# Functions 11

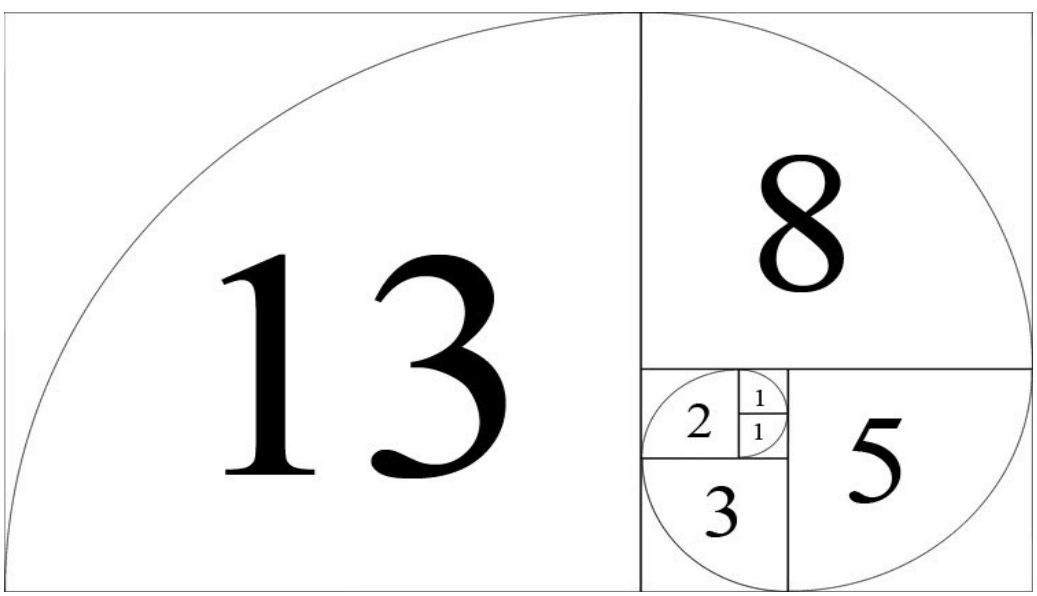
*Course Notes*

# Unit 7 – Sequences and Series

***AS EASY AS 1,1,2,3…***

*We will learn*

- *about the nature of a sequence and how to represent a sequence using recursive and general formulas*
- *about the Fibonacci Sequence and/or Pascal's Triangle*
- *about Arithmetic and Geometric Sequences and Series and how to use them in problem solving*

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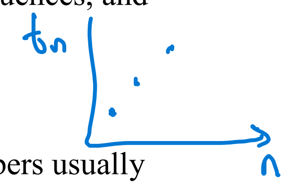
## 7.1 – Arithmetic Sequences

**Learning Goal:** We are learning to recognize the characteristics of arithmetic sequences, and express the general terms in a variety of ways.



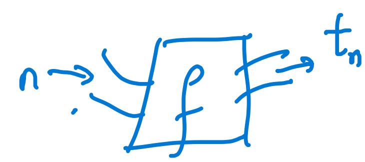

### Definition 7.1.1

A **mathematical sequence** is an ordered list of numbers, i.e. a list of numbers usually with some kind of order.

Each number is called a "term" of the sequence. *(Subscripts are used to identify the positions of the terms)*

e.g. 2, 4, 6, 8, 10, . . . . . .

Positions ($n$) $t_1$ $t_2$ $t_3$ $t_4$ $t_5$ . . . .




> Note: **Every Sequence is a Discrete Function**. Since each term is identified by its position in the list (1st, 2nd, and so on..); the domain is **N**={1,2,3,….} and the range is the set of all the terms of the sequence.

