3^{-1} x^{-2} y^{-1} \cdot x^3 y^{-4}}{x^{-6} y^{-4}} \right)^{-2} \\ &= \left(\frac{1}{3} x^{-2+3-(-6)} y^{-1+(-4)-(-4)} \right)^{-2} \\ &= \left(\frac{x^7 y^{-1}}{3} \right)^{-2} = \left(\frac{x^7}{3y} \right)^{-2} \\ &= \frac{3^2 y^2}{(x^7)^2} = \frac{9y^2}{x^{14}} \end{aligned}$$

$$\begin{aligned}
 \text{e) } & \left(\frac{\sqrt{16a^6}}{(a^3)^{-1}} \right)^{\frac{3}{2}} \\
 & = \left(\frac{4a^{\frac{6}{2}}}{a^{-3}} \right)^{\frac{3}{2}} \\
 & = (4a^{3-(-3)})^{\frac{3}{2}} \\
 & = (4a^6)^{\frac{3}{2}} \\
 & = 4^{\frac{3}{2}} (a^6)^{\frac{3}{2}} \\
 & = (\sqrt{4})^3 a^{6 \cdot \frac{3}{2}} \\
 & = 8a^9
 \end{aligned}$$



$$\begin{aligned}
 \text{f) } & \left(\frac{(6x^3)^2 (6y^3)}{(9xy)^6} \right)^{-\frac{1}{3}} \\
 & = \left(\frac{6^2 \cdot 6 \cdot y^3}{9^6 \cdot y^6} \right)^{-\frac{1}{3}} \\
 & = \left(\frac{216}{531441 y^3} \right)^{-\frac{1}{3}} \\
 & = \frac{\sqrt[3]{531441 y^3}}{\sqrt[3]{216}} \\
 & = \frac{27}{6} y \\
 & = \frac{27y}{2}
 \end{aligned}$$

HW- Section 4.4

Pg 236-237 #2bdf, 4bdf, 5cef, 6cef, 7ac, 9ad

Success Criteria:

- I can simplify algebraic expressions containing powers by using the exponent laws
- I can simplify algebraic expressions involving radicals

4.5-4.6 Properties and Transformations of Exponential Functions

Learning Goal: We are learning to identify the characteristics and transformations of the graphs and equations of exponential functions.

Exponential Functions are of the (basic) form:

$$f(x) = b^x \quad \text{where } b > 0 \text{ and } b \neq 0$$

(of course, we can apply transformations to this basic, or **parent function**!! Fun Times are – a – coming!! 😊)

In the parent exponential function $f(x) = b^x$, b is the base. **THE BASE OF AN EXPONENTIAL FUNCTION IS JUST A NUMBER.** For example, we might have the functions

Note that the exponent is the variable here in the Exponential Function!

What does the Base of an Exponential Function tell you?

Activity: Folding a Piece of Paper.

If you have ever tried to fold a piece of paper repeatedly, you probably already know that the more folds in the paper, the harder it is to get the next fold.

Take some time to think and fill in the table below:

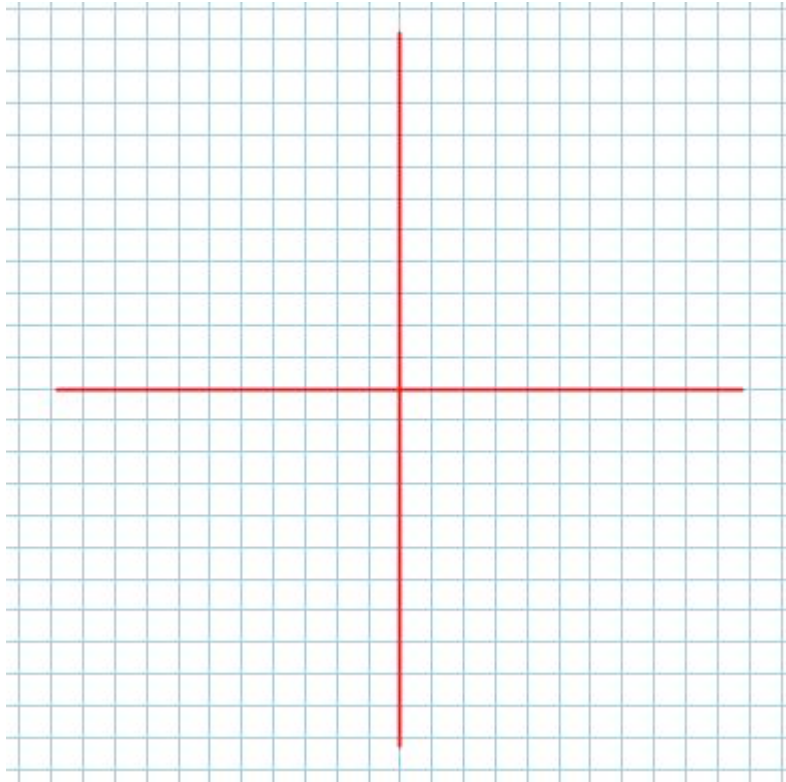
# of Folds in a Sheet of Paper	# of Layers of the Paper in each fold	Exponent form of the # of Layers
0		
1		
2		
3		
4		
5		
6		

