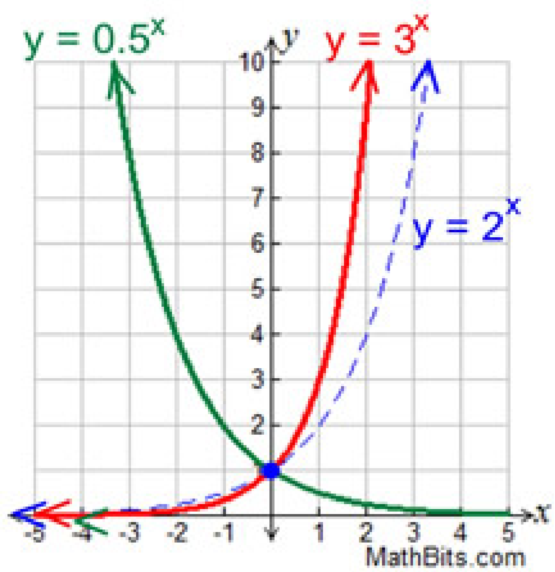
Functions 11

Course Notes

Unit 4 – Exponential Functions

We are learning to

- *describe the characteristics of exponential functions and their graphs*
- *evaluate powers with integer and rational exponents and simplify expressions involving them*
- *explore the transformation of exponential functions*
- *use exponential functions to solve problems involving exponential growth and decay*



4.2 Integer Exponents (Reviewing the known Power Laws and learning a few new ones 😊)

Learning Goal: We are learning to work with integer exponents.

Before beginning, we should quickly review: **THE POWER LAWS**

Consider a typical "power" a^n.

We call "***a***" the base. We call "***n***" the exponent and the entire expression a^n is called a power.

Let's recall ***The Known Power Laws***:

Given the powers a^m and a^n, with exponents *m* and *n*, and the number $\frac{a}{b}$, and a,b≠0, then

1) $1^n = 1$

2) $a^1 = a$

3) $a^0 = 1$

4) $a^m . a^n = a^{m+n}$

5) $\frac{a^m}{a^n} = a^{m-n}$

6) $(ab)^m = a^m b^m$

7) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

8) $(a^m)^n = a^{mn}$

Until now, for the most part, the exponents you've been working with have always been **Natural Numbers**. But, we now will examine some more **Integer Exponents**!!
i.e. we will also start dealing with Negative Exponents.
Let's explore Negative Exponents and create some additional Power Laws. 😊

Rule to remember for negative exponents: ***Negative Exponents reciprocate the Base***

Additional Power Laws:

9) $a^{-n} = \frac{1}{a^n}$

10) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$

11) $\frac{a^{-m}}{a^{-n}} = \frac{a^n}{a^m}$

Example 4.2.1

Write each expression as a **single power with a positive exponent**:

a) $(4)^{-5} = \frac{1}{4^5}$

b) $\left(\frac{3}{2}\right)^{-4} = \left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$

c) $\frac{7^3}{7^9} = 7^{3-9} = 7^{-6} = \frac{1}{7^6}$

Example 4.2.2

Simplify using exponent laws, then **evaluate** each expression and **state your answers in rational form:**

a) $3^5\left(3^{-2}\right)$

$= 3^{5+-2}$

$= 3^3$

$= 27$

b) $\left(2^{-3}\left(2^4\right)\right)^{-5}$

$= \left(2^{-3+4}\right)^{-5}$

$= \left(2^1\right)^{-5}$

$= 2^{-5}$

$= \frac{1}{2^5} = \frac{1}{32}$

c) $\frac{5^{-3}}{\left(5^2\right)^{-2}}$

$= \frac{5^{-3}}{5^{-4}}$

$= 5^{-3--4}$

$= 5^{-3+4}$

$= 5$

Example 4.2.3

Evaluate and express in rational form:

a) $3^2(6^{-3})$

$= \frac{3^2}{6^3}$

$= \frac{\cancel{3^2}^{1}}{2^3 \times \cancel{3^3}^{3}}$

$= \frac{1}{8 \times 3}$

$= \frac{1}{24}$

Alternately,

$\frac{3^2}{(2\times3)^3}$

$\frac{\cancel{3}^{1}\times\cancel{3}^{1}}{\cancel{6}_{2}\times\cancel{6}_{2}\times 6}$

