Functions 11

Course Notes

Unit 6 – Sinusoidal Functions

We will learn how to

- *identify situations that can be modelled by sinusoidal or periodic functions*
- *interpret the graphs of sinusoidal or periodic functions*
- *graph sinusoidal functions with transformations*
- *determine the equations of sinusoidal functions from real-world situations*

Chapter 6 – Trigonometric Functions   Name_____________________.

6.1 – Properties of Periodic Functions

Learning Goal: We are learning to interpret and describe graphs that repeat at regular intervals.

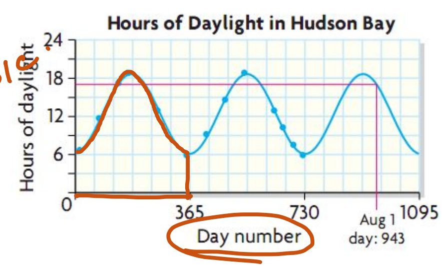
Definition 6.1.1

A **PERIODIC FUNCTION** is one in which ***the functional values repeat in a 'regular' way***. i.e. a function whose graph repeats at regular intervals.

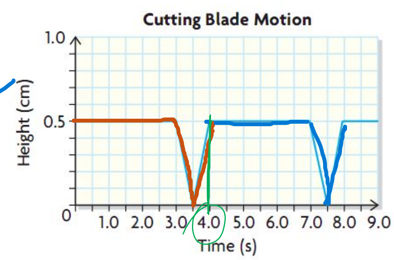
e.g. Consider the following pictures: Determine which are periodic.

Hint: Look for any repeating shape ☺

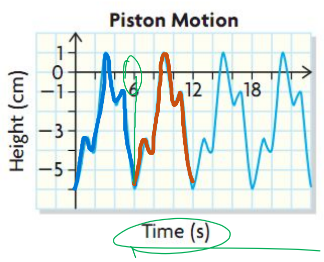
a) PERIODIC



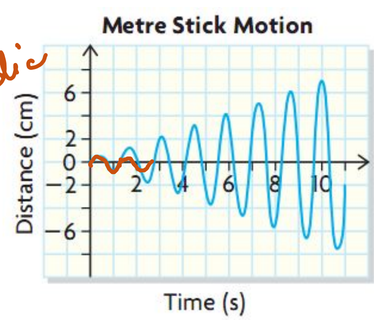
b) PERIODIC



c) PERIODIC



d) Not Periodic



You might be wondering….. why we are talking about Periodic Functions when the Unit is ***Sinusoidal Functions ?!!***.

Well, this is because **Sinusoidal Functions are a subset of Periodic Functions** i.e. Sinusoidal Functions are a group of some special Periodic Functions whose graph look like smooth symmetrical waves.

But why are we interested in Sinusoidal Functions? …. Well, that is because graphs of Sinusoidal Functions can be created by transforming the graph of the trigonometric functions $f(\theta) = \text{Sin}(\theta)$ and $f(\theta) = \text{Cos}(\theta)$.

The cool thing to note here is that the Sine and Cosine functions can be used as models to solve problems that involve many types of repetitive motions and trends and that's the reason we are interested in studying them!! ☺

Definition 6.1.2

The **Period** of a periodic function is the amount of the **domain values (x-values)** where **one cycle** takes place.

(*Note that a* ***cycle is the portion*** *of the periodic function graph* ***that repeats***.)

Example 6.1.1

Determine the periods of the graphs of periodic functions on the previous page:

① P = 365 days

② P = 4 sec.

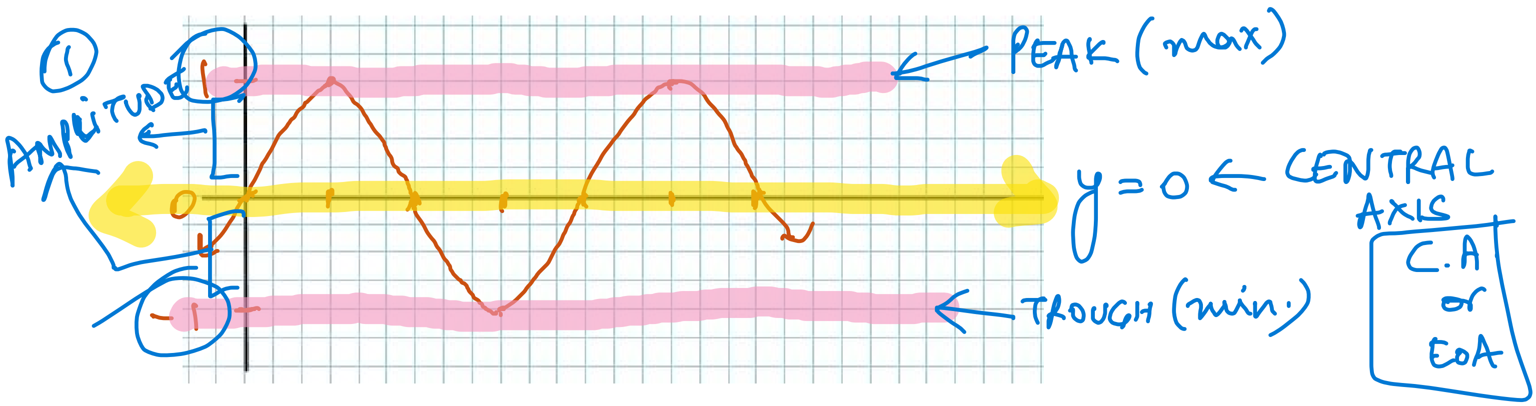
③ P = 6 sec

④ P → Not Periodic.

