Functions 11

Course Notes

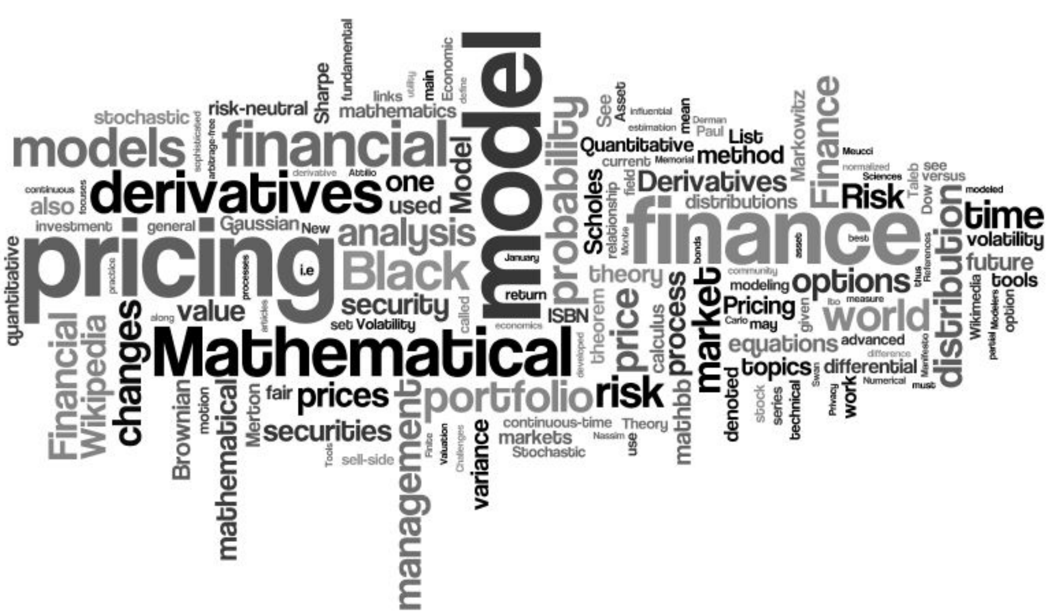
Unit 8 – Financial Math

Compound interest is the greatest Mathematical discovery of all time

Albert Einstein (maybe)

We will learn

- *understand the connections between simple interest and Arithmetic Sequences and linear growth (C3.1)*
- *understand the connections between compound interest and Geometric Sequences and exponential growth (C3.2)*
- *solve problems using the formulae, or using spreadsheets, for simple and compound interest, and for annuities/mortgages (C3.3 – C3.7)*



Chapter 8 – Financial Applications (Discrete Functions)

8.1 Simple Interest

This *should* be review, but it never hurts to review previously learned material.

Most people are *interested* in their personal financial situations. Obviously that's why we call the money earned on investments **interest**.

SI

$r \rightarrow$ decimal $\quad \% = \frac{1}{100}$ $\quad$ per annum p.a.

Simple **I**nterest is calculated using an interest **r**ate, ***r*** (% /a), over a period of time, ***t*** (in years).

We call the amount invested (borrowed) the **P**rincipal. P

The ~~Interest~~ SIMPLE Interest Formula $\quad SI = Prt$

The **A**mount (of money) Formula $\quad A = P + SI$

Example 8.1.1

From your text: Pg. 481 #5f

For each investment, calculate the interest earned and the total amount.

Prt

	Principal	Rate of Simple Interest per Year (decimals)	Time (years)	$SI = Prt$	A ($SI + P$)
a)	\$500	4.8% = 0.048	8 years	$500(0.048)(8) = \$192$	\$692
b)	\$3 200	9.8% = 0.098	12 years	$3200(0.098)(12) = \$3763.2$	\$6963.2
c)	\$5 000	3.9% = 0.039	16 months $= \frac{16}{12}$	$5000(0.039)\left(\frac{16}{12}\right) = \$260$	\$5260
d)	\$128	18% = 0.18	5 months $= \frac{5}{12}$	$128(0.18)\left(\frac{5}{12}\right) = \$9.6$	\$137.60
e)	\$50 000	24% = 0.24	17 weeks $= \frac{17}{52}$	$50000(0.24)\left(\frac{17}{52}\right) = \$3923.08$	\$53923.08
f)	\$4 500	12% = 0.12	100 days $= \frac{100}{365}$	$4500(0.12)\left(\frac{100}{365}\right) = \$147.95$	$147.95 + 4500 = \$4647.95$

Example 8.1.2

Jasmine invests $\$4850$ at $7.6\%/a$ simple interest. If she wants her money to increase to $\$8000$, for how long will she need to keep her money invested?