### Definition 7.1.2

An **Arithmetic Sequence** is a sequence where each term differs from the previous term by a common difference *d*.

e.g. 1, 2, 3, 4, 5, 6, . . . . . . ✓ ARITHMETIC

+1 +1 +1

+2 +4 +2

6, 8, 12, 15, . . . . . ✗ NOT ARITHMETIC

$t_1 \xrightarrow{+d} t_2 \xrightarrow{+d} t_3 \xrightarrow{+d} t_4 \xrightarrow{+d} t_5 \xrightarrow{+d} t_6$ . . . . . . . . $t_n$

$a$ $\quad a+d \quad a+2d \quad a+3d$ . . . . $\boxed{a+(n-1)d}$

> Note: **An arithmetic sequence is a recursive sequence** in which new terms are created by adding the same value (the common difference) each time. *A recursive sequence is a sequence for which one term(or more) is given and each successive term is determined from the previous term(s).*

$$\boxed{t_n = a + (n-1)d}$$

$a \rightarrow 1^{st}$ term
$d \rightarrow$ common difference

**Definition 7.1.3**

The general **term** of an arithmetic sequence, usually labelled $t_n$, is given by a formula. The subscript $n$ gives the position of the term in the sequence, with one exception: the first term of a sequence, $t_1$, is called a

e.g. In the sequence $4, \overset{-7}{-3}, \overset{-7}{-10}, \overset{-7}{-17}, -24, -31, \ldots$

$a = 4$

$t_2 = -3$

$t_5 = -24$

$d = -7$

## The General Term of an Arithmetic Sequence

The formula can be arrived at using some simple logic. Consider some arithmetic sequence with first term $a$ and common difference $d$. Then the sequence can be written:

| $t_1$ | $t_2$ | $t_3$ | . . . . | $t_n$ |
|---|---|---|---|---|
| $a$ | $a+d$ | $a+2d$ | . . . . | $\boxed{a+(n-1)d}$ |

**Example 7.1.1**

From your text: Pg. 424 # 1

Determine which sequences are arithmetic. For those that are, state the common difference.

a) 1, 5, 9, 13, 17, ...    (+4 +4 +4 +4)

b) 3, 7, 13, 17, 23, 27, ...    (+4 +6 +4)

c) 3, 6, 12, 24, ...    (+2 +6 +12)

d) 59, 48, 37, 26, 15, ...    (−11 −11 −11)

a) Arithmetic with $d = 4$

b) Not Arithmetic

c) Not Arithmetic

d) Arithmetic with $d = -11$

GENERAL FORMULA $t_n$

$t_n = a + (n-1)d$

RECURSIVE FORMULA $t_n$

$t_1 = a$

$t_n = t_{n-1} + d$

**Example 7.1.2**

From your text: Pg 424 #6

Determine the recursive formula and the general term for the arithmetic sequence in which

**a)** the first term is 19 and consecutive terms increase by 8

**b)** $t_1 = 4$ and consecutive terms decrease by 5

**c)** the first term is 21 and the second term is 26 $a = t_1 = 21$, $t_2 = 26$, $d = 26 - 21 = 5$

**d)** $t_4 = 35$ and consecutive terms decrease by 12

*(A recursive formula is a formula that defines each term of a sequence using preceding terms)*

So, a Recursive Formula requires two things:

1. A starting point
2. A way to get from one term to the next over and over and over again

a) $a = t_1 = 19$ ; $d = 8$

GENERAL $t_n = 19 + (n-1)8 = 19 + 8n - 8 = 11 + 8n$ LINEAR

RECURSIVE: $t_1 = 19$

$t_n = t_{n-1} + 8$

b) $t_1 = a = 4$, $d = -5$

GENERAL $t_n = 4 + (n-1)-5 = 4 - 5n + 5 = 9 - 5n$

RECURSIVE: $t_1 = 4$

$t_n = t_{n-1} - 5$

d) $t_4 = 35$

$d = -12$

$t_4 = a + 3d$

$t_4 = a + 3(-12)$

$t_4 = a - 36$

$35 + 36 = a$

$\boxed{71 = a}$

**Example 7.1.3**

From your text: Pg. 425 #9a

**i)** Determine whether each general term defines an arithmetic sequence.

**ii)** If the sequence is arithmetic, state the first five terms and the common difference.

a) $\boxed{t_n = 8 - 2n}$

$t_1 = 8 - 2(1) = 6$ $-2 = d$

$t_2 = 8 - 2(2) = 4$ $-2$

$t_3 = 8 - 2(3) = 2$ $-2$

$t_4 = 8 - 2(4) = 0$

$t_5 = -2$

Yes, it is ARITHMETIC with $d = -2$

> ***In Summary,***
>
> An arithmetic sequence can be defined
>
> 1. By the general term $t_n = a + (n - 1)\,d$
> 2. Recursively by $t_1 = a$ , $t_n = t_{n-1} + d$
> 3. By a discrete linear function $f(n) = dn + b$