Definition 6.1.3

a) The **Peak** of a periodic function is the maximum point on the graph.

b) The **Trough** of a periodic function is the minimum point on the graph.



Definition 6.1.4

$f(x) = |x|$

a) The **Central Axis** or the **Equation of the Axis** is half way between the peak (maximum value) and the trough (minimum value).

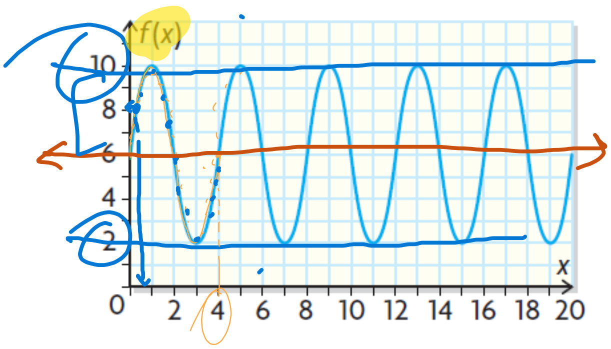
The equation of The Central Axis is given by: $\boxed{y = \frac{\max + \min}{2}}$.

b) The **Amplitude** of a periodic function is the **distance from the peak or trough to the central axis**. So, it is half of the distance between a peak (maximum value) and a trough (minimum value). Amplitude being distance is always POSITIVE! ☺

$|a| = \text{Amplitude} = \boxed{\frac{\max - \min}{2}}$

$a = \frac{1 - -1}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$

Example 6.1.1
Determine the range, period, equation of the axis, and amplitude of the function shown.



RANGE = $\{f(x) \in \mathbb{R} \mid 2 \le f(x) \le 10\}$

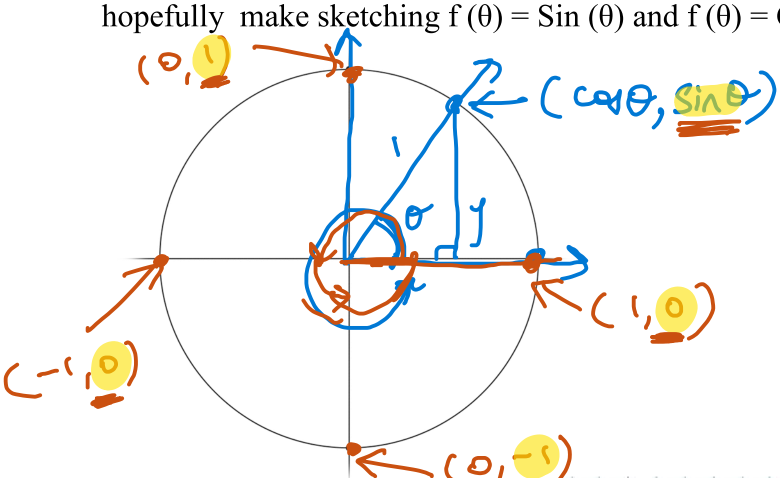
PERIOD = 4

EoA: $y = 6$

$|a| = 4$

So, What about the so-called Sinusoidal Functions? What do the graphs of functions f (θ) = Sin (θ) and f (θ) = Cos (θ) look like?

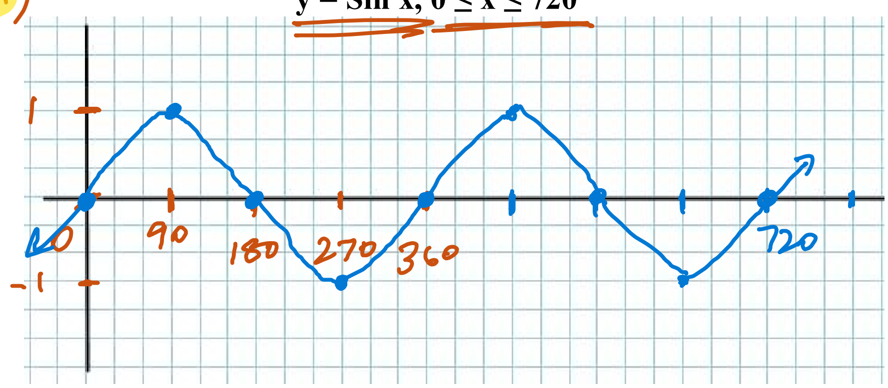
Let's try and sketch them …..but first consider our dear friend the unit circle which may hopefully make sketching f (θ) = Sin (θ) and f (θ) = Cos (θ) easy! ☺