If you could represent the paper folding situation as a function, what would it be?

(Hint: Take the independent variable as x and try defining $f(x)$ 😊)

_____.

Let's try plotting the function in the grid below:

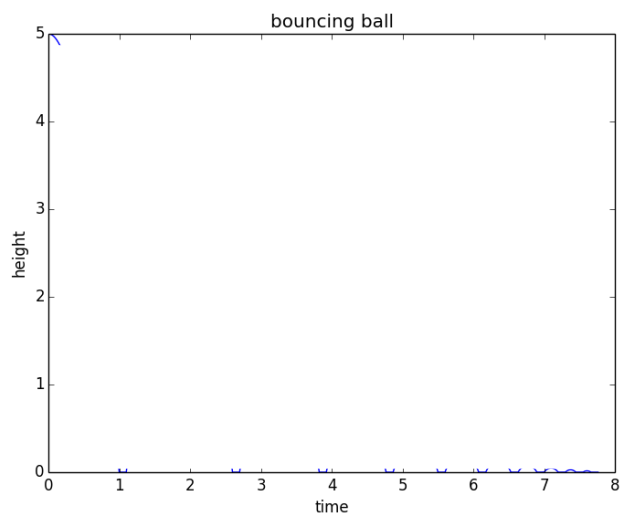


Let's explore another example:

Suppose you **drop freely** a bouncy ball from a certain height (say 1 metre), what do you think will happen?

We will notice that after the first bounce, the ball will bounce back only about _____ the height that it originally was dropped from.

After the second bounce, the peak height becomes _____ again and the process goes on until the ball stops bouncing completely.



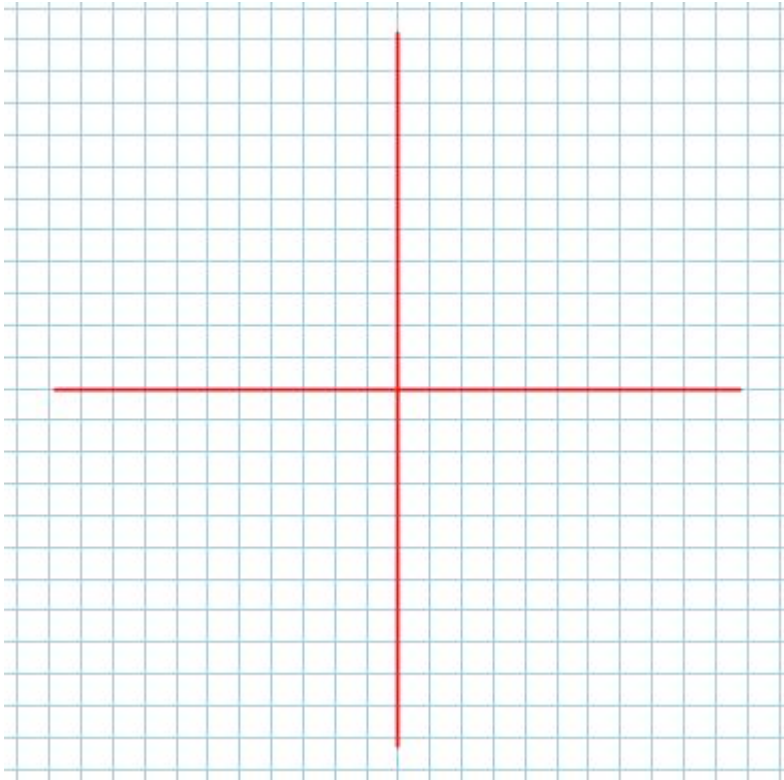
Take some time to think and fill in the table below:

# of bounces	Peak Height (in metres) <i>Use fractions</i>	Peak Height in exponential form
0		
1		
2		
3		
4		
5		
6		

If you could represent the situation as a function, what would it be?