GIVEN	WANT
$P = \$4850$	$t = ?$
$r = \frac{7.6}{100} = 0.076$	
$A = \$8000$	

$$\boxed{SI = Prt}$$

$$SI = A - P = 8000 - 4850 = \$3150$$

$$3150 = 4850(0.076)t$$

$$\frac{3150}{4850(0.076)} = t$$

$$t = \frac{3150}{368.6} \approx 8.5$$

$\therefore$ Jasmine needs to keep her money invested for about 8.5 years

Example 8.1.3

Philip (the unwise) borrows $\$1540$ for 90 days by taking a cash advance from the company YourCashIsOurCash. The interest rate Philip (unwisely) agrees to is $26\%/a$ (simple interest). How much money will Philip have to pay back at the end of 90 days, and how much interest does he pay?

SI A

Given	Want
$P = \$1540$	$A = ?$
$t = \frac{90}{365}$ years	$SI = ?$
$r = \frac{26}{100} = 0.26$	

$$SI = Prt = 1540(0.26)\left(\frac{90}{365}\right)$$

$$\boxed{I = \$98.73}$$

$$\therefore A = P + SI = 1540 + 98.73$$

$$\boxed{A = \$1638.73}$$

$\therefore$ Philip . . . . .

Chapter 8 – Financial Mathematics

8.2 – Compound Interest: Future Value and Present Value

$A = P + SI = P + Prt = P(1+rt)$

Last day we look at the idea of Simple Interest (with formula $\boxed{A = P(1+rt)}$). ~~Now we consider~~ the notion of Compound Interest. Compound Interest is such that your savings grows much more quickly than it could if you were just earning simple interest.

Compound Interest Formulae:

Future Value

$$\boxed{A = P(1+i)^n}$$

Present Value

$$\boxed{P = \frac{A}{(1+i)^n}}$$

where: $A =$ Future Value (value at the end of the compounding period)

$P =$ Present Value (Principal)

$i = \dfrac{r}{\#\text{ of compounding periods}}$

$n = t \times \#$ of compounding periods. (t usually in years.)

Today we are looking at **Future Value** – planning today for financial health later in life!

Example 8.2.1

a) You deposit \$10 000 into an account which pays 2.4% per year, compounded annually. What is the amount of money in the account in 10 years?

$\frac{2.4}{100} = 0.024$

Given	Want
$P = \$10,000$	$A = ?$
$r = 0.024 = i$	
$n = 10 = t$	

$A = P(1+i)^n$

$= 10,000(1.024)^{10}$

$= \$12,676.51$

b) You deposit \$10 000 into an account which pays 2.4% per year, compounded monthly. What is the amount of money in the account in 10 years?

12

Given	Want
$P = \$10,000$	$A = ?$
$r = 0.024$	
$i = \frac{0.024}{12}$	

$i = 0.002$

$t = 10$ years

$\therefore n = 10(12) = 120$

$A = P(1+i)^n$

$= 10,000(1.002)^{120}$

$= \$12,709.44$

Example 8.2.2

Beth deposits \$500 into an account which pays 6% compounded monthly. Fred deposits \$500 into an account which pays 6% **simple interest**. What is the difference in the value of their accounts after 5 years?

Beth

Given	Want
$\$500 = P$ $r = 0.06$ $i = \frac{0.06}{12} = 0.005$ $t = 5$ $n = 5(12) = 60$	$A = ?$

Beth : $A = P(1+i)^n$

$= 500(1.005)^{60}$

$= \$674.43$

Fred

Given	Want
$P = \$500$ $r = 0.06$ $t = 5$	$A = ?$

Fred : $A = P + Prt$

$= 500 + 500(0.06)(5)$

$= 500 + 150$

$= \$650$

$\therefore$ Difference $= 674.43 - 650 = \$24.43$

Example 8.2.3

On her 15th birthday, being very wise, Susan invests \$10,000 in an account which pays 2.4% compounded monthly. The not-so-wise John waits until his 45th birthday to invest \$10,000 in an account which pays 2.4% compounded monthly. How much is each account worth when they reach 65 years old?