x	y=Sin x
0°	0
90	1
180°	0
270°	−1
360	0

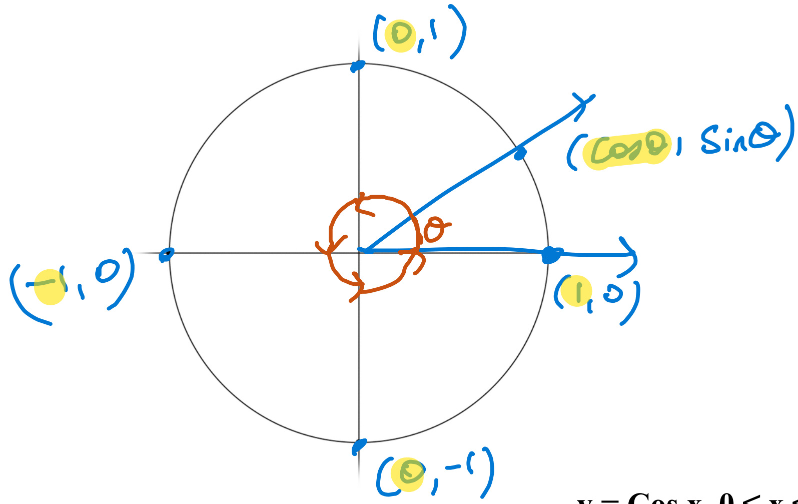
Sin0
Sin90

y = Sin x, 0 ≤ x ≤ 720



Notice the Axis Angles 0°, 90°, 180°, 270°, 360° in the unit circle above to make the table of values.

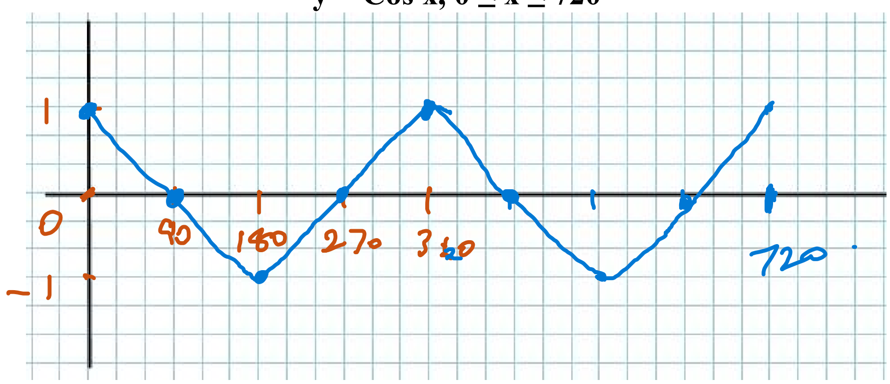
Do you think considering these special axis angles could make sketching the Sine and Cosine graphs easier?



x	y=Cos x
0	1
90	0
180°	-1
270	0
360	1

Sine
Cos

y = Cos x, 0 ≤ x ≤ 720



So,

1. The graph of f (θ) = Sin (θ) has these characteristics:
   - The period is 360°.
   - The amplitude is 1, the max value is 1, and the min value is -1.
   - The domain is $\{\theta \in R\}$, and the range is $\{f(\theta) \in R \mid -1 \leq f(\theta) \leq 1\}$.
   - The zeros are located at 0°, 180°, 360°, ...
2. The graph of f (θ) = Cos (θ) has these characteristics:
   - The period is 360°.
   - The amplitude is 1, the max value is 1, and the min value is -1.
   - The domain is $\{\theta \in R\}$, and the range is $\{f(\theta) \in R \mid -1 \leq f(\theta) \leq 1\}$.
   - The zeros are located at 90°, 270°, ...

Example 6.1.2

3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.

a) What is the period of one complete cycle?

b) What is the maximum distance between the device and the appliance?

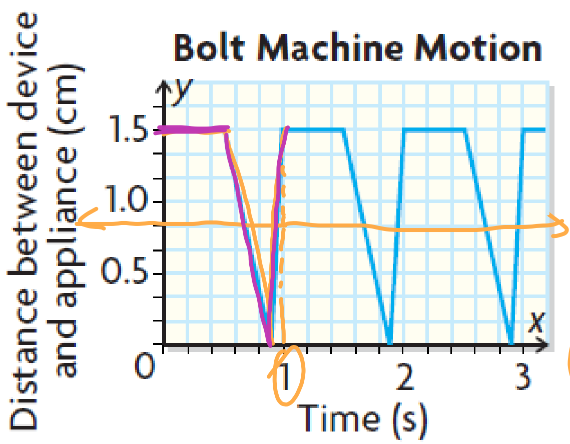
c) What is the range of this function?

d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.

e) Determine the equation of the axis. CA.

f) Determine the amplitude.

g) There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of "attaching the bolt."



(a) $P = 1$ sec.

(b) max dist $= 1.5$ cm

(c) $R = \{f(x) \in \mathbb{R} \mid 0 \le f(x) \le 1.5\}$

(d) $D = \{x \in \mathbb{R} \mid 0 \le x \le 5\}$

(e) EOA: $y = \frac{0+1.5}{2} = 0.75$

$\boxed{y = 0.75}$

(f) $a = \frac{1.5-0}{2} = 0.75$

g (i) H. piece represents the (slope $= 0$)
Bolt machine's waiting time for the appliance

(ii) piece with negative slope represents the bolt machine getting closer to the appliance

(iii) piece with positive slope represents the bolt machine returning to its original location after bolting the appliance

HW - Section 6.1

Pg. 352 – 355 #4, 5, 7 – 10

Success Criteria:

- I can find the range, period, central axis, and amplitude of a periodic function
- I can determine IF a function is periodic

6.5 – Sketching Sinusoidal Functions

Learning Goal: We are learning to sketch the graphs of sinusoidal functions using transformations.