(Hint: Take the independent variable as x and try defining $f(x)$ 😊)

Let's try plotting the function in the grid below:



From the above activities, it is pretty clear that the base of the exponential function i.e. the constant **b** determines

the RATE of CHANGE of the exponential function.

- i. If the base $b > 1$, then the Exponential function defines _____.

For example, the growth of population, bacteria, money in some investment, etc.

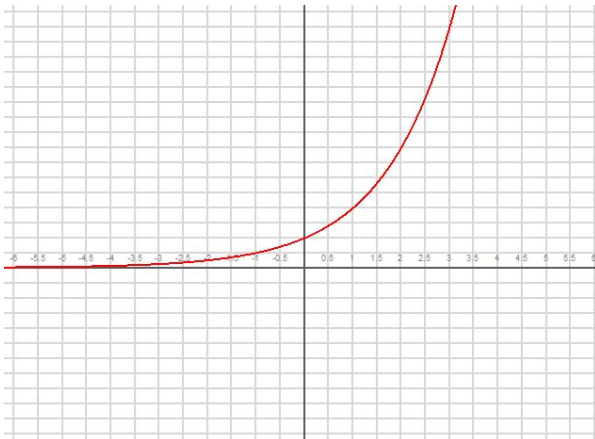
- ii. If the base $0 < b < 1$, then the Exponential function defines _____.

For example, the death of populations, radioactive decay, the temperature of a hot cup of coffee left on the desk, depreciation of a car, etc.

Note that if exponential functions are some kind of Growth or Decay, then we need to have something to begin with. Hence, we will not consider the **b** to be negative and choose to consider only **positive bases 'b'**.

Domain and Range of the Basic Exponential Function

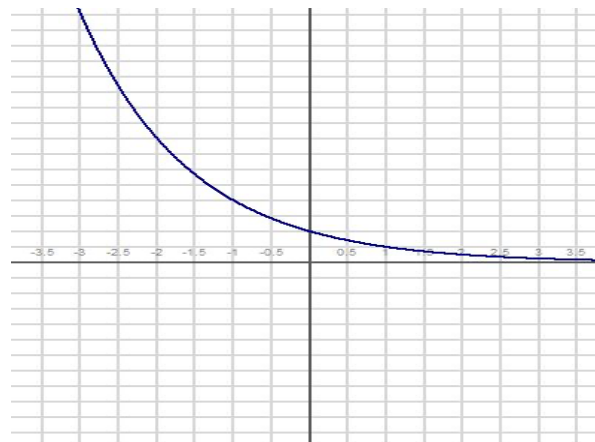
Consider the sketches:



Growth Function

D = _____.

R = _____.



Decay Function

D = _____.

R = _____.

ALL Exponential Functions have a Horizontal **ASYMPTOTE** (Basic Exponential Functions have $y = 0$ i.e. x-axis as their Horizontal Asymptote.

The Horizontal Asymptote of a Transformed Exponential Function, however, depends on the _____.)

Example. $f(x) = 2^x + 1$

ALL BASIC Exponential Functions pass through the point $(0,1)$ which is the y-intercept. Transformed Exponential Functions will have a y-intercept too, but depends on the vertical shift/stretch and horizontal shift/stretch. So, we can find the y-intercept by taking _____.