**Example 7.1.4**

From your text: Pg 424 #13b

Determine the number of terms in the arithmetic sequence

$$-20, -25, -30, \ldots, -205$$

$-5 = d$

$a = t_1 \quad t_2 \quad t_3 \qquad t_n$

$a = -20,\ d = -5,\ t_n = -205$

$t_n = a + (n-1)d$

$-205 = -20 + (n-1)(-5)$

$-205 = -20 - 5n + 5$

$-205 + 20 - 5 = -5n$

$\frac{-190}{-5} = n \Rightarrow \boxed{n = 38}$

$\therefore$ The A. Sequence has 38 terms

**Example 7.1.5**

Given an arithmetic sequence with $t_7 = 25$ and $t_{20} = 77$ determine the general term for the sequence, and also determine $t_{150}$. $= a + 149d$

$\boxed{t_n = a + (n-1)d}$

$t_7 = a + 6d = 25$

$t_{20} = -a + 19d = -77$

$-13d = -52$

$d = \frac{-52}{-13} = 4$

$\boxed{d = 4}$

$a + 6d = 25$

$a + 6(4) = 25$

$a + 24 = 25$

$a = 25 - 24 = 1$

$\boxed{a = 1}$

$t_n = 1 + (n-1)4$

$= 1 + 4n - 4$

$\boxed{t_n = 4n - 3}$

$t_{150}$

$= 4(150) - 3$

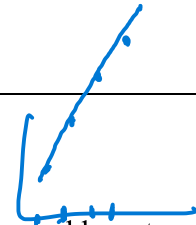
$= 597$

## HW Section 7.1:

Pg. 424 – 425 #2 – 5 (general terms only), 8i) iii), 9bcd, 10, 11, 13, 15

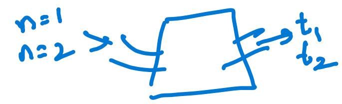
**Success Criteria:**

- I can identify when a sequence is arithmetic, by seeing if it has a common difference
- I can use the General Term Formula to develop an equation for an arithmetic sequence
- I can use the General Term to find any term in a sequence OR to find out how many terms are in a sequence
- I can recognize that an arithmetic sequence is always a linear function



## 7.2 – Geometric Sequences

**Learning Goal:** We are learning the characteristics of geometric sequences and how to express the general terms in a variety of ways.




In the last lesson we considered sequences of the form:

$$\overset{+4\ +4\ +4 \leftarrow d}{3, 7, 11, 15, 19, 23, \ldots}$$

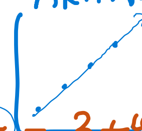
$a = 3$; $t_1, t_2, t_3, t_4$

ARITHMETIC SEQUENCE

This sequence is arithmetic because there is a common **difference** between successive terms. We can write the general term of the above sequence because we know the first term $(a = 3)$, and the common difference $(d = 4)$.

$$\boxed{t_n = a + (n-1)d}$$

ARITHMETIC



$t_n = 3 + (n-1)4 = 3 + 4n - 4 = 4n - 1$

Note that if we "simplify" the general term, we can actually consider that simplification as a **function** of $n$!! DISCRETE FUNCTION.

i.e. $\boxed{f(n) = 4n - 1}$

Consider the following sequence:

$$\overset{\times 2\ \times 2\ \times 2\ \times 2}{3, 6, 12, 24, 48, \ldots}$$

GEOMETRIC SEQUENCE

COMMON RATIO: $r = \dfrac{t_n}{t_{n-1}}$

$\dfrac{6}{3} = \dfrac{12}{6} = \dfrac{24}{12} = 2 \leftarrow r$

This sequence is not arithmetic, but there is a discernible pattern as we look at moving from one term to the **next** term. In this case, we see that each new term is generated by multiplying the previous term by 2.

Such a sequence is called a **Geometric Sequence**. There isn't a common difference between two successive terms, but there is a **common ratio** $(r)$ between two successive terms.