By this point in your illustrious High School careers, you have a solid understanding of Transformations of Functions in general. In terms of the sinusoidal functions Sine and Cosine in particular, the concepts are as you expect

$g(x) = a f(k(x-d)) + c$ $\quad$ $f(x)$ $(x, y) \to \left(\frac{1}{k}x + d, ay + c\right)$ $g(x)$

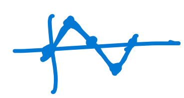
i.e. the graphs of the functions $\boxed{g(x) = a \sin(k(x-d)) + c}$ and $\boxed{g(x) = a \cos(k(x-d)) + c}$ are obviously periodic in the same way that the graphs of f(x) = Sin x and f(x) = Cos x are, and the differences are only in the placement of the graph(ie the shifts) and how stretched it is.

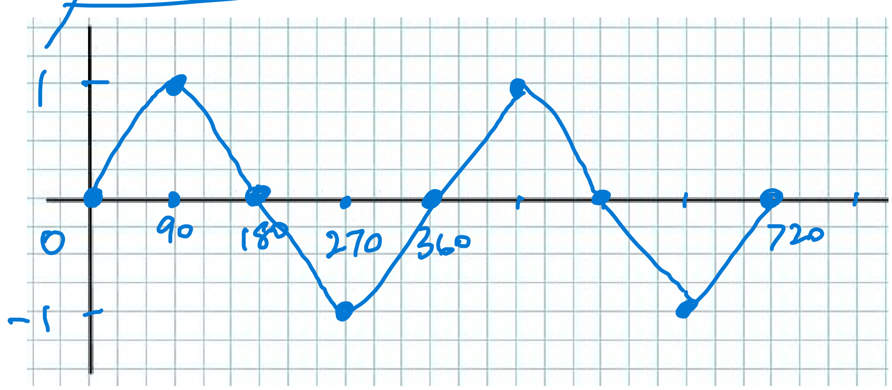
However, what is new is that **the transformations have specific meanings relating to the nature of the sinusoidal "wave".**

But first,

Using the special angles of 0º, 90º, 180º, 270º, and 360º, let's graph the following functions again:

$\boxed{f(\theta) = \sin(\theta)}$ $\quad 0^\circ \le \theta \le 720^\circ$





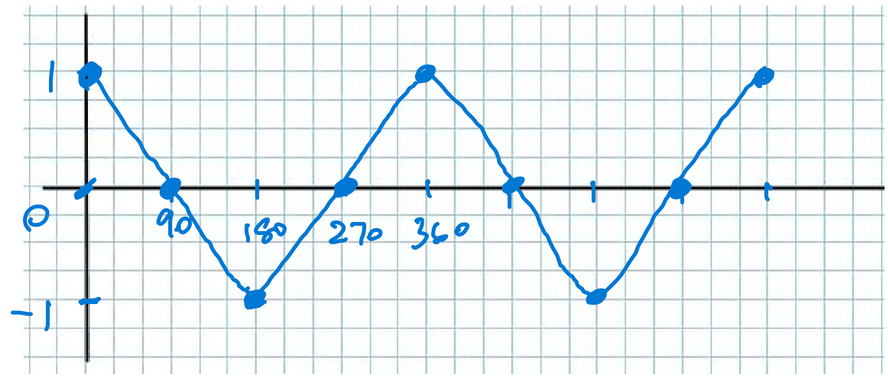
So,

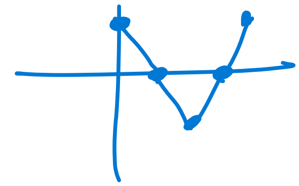
- Key points for f(x) = Sin x

(0º, 0) , (90º, 1) , (180º, 0) , (270º, -1) , (360º, 0)

Note that from 0 to 720, we see 2 periods.

$g(\theta)=\cos(\theta) \qquad 0^\circ \le \theta \le 720^\circ$





So,

- Key points for f(x) = Cos x

(0º, 1) , (90º, 0) , (180º, -1) , (270º, 0) , (360º, 1)

Note that from 0 to 720, we see 2 periods in the base Cosine graph too!

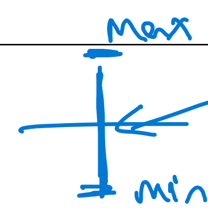
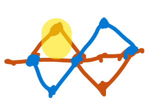
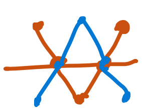
It's finally time to transform the above graphs!! 😊

NOTE that to graph the transformed function g(x), you need to apply the transformations to only the key points of f(x) = Sin x and f(x) = Cos x and NOT to every point on f(x).

The key points are the points corresponding to the special angles.