$= \frac{1}{24}$

b) $2^{-3} + 10^{-3} - 3(5^{-3})$

$= \frac{1}{2^3} + \frac{1}{10^3} - \frac{3}{5^3}$

$= \frac{1^{\times 125}}{8_{\times 125}} + \frac{1}{1000} - \frac{3^{\times 8}}{125_{\times 8}}$

$= \frac{125+1-24}{1000}$

$= \frac{\cancel{102}^{51}}{\cancel{1000}_{500}}$

$= \frac{51}{500}$

c) $13^{-5} \times \left(\frac{13^2}{13^8}\right)^{-1}$

$= \frac{13^8}{13^5 \times 13^2}$

$= 13^{8-5-2}$

$= 13^1$

$= 13$

Alternately,

$13^{-5} \cdot \frac{13^{-2}}{13^{-8}}$

$= 13^{-5+-2--8}$

$= 13^1$

Example 4.2.4

Evaluate using the laws of exponents (the power rules):

a) $3^2 \times 9^{-3} \div 3^{-7}$

$= 3^2 \times \frac{1}{9^3} \div \frac{1}{3^7}$

$= 3^2 \times \frac{1}{9^3} \times 3^7$

$= \frac{3^9}{(3^2)^3}$

$= \frac{3^9}{3^6}$

$= 3^{9-6}$

$= 3^3$

$= 27$

b) $\frac{(4^{-2}+3^{-1})}{(5^{-1}+2^{-2})}$

$= \left(\frac{1}{4^2}+\frac{1}{3}\right) \div \left(\frac{1}{5}+\frac{1}{2^2}\right)$

$= \left(\frac{1}{16}+\frac{1}{3}\right) \div \left(\frac{1}{5}+\frac{1}{4}\right)$

$= \left(\frac{3+16}{48}\right) \div \left(\frac{4+5}{20}\right)$

$= \frac{19}{48} \div \frac{9}{20}$

$= \frac{19}{\cancel{48}_{12}} \times \frac{\cancel{20}^{5}}{9}$

$= \frac{95}{108}$

HW- Section 4.2 Pg. 222 – 223 #5 – 8, 13

Success Criteria:

- I can apply the exponent laws
- I can recognize that a negative exponent represents a reciprocal expression

Chapter 4 – Exponential Functions Name____________________.

4.3 Rational Exponents (Learning a few more new power laws 😊)

Learning Goal: We are learning to work with powers involving rational (fractional) exponents and to evaluate expressions containing them.

A **RATIONAL EXPONENT** can be a **FRACTION**.

For example, we can consider the number $(16)^{\frac{3}{4}}$. Of course, the question we need to ask is:

What the rip is that thing??

Let's put our calculators to some good use. Use a calculator to complete the following:

i. $9^{\frac{1}{2}}$ = 3 and $\sqrt{9}$ = 3. So, $9^{\frac{1}{2}} = \sqrt{9}$.

ii. $64^{\frac{1}{3}}$ = 4 and $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$. So, $64^{\frac{1}{3}} = \sqrt[3]{64}$.

iii. $16^{\frac{1}{4}}$ = 2 and $\sqrt[4]{16} = \sqrt[4]{2^4} = 2$. So, $16^{\frac{1}{4}} = \sqrt[4]{16}$.

Can you generalize the rule from the above examples? Note that each of the powers in the examples were **unit fractions.**

General Rule: $a^{\frac{1}{n}} = \sqrt[n]{a}$

Yes, that's right!! But what if the power is not a unit fraction like $(16)^{\frac{3}{4}}$**?**

As you know, a fraction has two parts: a numerator, and a denominator. When a fraction is used as an exponent, the two parts of the fraction carry two related (but different) meanings in terms of "powers".

Definition 4.3.1

Given a power with a "rational" (fractional) exponent $a^{\frac{m}{n}}$, the **numerator** of the exponent is a "**power**" in the usual sense, and the **denominator** represents a "**root**" or "**radical**".

So, with this knowledge let's come back to our example in the beginning.

e.g. For the number $16^{\frac{3}{4}}$ $= \left(16^3\right)^{\frac{1}{4}} = \left(16^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{16}\right)^3 = (2)^3 = 8$

So, ***General Rule:*** $a^{\frac{m}{n}} = \sqrt[n]{(a^m)} = \left(\sqrt[n]{a}\right)^m$

Also, remember that **the exponent laws that apply to powers are the same for rational exponents too.**

Example 4.3.1

From your text: Pg. 229 #2.