### Comparing Two Successive Terms

$a,\ a+d,\ a+2d, \ldots$ ($+d$, $+d$, $+d$); $t_1, t_2, t_3$

$a,\ ar^1,\ ar^2,\ ar^3$ ($\times r$, $\times r$, $\times r$); $t_1, t_2, t_3, t_4$

| | **Arithmetic** | **Geometric** |
|---|---|---|
| | $\boxed{t_n - t_{n-1} = d}$ | $\boxed{\dfrac{t_n}{t_{n-1}} = r}$ |
| GENERAL TERM | $t_n = a + (n-1)d$ | $t_n = ar^{n-1}$ |
| RECURSIVE | (i) $t_1 = a$<br>(ii) $t_n = t_{n-1} + d$ | (i) $t_1 = a$<br>(ii) $t_n = r\,t_{n-1}$ |

$t_n = ar^{n-1}$

# The General Term of a Geometric Sequence

Again, using simple logic will allow us to arrive at a formula (or even a function depending on how you interpret things). Consider some geometric sequence with first term $a$ and common ratio $r$. The sequence can be written:

| $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n$ |
|---|---|---|---|---|---|
| $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_n$ |
| $a$ | $ar$ | $ar^2$ | $ar^3$ | $ar^4$ | $ar^{n-1}$ |

$$\boxed{t_n = ar^{n-1}}$$

**Example 7.2.1**

From your text: Pg. 430 #1

Determine which sequences are geometric. For those that are, state the common ratio.

a) 15, 26, 37, 48, ... ($+11$, $+11$)

b) 5, 15, 45, 135, ... ($\times 3$, $\times 3$, $\times 3$)

c) 3, 9, 81, 6561, ... ($\times 3$, $\times 9$, $\times 81$)

d) 6000, 3000, 1500, 750, 375, ... ($\times \frac{1}{2}$, $\times \frac{1}{2}$, $\times \frac{1}{2}$)

a) Not Geometric but Arithmetic with $d = 11$

b) Geometric with $r = 3$

c) Not Geometric

d) Geometric with $r = \frac{1}{2} = 0.5$

**Example 7.2.2**

Determine the general term ($t_n$) and $t_{10}$ of the geometric sequence

$a = 81$; 81, 27, 9, 3, ...

$$r = \frac{27}{81} = \frac{3}{9} = \frac{1}{3}$$

$$t_n = ar^{n-1}$$

$$\boxed{t_n = 81\left(\frac{1}{3}\right)^{n-1}}$$

$$t_{10} = 81\left(\frac{1}{3}\right)^9$$

$$t_{10} = \frac{81}{19683} = \frac{1}{243}$$

**Example 7.2.3**

From your text: Pg. 430 #8

Determine the recursive formula and the general term for the geometric sequence in which

a) the first term is 19 and the common ratio is 5

b) $t_1 = -9$ and $r = -4$

c) the first term is 144 and the second term is 36 $\quad a = 144,\ r = \frac{36}{144} = \frac{1}{4}$

d) $t_1 = 900$ and $r = \frac{1}{6}$

a

a) $a = 19,\ r = 5$

GENERAL: $t_n = 19(5)^{n-1}$

RECURSIVE: (i) $t_1 = 19$
(ii) $t_n = 5(t_{n-1})$

b) $t_1 = a = -9,\ r = -4$

GENERAL: $t_n = (-9)(-4)^{n-1}$

RECURSIVE: (i) $t_1 = -9$
(ii) $t_n = -4\, t_{n-1}$

**Example 7.2.4**

Given a geometric sequence with $t_6 = -486$ and $t_9 = 13122$, determine the general term and the first 4 terms of the sequence.

$t_6 = ar^5 = -486$

$t_9 = ar^8 = 13122$

$\frac{t_9}{t_6} = \frac{\cancel{a}r^8}{\cancel{a}r^5} = \frac{13122}{-486} \Rightarrow r^3 = -27$