General Form of the Sine and Cosine Functions

$$f(\theta)=a\sin\left(k(\theta-d)\right)+c \qquad g(\theta)=a\cos\left(k(\theta-d)\right)+c$$

Transformation	**Properties**
a = V. Stretch a>0 No Flip a<0 V. Flip.	$\boxed{a=\frac{\max-\min}{2}}$  max − min / 2
 Pattern of Sin: CA-max-CA-min-CA If a < 0, pattern: CA - min - CA - max - CA	|a| is the AMPLITUDE of the graph ← a is Positive
 Pattern of Cos: max-CA-min-CA-max If a < 0, pattern: Min - CA - Max - CA - min.	*Remember that Amplitude is the distance from the peak (or trough) to the central axis.*

$k =$ H. Stretch by $\frac{1}{k}$ * k must always be factored out	Period Factor $= \boxed{\dfrac{360^\circ}{\lvert k \rvert}}$ *Remember that Period is the set of all the x-values in one cycle of the repeating graph*
$d =$ H shift left/right	Note: To determine d you MUST first FACTOR k from the θ eg $f(x) = 5\sin\left(\frac{2x}{2} + \frac{90}{2}\right) + 5$ $= 5\sin(2(x+45)) + 5$ $d = -45^\circ$ *The Horizontal translation(shift) of a Sinusoidal Function is called a* ***PHASE SHIFT***
$c =$ V. shift up/down	$\boxed{c = \dfrac{\max + \min}{2}}$ → CENTRAL Axis *Note that x-axis is the central axis of our base Sine and Cosine function graphs and the vertical shift of the base graph is the same as the vertical shift of the central axis.* So, Equation of Central Axis : $y = c$

Example 6.5.1

Determine the amplitude, period, phase shift and the equation of the central axis for:

a) $f(\theta) = 2\sin(\theta + 60^\circ) + 1$

$= 2\sin(1(\theta + 60)) + 1$

AMPLITUDE $= |a| = 2$ $\quad k = 1$

PERIOD $= \frac{360}{k} = 360$

PHASE SHIFT $= 60^\circ$ to the left
(d)

EoA : $y = 1$

b) $g(\theta) = 3\cos(2\theta - 90^\circ)$

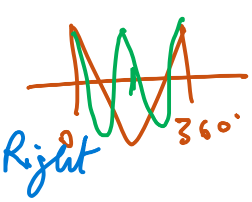
$= 3\cos(2(\theta - 45)) + 0$

AMPLITUDE $= |a| = 3$

PERIOD $= \frac{360}{2} = 180^\circ$

PHASE SHIFT

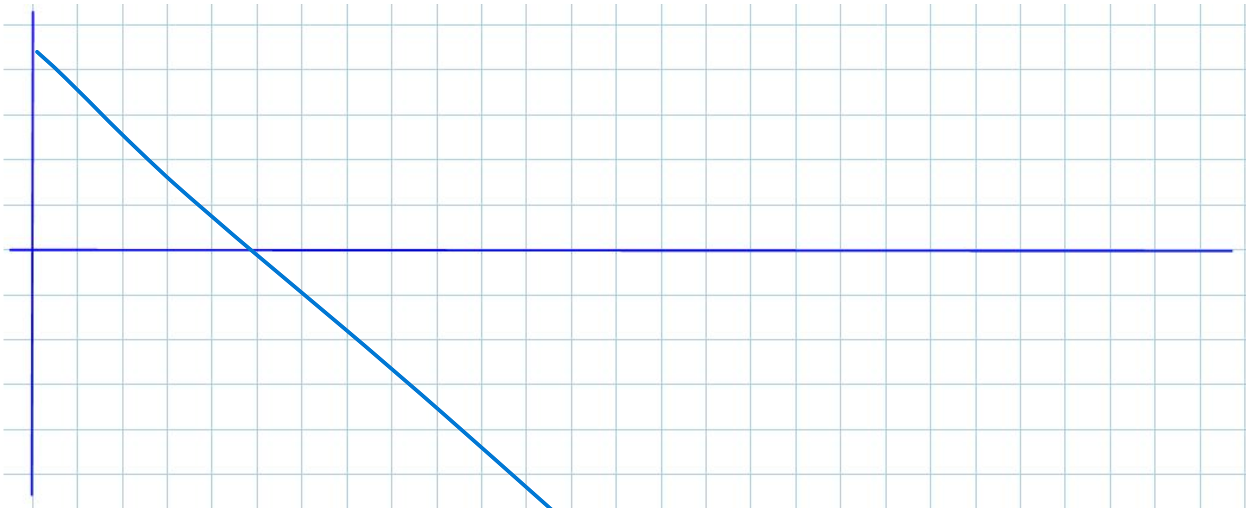
$= 45^\circ$ to the Right



EoA : $y = 0$

Example 6.5.2

Sketch $f(x) = \sin(x)$ and $g(x) = \sin(2x)$ for $0^\circ \le x \le 360^\circ$ on the same set of axes.



Analyzing g (x) :

Now…what would $\sin\left(\frac{1}{2}x\right)$ look like?

Notes about Domain and Range: Consider the function $f(x) = -2\cos\left(3x + 90^\circ\right) + 3$.

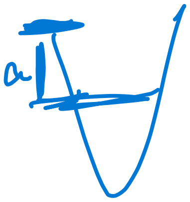
Determine all the transformations for this function. Without graphing, determine the range of the function. Determine the domain of the function for: 1 cycle; 2 cycles; 3 cycles.