Write in exponent form, and then evaluate:

a) $\sqrt[9]{512}$

$= (512)^{\frac{1}{9}}$

$= 2$

c) $\sqrt[3]{27^2}$

$= (27)^{\frac{2}{3}}$

$= 9$

f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$

$= \left(\frac{16}{81}\right)^{-\frac{1}{4}}$

$= \left(\frac{81}{16}\right)^{\frac{1}{4}}$

$\sqrt[4]{\frac{81}{16}} = \frac{3}{2}$

$= 1.5$

b) $\sqrt[3]{-27}$

$= \sqrt[3]{(-3)(-3)(-3)}$

$= (-3)$

* Note

All odd radicals can have negative radicands.

Note: We ***CANNOT*** take an **Even Root** of a **negative radicand**.
We ***CAN*** take an **Odd Root** of a negative radicand, however.
Why ?

Example 4.3.2

From your text: Pg. 229 #3

Write as a single power:

a) $\left(8^{\frac{2}{3}}\right)\left(8^{\frac{1}{3}}\right)$

$= (8)^{\frac{2}{3}+\frac{1}{3}}$

$= 8^{\frac{3}{3}}$

$= 8$

b) $8^{\frac{2}{3}} \div 8^{\frac{1}{3}}$

$= 8^{\frac{2}{3}-\frac{1}{3}}$

$= 8^{\frac{2-1}{3}}$

$= 8^{\frac{1}{3}} \longrightarrow \sqrt[3]{8} = 2$

Example 4.3.3

From your text: Pg. 229 #4

Write as a single power, then **evaluate**. Express answers in **rational form**.

a) $\sqrt{5}\cdot\sqrt{5}$

$= (\sqrt{5})^2$

$= 5^{\frac{2}{2}}$

$= 5$

b) $\dfrac{\sqrt[3]{-16}}{\sqrt[3]{2}}$

$= \dfrac{(-16)^{\frac{1}{3}}}{(2)^{\frac{1}{3}}}$

$= \left(\dfrac{-\cancel{16}^{\,8}}{\cancel{2}_{\,1}}\right)^{\frac{1}{3}}$

$= (-8)^{\frac{1}{3}}$

$= \sqrt[3]{-8} = \sqrt[3]{(-2)(-2)(-2)} = \sqrt[3]{(-2)^3}$

$= \boxed{(-2)}$

HW- Section 4.3

Pg. 229: #2de, 3cdef, 4cd, 5, 6, 8, 10 (*a question of awesomeness*), 12 (*we will take up next day*)

Success Criteria:

- I can understand that the numerator of a fractional exponent is the power, while the denominator is the root.

Chapter 4 – Exponential Functions Name______________________.

4.4 Simplifying Expressions involving Exponents

Learning Goal: We are learning to simplify algebraic expressions involving powers and radicals.

Keep the **EXPONENT RULES** in your mind at all times.

One of the Keys of the exponent rules is "**SAMENESS**".

- When you have the **SAME BASE**, (but possibly different exponents) you can combine powers.

  e.g. $\dfrac{x^3 \times x^4}{x^7} = x^{3+4-7} = x^0 = 1$

- When you have the **SAME EXPONENT** (but possibly different bases) you can "combine the bases under the same exponent".

  e.g. $\dfrac{\sqrt[3]{12} \times \sqrt[3]{36}}{\sqrt[3]{16}} = \sqrt[3]{\dfrac{12 \times 36}{16}}$

  $= \left(\dfrac{\overset{3}{\cancel{12}} \times \overset{9}{\cancel{36}}}{\underset{\overset{\cancel{4}}{1}}{\cancel{16}}}\right)^{\frac{1}{3}}$

  $= (27)^{\frac{1}{3}} = \sqrt[3]{27}$

  $= 3$

Now we turn to problems involving both numbers and variables being exponentized (not a word, but it should be because of how awesome it sounds).