$r = \sqrt[3]{-27}$

$\boxed{r = -3}$

$a(-3)^5 = -486$

$a = \frac{-486}{(-3)^5}$

$a = \frac{-486}{-243}$

$\boxed{a = 2}$

$t_n = 2(-3)^{n-1}$

$t_1 = 2(-3)^0 = 2$

$t_2 = 2(-3)^1 = -6$

$t_3 = 2(-3)^2 = 18$

$t_4 = 2(-3)^3 = -54$

**HW Section 7.2**: Pg. 430 – 432 #2bc, 4, 5, 6, i) iii), 7, 9bcd, 10, 11, 13, ~~15~~ (interesting question…make sure you understand the text answer…ask for help if you need it), ~~16~~

bdf bdf bdf

27/2

**Success Criteria:**

- I can identify when a sequence is geometric, by seeing if it has a common ratio
- I can use the General Term Formula to develop an equation for an geometric sequence
- I can use the General Term to find any term in a sequence OR to find out how many terms are in a sequence
- I can recognize that an geometric sequence is always an exponential function

$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots + a + (n-1)d$



## 7.5 – Arithmetic Series

**Learning Goal:** We are learning to calculate the sums of the terms of an arithmetic sequence.

We have been studying sequences (which are ordered lists of numbers). We examined two types of sequences: Arithmetic and Geometric. We now turn our attention to a concept very closely related to sequences – **Series.**

**Definition 7.5.1**

A **Series** is constructed by **adding together the terms of a sequence**.

So an Arithmetic Series arises when we add together the terms of an Arithmetic Sequence.

**Example 7.5.1**

Given the 8 term arithmetic sequence $3, 7, 11, 15, 19, 23, 27, 31$ determine the associated series. Determine the PARTIAL SUM $S_4$.

$S_n = 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 = 136$

$S_4 = 3 + 7 + 11 + 15 = 36$

A partial sum occurs when you add up ***PART* of a series**

## Obtaining a Partial Sum Formula

Karl Friedrich Gauss was really smart. He found a cool and quick way to add up the numbers from 1 to 100.

$1 + 2 + 3 + 4 + \ldots\ldots\ldots + 100 = S$

$100 + 99 + 98 + 97 + \ldots\ldots\ldots + 1 = S$

$100(101) = 2S$

$50(101) = S$

$\boxed{5050 = S}$

Consider now the arithmetic sequence $t_n = a + (n-1)d$ .

We can write the sequence:

a, a + d, a + 2d, a + 3d, a + 4d, ..., $t_n$ ,...

Thus, we can consider the associated series:

$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + \ldots + (t_n) + \ldots$$

Now, if the series is infinite (that is if the associated sequence has no last term (there are infinitely many terms)), then we cannot find the sum of the series without some high power mathematics. However, we CAN find a partial sum (using Gauss' trick):

$$S_n = \quad a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(n-1)d)$$

$$+ \quad S_n = a+(n-1)d + a+(n-2)d + \ldots + a$$

$$2S_n = n[2a+(n-1)d]$$

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d] \qquad \frac{n}{2}[a+a+(n-1)d]}$$

$$S_n = \frac{n}{2}[a+l] = \frac{n}{2}[t_1+t_n]$$

**Example 7.5.2**

Determine the sum of the first 20 terms of the arithmetic sequence: n

$a=5, 2, -1, -4, \ldots$

$-3 \quad -3 \quad -3 = d$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_{20} = \frac{20}{2}[2(5)+19(-3)]$$

$$= 10[10-57]$$

$$\boxed{S_{20} = -470}$$

**Example 7.5.3**

From your text: Pg. 452 #5d

For the given arithmetic series determine $t_{12}$ and $S_{12}$: $\frac{1}{5}+\frac{7}{10}+\frac{6}{5}+\frac{17}{10}+\ldots$

$$\boxed{t_n = a+(n-1)d}$$

$$\boxed{\begin{aligned} S_n &= \frac{n}{2}\left[2a+(n-1)d\right] \\ &= \frac{n}{2}\left[t_1+t_n\right] \end{aligned}}$$

$$t_1 = a = \frac{1}{5}$$

$$d = \frac{7}{10}-\frac{1}{5} = \frac{7}{10}-\frac{2}{10} = \frac{5}{10} = \frac{1}{2}$$

$$t_n = t_{12} = \frac{1}{5}+11\left(\frac{1}{2}\right) = \frac{1}{5}+\frac{11}{2} = \frac{2+55}{10} = \boxed{\frac{57}{10} = t_{12}}$$

$$S_n = \frac{n}{2}(t_1+t_n)$$

$$S_{12} = 6\left(\frac{1}{5}+\frac{57}{10}\right)$$

$$= 6\left(\frac{2+57}{10}\right) = \frac{\overset{3}{\cancel{6}}}{1}\left(\frac{59}{\underset{5}{\cancel{10}}}\right) = \frac{177}{5}$$

$$\boxed{S_{12} = \frac{177}{5}}$$

**Example 7.5.4**

From your text: Pg 452 #7e

Calculate the sum of the series: $\overset{-7}{} \; \overset{-7}{}$ $-31-38-45-\ldots-136$