Example 6.5.3

Sketch $f(\theta) = -2\cos(\theta - 60^\circ) + 1$ on $0^\circ \le \theta \le 360^\circ$. State transformations, create tables, and state domain and range of the function.

$a = -2$ V. stretch by 2
V. flip

AMPLITUDE = 2 $2 = a$



$k = 1$ No H. stretch

PERIOD $= \frac{360}{k} = 360^\circ$

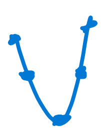
$d = 60^\circ$ H. shift 60° right

you can think of d as the starting point.

PHASE SHIFT $= 60^\circ$ right

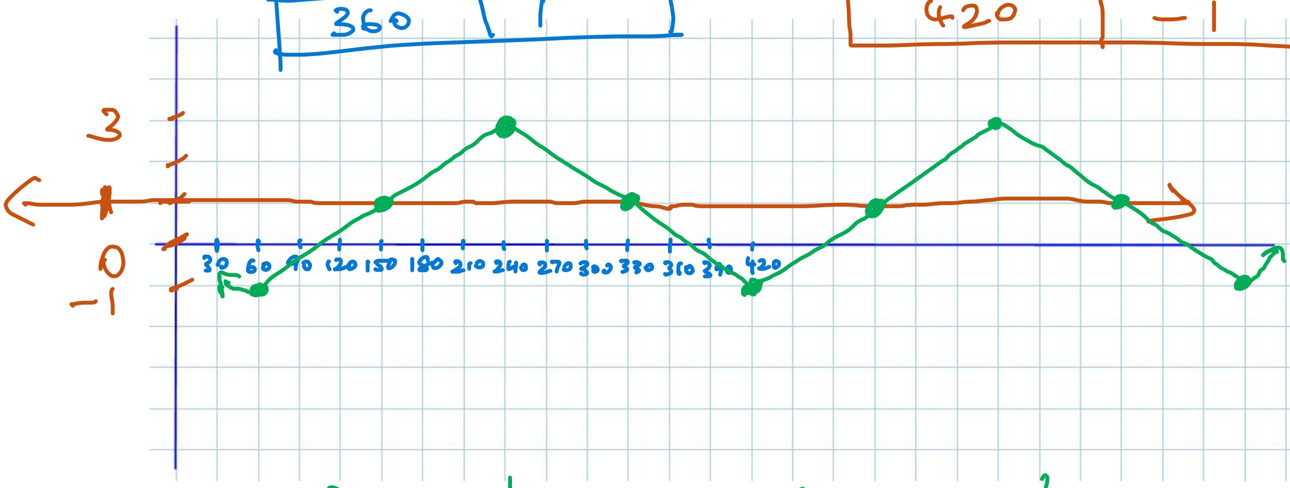
$c = 1$ V. shift 1 unit up

EoA: $y = 1$



x	$\cos x$ (y)
0	1
90	0
180	-1
270	0
360	1

$x + 60$	$-2y + 1$
60	-1
150	1
240	3
330	1
420°	-1



$D_{2\text{cycles}} = \{\theta \in R \mid 0 \le \theta \le \cancel{2(360)} = 720\}$

$0 \le \theta \le 720$

$R = \{f(\theta) \in R \mid -1 \le f(\theta) \le 3\}$

Example 6.5.4

Sketch $f(\theta) = 3\sin(2\theta - 90) - 1$. State transformations, create tables, and state domain and range of the function.

~~**HW Section 6.5**~~

~~Pg. 383 – 3385 #1, 2, 4 – 7, 9~~ HANDOUT

Success Criteria

- I can sketch the graph of a sinusoidal function by applying the transformations to the parent function.

6.6 – Models of Sinusoidal Functions

Learning Goal: We are learning to create a sinusoidal function from a graph or table of values.

In this section we will look at how to develop a sinusoidal function which can explain given information. In essence we will be writing sine or cosine functions based on given transformations.

Just as a reminder:

General Form of the Sine and Cosine Functions

$f(\theta) = a\sin(k(\theta - d)) + c$ $g(\theta) = a\cos(k(\theta - d)) + c$

Another reminder (about the pattern of sinusoidal functions):

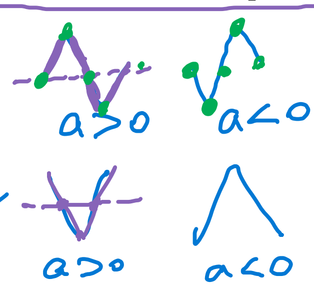
Sine functions "start" at the central axis and go up to a **max** if "*a*" is **positive**, or down to a **min** if "*a*" is **negative**.

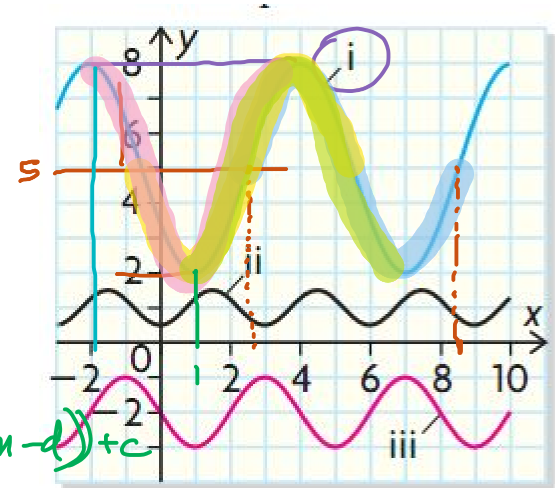
Cosine functions "start" at a **max** if "*a*" is **positive**, or at a **min** if "*a*" is **negative**.