The Transformed Exponential Function

The general form of an exponential function is:

$$f(x) = a \cdot b^{k(x-d)} + c$$

Where:

Example 4.6.1

State the transformations applied to the parent function $f(x) = 3^x$. Also state the y-intercept, and the equation of the horizontal asymptote of the transformed function.

$$g(x) = -2 \cdot 3^{3x+3} + 4$$

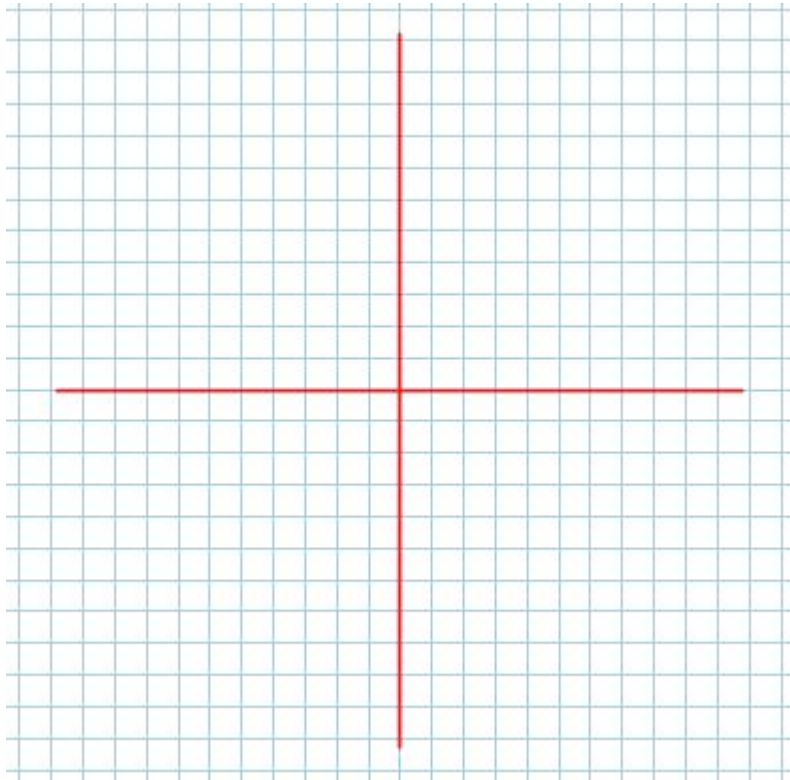
Example 4.6.2

From your Text: Page 252 #7

7. A cup of hot liquid was left to cool in a room whose temperature was 20°C .

C The temperature changes with time according to the function

$T(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{30}} + 20$. Use your knowledge of transformations to sketch this function. Explain the meaning of the y -intercept and the asymptote in the context of this problem.



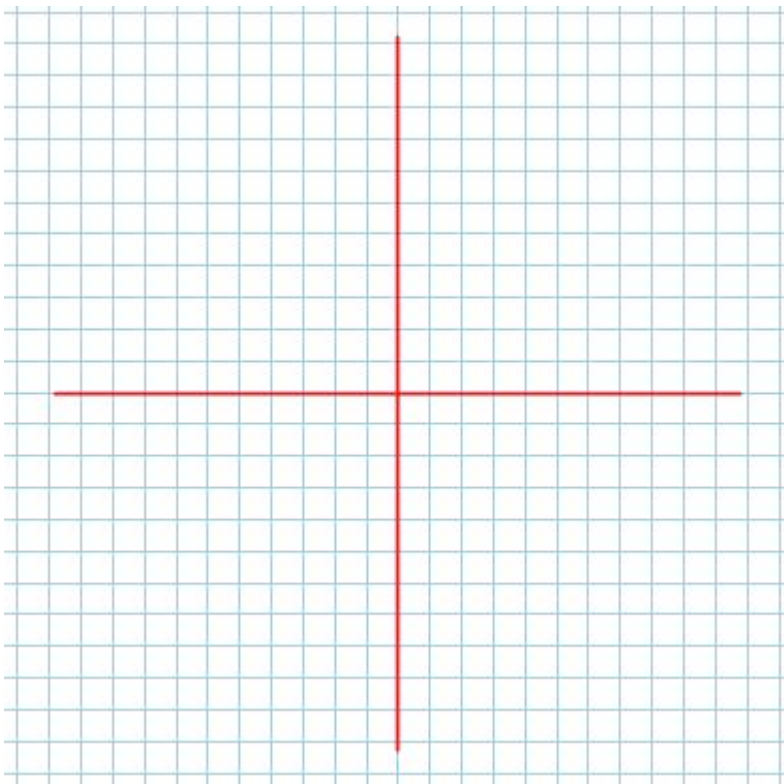
Example 4.6.3

From your Text: Page 252 #5a

Let $f(x) = 4^x$. For the function which follows,

- State the transformations applied to $f(x)$
- State the y-intercept, and the horizontal asymptote
- Sketch the transformed function, and write the function “properly”
- State the domain and range of the transformed function

$$g(x) = 0.5f(-x) + 2$$



HW- Section 4.5/4.6

Pg 251#2, 3, 5bcd(write each transformed function “properly), 8, 9, 10 (for #10 please see example 4 on page 250)