$d = -7$ $\quad t_n = -136$

$a = -31$

$S_n = \frac{n}{2}[t_1 + t_n]$

$S_{16} = 8(-31 + -136)$

$\boxed{S_{16} = -1336}$

$t_n = a + (n-1)d$

$-136 = -31 + (n-1)(-7)$

$-136 = -31 - 7n + 7$

$-136 + 31 - 7 = -7n$

$-112 = -7n$

$\frac{-112}{-7} = n$

$\boxed{16 = n}$

~~**HW** **Section 7.3: Pg. 452 – 453 #1bc, 2 , 4ace, 5abe, 6, 7abcf, 11, 13, 15**~~

1st page HANDOUT

**Success Criteria:**

- I can calculate the sum of the first *n* terms of a arithmetic sequence by using one of the two formulas we learnt
    - $S_n = \frac{n[2a+(n-1)d]}{2}$
    - $S_n = \frac{n[t_1+t_n]}{2}$
- I can recognize when each formula is the most appropriate one to use

$a + ar + ar^2 + ar^3 \ldots$

Chapter 7 – Sequences and Series (Discrete Functions) Name______________________.

## 7.6 – Geometric Series

**Learning Goal:** We are learning to calculate the sum of the terms of a geometric sequence.

Again, a series is associated with a sequence. A series arises by adding together the terms of a sequence, so a Geometric Series arises by adding together the terms of a geometric sequence.

### The Partial Sum Formula for a Geometric Series

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}, \quad r \neq 1, \qquad \text{OR} \qquad \boxed{S_n = \frac{t_{n+1} - t_1}{r - 1}}, \quad r \neq 1$$

$t_n = ar^{n-1}$

Remember that $r$ is the common ratio between successive terms!

**Example 7.6.1**

Given the geometric series, determine $t_7$ and $S_7$

$3, 12, 48, \ldots$ ($\times 4$, $\times 4$)

$t_7 = 3(4)^6 = 12{,}288$

$S_7 = \frac{3(4^7 - 1)}{(4-1)} = \frac{3(16384 - 1)}{3}$

$= \frac{\cancel{3}(16383)}{\cancel{3}}$

$= \boxed{16383}$

**Example 7.6.2**

Determine $S_{10}$ for the geometric series 1.3, 3.25, 8.125, 20.3125,...

$$\boxed{S_n = \frac{a(r^n - 1)}{(r-1)}}$$

$r = \frac{3.25}{1.3} = 2.5$

$a = 1.3$

$S_{10} = \frac{1.3(2.5^{10} - 1)}{1.5} = 8264.31$

**Example 7.6.3**

Calculate the sum of the geometric series $2-6+18-54+...+13122$

$2, -6, 18, -54, \ldots, 13122$

$r = \frac{-6}{2} = -3$

$a = 2$

$\times(-3)$

$t_n = 13122 \quad -39366$

$t_{n+1} =$

$$S_n = \frac{(t_{n+1} - t_1)}{r-1} = \frac{-39366 - 2}{-3-1} = \frac{-39368}{-4}$$

$$\boxed{S_n = 9842}$$

2nd page HANDOUT

~~**HW Section 7.6:** Pg. 459 – 461 #1abc, 3abde , 5 – 7 (what would the common ratio be?), 11~~

**Success Criteria:**

- I can add the first $n$ terms of a geometric sequence using:

$$S_n = \frac{a(r^n - 1)}{r-1}, \quad r \neq 1, \qquad \text{OR} \qquad S_n = \frac{t_{n+1} - t_1}{r-1}, \quad r \neq 1$$

- I can recognize when each formula is the most appropriate one to use