Example 6.6.1

From your text: Pg. 391 #4a

Determine a sinusoidal equation for each function:





(i) $f(x) = a\sin(k(x-d))+c$ $f(x) = a\cos(k(x-d))+c$

max = 8
min = 2

$a = 3$

$k = 6$

$P = \frac{360}{k} \Leftrightarrow k = \frac{360}{P}$

$d = 2.5$; + Sine $(-0.5, -\text{Sine})$ $(-2, +\text{Cos})$ $(+1, -\text{Cos})$

$c = 5$

$f(x) = 3\sin(6(x-2.5))+5$ $f(x) = -3\sin(6(x+0.5))+5$ $f(x) = 3\cos(6(x+2))+5$ $f(x) = -3\cos(6(x-1))+5$

Example 6.6.2

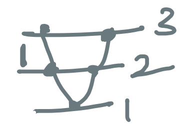
From your text: Pg. 392 #5a)

5. For each table of data, determine the equation of the function that is the simplest model.

$d = 0$

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1



$a = 1$

$d = 0, +\cos$

$k = \frac{360}{p} = \frac{360}{120} = 3$

$c = 2$

$f(x) = 1\cos(3(x-0)) + 2$

$f(x) = \cos(3x) + 2$

Example 6.6.3

From your text: Pg. 392 #6b)

6. Determine the equation of the cosine function whose graph has each of the following features.

K

	Amplitude (a)	Period	Equation of the Axis (c)	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	30°

a) $a = 3$, $p = 360 \Rightarrow k = \frac{360}{360} = 1$, $c = 11$, $d = 0$

$f(x) = 3\cos(1(x-0)) + 11 = 3\cos x + 11$

b) $a = 4$, $p = 180 \Rightarrow k = \frac{360}{180} = 2$, $c = 15$, $d = 30$

$f(x) = 4\cos(2(x-30)) + 15$

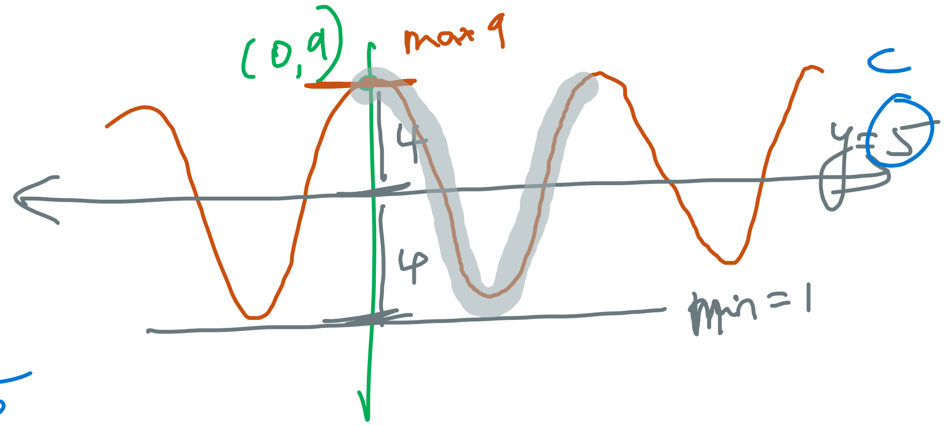
$= 4\cos(2x - 60) + 15$

Example 6.6.4
A sinusoidal function has an amplitude of 4 units, a period of 120º, and a maximum at (0,9). Determine the equation of the function.

$P = 120$

$k = \dfrac{360}{120} = 3$

$d = 0$ + Cos.



$f(x) = 4\cos(3x) + 5$

HW Section 6.6
Pg. 391 – 393 #4b, 5bcd, 6acd, 7, 11

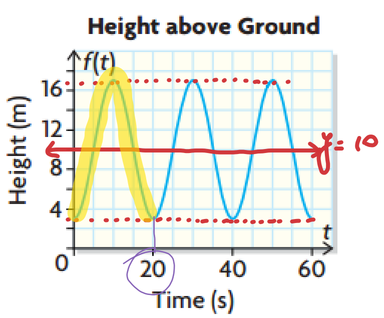
Success Criteria:
- I can create an sinusoidal function based on information from a graph or table
- I can recognize when it is best to use a sine or cosine function

6.7 – Problem Solving with Sinusoidal Functions

Learning Goal: We are learning to solve problems related to real-world applications of sinusoidal functions.

We can **use the sinusoidal properties** of **Period**, **Central Axis**, **Amplitude** and **Phase Shift** to describe and solve "real world" problems.