Example 4.4.1

Simplify, leaving you answer with only positive exponents:

a) $\left(x^{3}\right)^{2}\left(x^{-8}\right)$

$= x^{6}x^{-8}$

$= x^{6+-8}$

$= x^{-2}$

$= \frac{1}{x^{2}}$

b) $\frac{z^{4}\left(z^{-2}\right)}{z^{-3}}$

$= z^{4+(-2)-(-3)}$

$= z^{4-2+3}$

$= z^{5}$

c) $\sqrt{36x^{-6}} = \left(36x^{-6}\right)^{\frac{1}{2}} = 36^{\frac{1}{2}}\ x^{-\frac{6}{2}}$

$= \sqrt{36}\ x^{-\cancel{\frac{6}{2}}\,\frac{3}{1}}$

$= 6x^{-3}$

$= \frac{6}{x^{3}}$

d) $\left(\frac{\left(3x^{2}y^{1}\right)^{-1}\left(x^{3}y^{-4}\right)}{\left(x^{3}y^{2}\right)^{-2}}\right)^{-2}$

$(x^{3}y^{2})^{-2}$

$(x^{3})^{-2}(y^{2})^{-2}$

$= \left(\frac{3^{-1}x^{-2}y^{-1}\ x^{3}y^{-4}}{x^{-6}y^{-4}}\right)^{-2}$

$= \frac{3^{2}x^{4}y^{2}x^{-6}y^{8}}{x^{12}y^{8}}$

$= 9x^{4+-6-12}y^{2+8-8}$

$= 9x^{-14}y^{2}$

$= \frac{9y^{2}}{x^{14}}$

e) $\left(\dfrac{\sqrt{16a^6}}{\left(a^3\right)^{-1}}\right)^{\frac{3}{2}}$

$= \left(\dfrac{\sqrt{16}\, a^{\cancel{\frac{6}{2}}\,3}}{a^{-3}}\right)^{\frac{3}{2}}$

$= \left(4a^{3-(-3)}\right)^{\frac{3}{2}}$

$= \left(4a^6\right)^{\frac{3}{2}}$

$= 4^{\frac{3}{2}} \cdot a^{\cancel{6}\,3 \times \frac{3}{\cancel{2}\,1}}$

$= \left(\sqrt{4}\right)^3 \cdot a^9$

$= 8a^9$

f) $\left(\dfrac{\left(6x^3\right)^2\left(6y^3\right)}{\left(9xy\right)^6}\right)^{-\frac{1}{3}}$

$= \left(\dfrac{6^2 x^6 \cdot 6y^3}{9^6 x^6 y^6}\right)^{-\frac{1}{3}}$

$= \left[\dfrac{\cancel{6}\times\cancel{6}\times\cancel{6}}{\cancel{9}\times\cancel{9}\times\cancel{9}\times 9\times 9\times 9}\left(\dfrac{\cancel{x^6}}{\cancel{x^6}}\right)\left(\dfrac{y^3}{y^6}\right)\right]^{-\frac{1}{3}}$

$= \left[\dfrac{8}{19683}\, y^{3-6}\right]^{-\frac{1}{3}}$

$= \left[\dfrac{8}{19683}\, y^{-3}\right]^{-\frac{1}{3}}$

$= \left(\dfrac{8}{19683}\right)^{-\frac{1}{3}} \cdot \left(y^{-3}\right)^{-\frac{1}{3}}$

$= \sqrt[3]{\dfrac{19683}{8}}\; y^{(-)\cancel{3}\,1 \times (-)\frac{1}{\cancel{3}\,1}}$

$= \dfrac{\sqrt[3]{19683}}{\sqrt[3]{8}}\, y^1$

$= \dfrac{27}{2}y$

HW- Section 4.4

Pg. 236 – 237 #2acef, 4acdf, 5, 6, 7ac (simplify BEFORE substituting!), 9ad

Success Criteria:

- I can simplify algebraic expressions containing powers by using the exponent laws
- I can simplify algebraic expressions involving radicals

 Name_____________________.

4.5-4.6 Properties and Transformations of Exponential Functions

Learning Goal: We are learning to identify the characteristics and transformations of the graphs and equations of exponential functions.

Exponential Functions are of the (basic) form:

$$\boxed{f(x) = b^x}$$ where b > 0 and b ≠ 0

(of course, we can apply transformations to this basic, or **parent function**!! Fun Times are – a – coming!! ☺)

In the parent exponential function $f(x) = b^x$, b is the base. **THE BASE OF AN EXPONENTIAL FUNCTION IS JUST A NUMBER.** For example, we might have the functions

$f(x) = 2^x$ $g(x) = 0.5^x + 3$

$h(x) = \left(\frac{2}{3}\right)^{2x+5} - 1$

> *Note that **the exponent is the variable here in the Exponential Function!***

What does the Base of an Exponential Function tell you?

Activity: <u>Folding a Piece of Paper</u>.
If you have ever tried to fold a piece of paper repeatedly, you probably already know that the more folds in the paper, the harder it is to get the next fold.