Susan

Given	Want
$P = \$10{,}000$ $i = \frac{0.024}{12} = 0.002$ $n = 50(12)$ $= 600$	$A = ?$

$A = P(1+i)^n$

$A = 10000(1.002)^{600}$

$\boxed{A = \$33{,}161.40}$

John

Given	Want
$P = \$10{,}000$ $i = \frac{0.024}{12} = 0.002$ $n = 20(12)$ $= 240$	$A = ?$

$A = P(1+i)^n$

$A = 10000(1.002)^{240}$

$\boxed{A = \$16{,}152.99}$

Example 8.2.4

You find yourself in a furniture store. Looking around you become dazzled by the adverts promising a better and happier life if you only had one of their beautiful couches. The advertisement reads:

No Money Down!

No Payments for 3 years!!

Take this couch home today!!!!!

Only $1599*!!!!!!!!!!!!

You decide it's a good deal, but you neglect to read the asterisk until it's too late. You've already signed on the dotted line. After signing you decide to finally read the fine print which says:

**Financed at 18% compounded monthly*

How much do you have to pay after 3 years?

Example 8.2.5

You want $10 000 in your bank account 20 years from now. Your account pays 1.8% per year, compounded annually. What is the amount of money you have to deposit today?

Given	Want
$A = \$10,000$	$P = ?$
$t = 20 = n$	
$r = 0.018 = i$	

$$A = P(1+i)^n$$

$$P = \frac{A}{(1+i)^n} = \frac{10000}{\boxed{(1.018)^{20}}} = \$6999.14$$

Example 8.2.6

You want to buy a house at 30 years old (14 years from now). You estimate that you will need a down payment of $150 000. You find a bond which matures in 14 years paying 3.6% interest compounded monthly. How much do you need to invest in the bond today?

Given	Want
$A = \$150,000$	$P = ?$
$t = 14$	
$n = 14(12) = 168$	
$r = 0.036$	
$i = \frac{0.036}{12} = 0.003$	

$$P = \frac{A}{(1+i)^n} = \frac{150\,000}{(1.003)^{168}}$$

$$P = \$\ 90,684.80$$

Example 8.2.7 Scott.

Today ~~Henry~~ invests some money in an account which pays 2% compounded quarterly. 10 years from now he takes the money in his account and reinvests it in an account which pays 4.8% compounded monthly. After an additional 15 years Scott has $125,345 in his account. How much did he originally invest?

P_1 = ?

$P_1 \xrightarrow[\text{2\% quarterly}]{t=10} P_2 = A_1 \xrightarrow[\text{4.8\% monthly}]{t=15} A_2 = \$125,345$

Given	Want
$A = 125,345$	$P = ?$
$i = \frac{0.048}{12} = 0.004$	
$n = 15(12) = 180$	

$$\boxed{P = \frac{A}{(1+i)^n}} = \frac{125\,345}{(1.004)^{180}}$$

$$P_2 = \$61,099.65 = A_1$$

Given	Want
$A = 61,099.65$	$P = ?$
$i = \frac{0.02}{4} = 0.005$	
$n = 10(4) = 40$	

$$P = \frac{61,099.65}{(1.005)^{40}}$$

$$\boxed{P_1 = \$\ 50,049.10}$$

$\therefore$ Scott originally invested

Chapter 8 – Financial Mathematics

8.3 – Annuities: Future Value and Present Value

The problem with the examples in 8.2 is that not many people have large amounts of money to be depositing into savings accounts. People usually make **regular deposits of smaller amounts of money**. **An account into which (or out of!) regular payments are made is called AN ANNUITY**. We will study two aspects of annuities: Future Value and Present Value.

Future Value of an Annuity

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

Present Value of an Annuity

$$PV = \frac{R[1-(1+i)^{-n}]}{i}$$

where: FV = FUTURE VALUE

n = $t \times$ (# of compounding periods)

R = REGULAR PAYMENT

PV = PRESENT VALUE

i = $\frac{r}{\text{\# of compounding periods}}$ $r \rightarrow$ rate in decimals

Example 8.3.1

Dylan decides to deposit $200 monthly into an account which pays 3.6% per year, compounded monthly. What is the value of his annuity after 25 years? How much interest is earned? If Dylan leaves the money in his account for another 20 years, but makes no more regular payments, how much money will be in the account at the end of 45 years?