Example 6.7.1 (*From the text: Pg. 398 #2*)



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

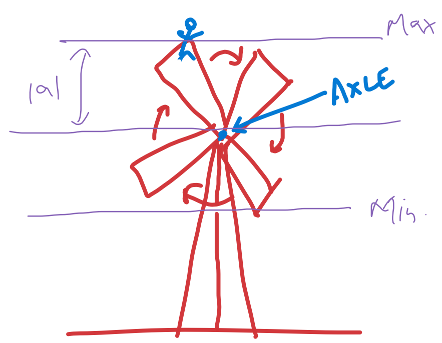
b) What is the amplitude of the function, and what does it represent in this situation?

c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?



a) max = 17 min = 3

EoA : $y = \frac{17+3}{2} = \frac{20}{2} = 10 = c$

The EoA in this situation represents the height of Axle

b) $|a| = \frac{17-3}{2} = \frac{14}{2} = 7$

The amplitude represents the length of the blades

c) Period = 20 sec.

It is one cycle i.e. the time taken by Quixote to cover one rotation around the axle

d) $D_{1\,cycle} = \{t \in \mathbb{R} \mid 0 \le t \le 20\}$ $R = \{f(t) \in \mathbb{R} \mid 3 \le f(t) \le 17\}$

$D_{7\,cycles} = \{t \in \mathbb{R} \mid 0 \le t \le 140\}$

e) $d = 0$ (−ve Cos.)

$a = 7$

$c = 10$

$k = \frac{360}{p} = \frac{360}{20} = 18$

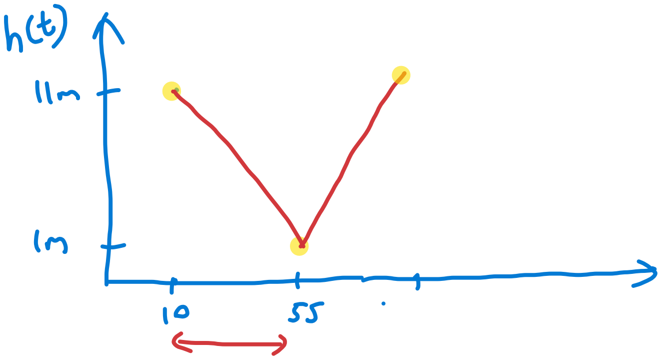
$f(t) = -7\cos(18(t-0)) + 10$

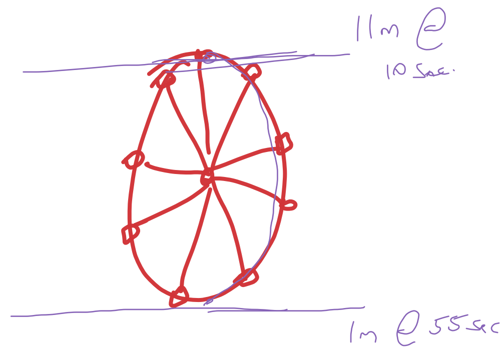
$\boxed{f(t) = -7\cos 18t + 10}$

f) If the wind speed decreased, the Period will increase.

Example 6.7.2

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches a maximum height of 11m at 10s and then reaches a minimum height of 1m at 55s. How high is John after 2 minutes?





Period of ½ cycle = 55 − 10 = 45 sec.

∴ Period of 1 cycle = 45 × 2 = 90 sec.

max = 11 min = 1

$|a| = \frac{11-1}{2} = \frac{10}{2} = 5m.$

$c = \frac{11+1}{2} = \frac{12}{2} = 6m.$

$k = \frac{360}{90} = 4$

$d = 10$ (+ Cos)

$\therefore h(t) = 5\cos(4(t-10)) + 6$

$\boxed{h(t) = 5\cos(4t-40) + 6}$

2 min = 2×60
= 120 sec.

$\therefore h(120)$

$= 5\cos(4(120)-40) + 6$

$= 5\cos(440) + 6$

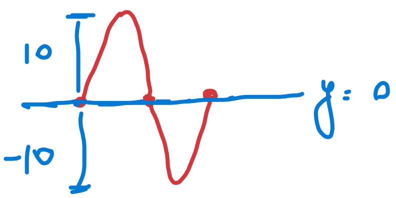
$\approx 6.87m$

∴ John is 6.87m high approx after 2 min.

Example 6.7.3 (Text pg. 396)

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right (+10 cm) and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At t = 0, the pole was momentarily at its resting position. Then it started moving to the right.

a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.



$|a| = 10$

$c = 0$

$d = 0$ +Sine

$k = \frac{360}{p} = \frac{360}{0.25} = 1440$

$\therefore f(t) = 10\sin(1440(t-0)) + 0$

$f(t) = 10\sin(1440t)$

1 min = 240 sways.

$\therefore$ 1 sway $= \frac{1\text{ min}}{240}$

$= \frac{60\text{ sec.}}{240}$

$\boxed{p = 0.25\text{ sec.}}$

b) How does the situation affect the domain and range?

The domain here represents time which cannot be negative and the range must be between −10 and 10.

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Wind speed changes the amplitude

$\therefore$ New $|a| = 80\% \times 10 = \frac{80}{100} \times 10 = 8$

$\therefore$ New equation $\Rightarrow f(t) = 8\sin(1440t)$

HW Section 6.7

Pg. 398 – 401 #4 – 6, 8, 10 (*a question of* ***beauty***)

Success Criteria:

- I can create a sinusoidal function that represents information from a real-life scenario
- I can use the function to solve further problems