Success Criteria:

- I can identify the graph of an exponential function
- I can identify and apply the four transformations (a, k, d, c) to the equation of an exponential function

4.7 Applications of Exponential Functions

Learning Goal: We are learning to use exponential functions to solve problems involving exponential growth and decay.

Anything in the real world which grows, or decays can be “**MODELED**” (or in some sense “**DESCRIBED**”) with words, or pictures or **mathematics**. **Mathematical models** are useful for getting solutions to problems, and making predictions.

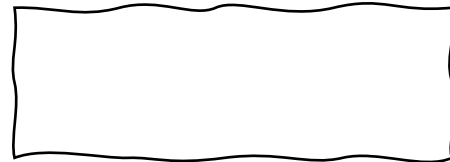
So far in Mathematics 11U we have studied the basics of functions in general (chapter 1), we’ve done some algebra (chapter 2), and we’ve examined Quadratic functions (chapter 3). Part of our study of Quadratics was learning how to use the vertex of a parabola to answer questions about maxima and minima for some real word problems.

We now know that Quadratic **MODELS** are very useful for solving max/min problems.

In this lesson we want to work on **LEARNING HOW TO SOLVE PROBLEMS DEALING WITH GROWTH AND DECAY**. We have to decide what type of function will best model (or describe) the type of growth/decay seen in the problem (hint: for this lesson we’ll be examining Exponential Growth and Decay, and therefore we expect that exponential functions will be used...shocking, I know :P)

Recollect:

The general equation for an exponential function is:



It is different than our other functions because we can have many different numbers for ____.

For example:

So essentially the exponential function includes many, many parent functions.

What the base of an exponential function tells you...

It tells you if the function is describing a _____ or a _____ phenomenon.

If b is _____, then it is a _____ function.

If b is _____, then it is a _____ function.

Note: The base is always _____ and never _____.

Definition 4.7.1

A function describing Exponential Growth is of the form:

$$A(x) = A_0(1 + R)^x$$

A₀ is the INITIAL AMOUNT.

x is # GROWTH PERIODS

GROWTH RATE
and 0 < R < 1

A function describing Exponential Decay is of the form:

$$A(x) = A_0(1 - R)^x$$

x is # DECAY PERIODS

R is the DECAY RATE

Example 4.7.2

From your text, Pg. 263

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.
- the percent of colour left if blue jeans lose 1% of their colour every time they are washed
 - the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for t years
 - the population of a colony if a single bacterium takes 1 day to divide into two; the population is P after t days

Example 4.7.3 From your text, Pg. 263

11. A population of yeast cells can double in as little as 1 h. Assume an initial population of 80 cells.
- What is the growth rate, in percent per hour, of this colony of yeast cells?
 - Write an equation that can be used to determine the population of cells at t hours.
 - Use your equation to determine the population after 6 h.
 - Use your equation to determine the population after 90 min.
 - Approximately how many hours would it take for the population to reach 1 million cells?
 - What are the domain and range for this situation?

Example 4.7.4

A new car depreciates at a rate of 20% per year. Steve bought a new car for \$26,000.

- Write the equation that models this scenario.
- How much will Steve's car be worth in 3 years?
- When will Steve's car be worth \$4000?

Example (Half-Life)

A 200g sample of radioactive material has a half-life of 138 days. How much will be left in 5 years?

HW- Section 4.7

Pg. 261 – 263 #1, 3 – 9, 11 – 16

Success Criteria:

- I can differentiate between exponential growth and exponential decay
- I can use the exponential function $f(x) = ab^x$ to model and solve problems involving exponential growth and decay
 - Growth rate is $b = 1 + r$. Decay rate is $b = 1 - r$.
 - r is a DECIMAL, not a percent!!!!