Given	Want
$R = \$200$	$F.V. = ?$
$r = 0.036$	
$i = \frac{0.036}{12} = 0.003$	
$t = 25$	
$n = 25(12) = 300$	

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

$$= \frac{200[(1.003)^{300} - 1]}{0.003}$$

$$\boxed{F.V. = \$97086.10}$$

$P = 200 \times 25 \times 12 = \$60,000.$

$I = F.V. - P = 97,086.10 - 60,000 = \$37,086.10$

Given	Want
$P = 97,086.10$	$A = ?$
$t = 20$	$A = 97086.10(1.003)^{240}$
$\therefore n = 20(12) = 240$	$\$199242.04$
$i = 0.003$	

$$\boxed{A = P(1+i)^n}$$

Example 8.3.2

Tingyi invests \$300, every three months (i.e. quarterly) into an RRSP which pays (on average) 12% compounded quarterly. What is the value of her RRSP after 45 years?

Given	Want
$R = \$300$	F.V.
$r = 0.12$	
$\therefore i = \frac{0.12}{4} = 0.03$	
$t = 45$	
$n = 45(4) = 180$	

$$F.V. = \frac{R[(1+i)^n - 1]}{i}$$

$$F.V. = \frac{300[(1.03)^{180} - 1]}{0.03}$$

$$\boxed{F.V. = \$2035033.60}$$

$\therefore$ Ans. statement.

Example 8.3.3

An absolutely wonderful student decides to give his teacher Mr. Benett, \$100,000 when he retires in 15 years. The student finds an investment which pays 9% compounded monthly. What is the regular payment s/he would need to make to have \$100,000 in 15 years?

Given	Want
$FV = \$100,000$	$R = ?$
$r = 0.09$	
$i = \frac{0.09}{12} = 0.0075$	
$t = 15$	
$n = 15(12) = 180$	

$$F.V = \frac{R\,\boxed{[(1+i)^n - 1]}}{i}$$

$$\Rightarrow \frac{(F.V.)i}{[(1+i)^n - 1]} = R$$

$$R = \frac{(100,000)(0.0075)}{[(1.0075)^{180} - 1]} \approx \$264.3$$

$\therefore$ statement.

Example 8.3.4

While in college you want an annuity to pay you \$250 a month, every month over the four years you are studying. How much money would you need to invest today in an account which pays 2.4% compounded monthly to guarantee the monthly payment? What is the total amount of money you receive from the annuity? How much interest do you earn?

Given	Want
$R = \$250$	$PV = ?$
$r = 0.024$	
$i = \frac{0.024}{12} = 0.002$	
$n = 4(12) = 48$	

$$PV = R\frac{\left[1-(1+i)^{-n}\right]}{i}$$

$$P.V. = 250\frac{\left[1-(1.002)^{-48}\right]}{0.002}$$

$$= \$11,431.11$$

Total Money recieved from the annuity $= 250 \times 4 \times 12 = \$12,000$

$\therefore$ Interest earned $= 12,000 - 11,431.11 = \$568.89$

Example 8.3.5

Henry borrows \$25,000 to buy a car. He pays \$3,000 down. He will take 6 years to pay off the loan. The bank charges 2.6% compounded biweekly. What are the regular payments Henry has to make to pay off the car? How much interest does he pay?

LOAN VALUE $= 25000 - 3000 = 22,000$

Given	Want
$P.V. = \$22,000$	$R = ?$
$t = 6$	
$n = 6(26)$	
$n = 156$ biweeks	
$r = 0.026$	
$i = \frac{0.026}{26}$	
$i = 0.001$	

$$PV = R\frac{\left[1-(1+i)^{-n}\right]}{i}$$

$$R = \frac{(PV)i}{\left[1-(1+i)^{-n}\right]} = \frac{22000(0.001)}{\left[1-(1.001)^{-156}\right]}$$

$$R \approx \$152.38$$

$\therefore$ over 156 biweeks, Henry pays $= 152.38 \times 156 = \$23,771.28$

$\therefore$ interest $= 23,771.28 - 22,000 \to \$1,771.28$