Take some time to think and fill in the table below:

x # of Folds in a Sheet of Paper	y # of Layers of the Paper in each fold	Exponent form of the # of Layers
0	1	2^0
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
6	64	2^6
		⋮
x		2^x

×2 ×2 ×2

If you could represent the paper folding situation as a function, what would it be?

***(Hint:** Take the independent variable as x and try defining f(x) ☺)*

$f(x) = 2^x$

$2^{-3} = \frac{1}{8}$

$2^{-2} = \frac{1}{4}$

$2^{-1} = \frac{1}{2} = 0.5$

$2^{0} = 1$

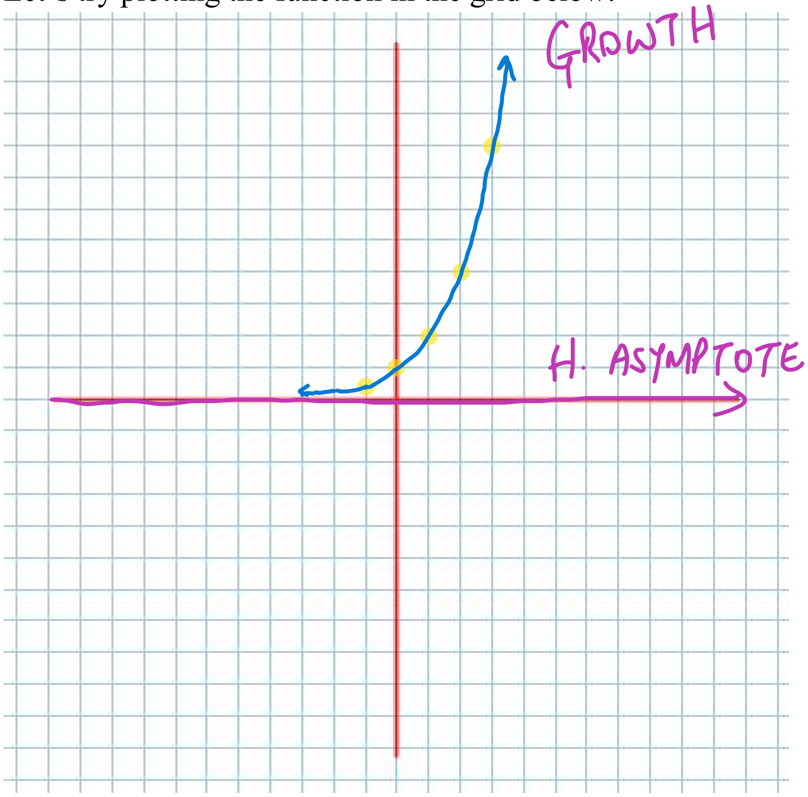
$2^{1} = 2$

$2^{2} = 4$

$2^{3} = 8$

$\vdots$

Let’s try plotting the function in the grid below:

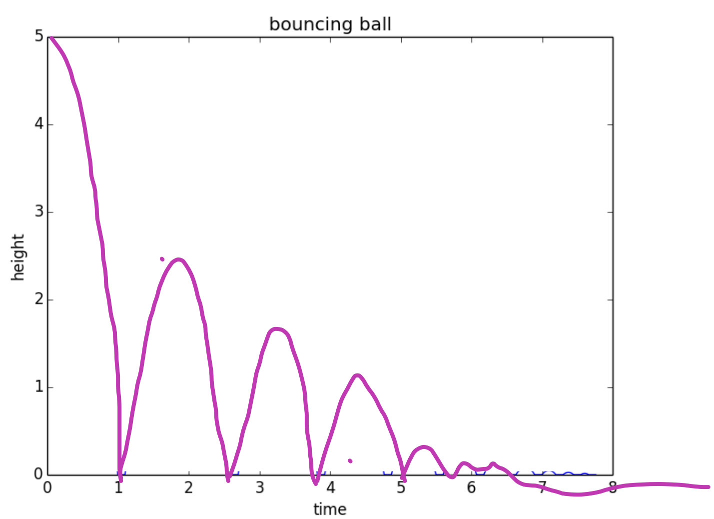


Let’s explore another example:

Suppose you **drop freely** a bouncy ball from a certain height (say 1 metre), what do you think will happen?

We will notice that after the first bounce, the ball will bounce back only about ___half___ the height that it originally was dropped from.

After the second bounce, the peak height becomes ___half___ again and the process goes on until the ball stops bouncing completely.



Take some time to think and fill in the table below:

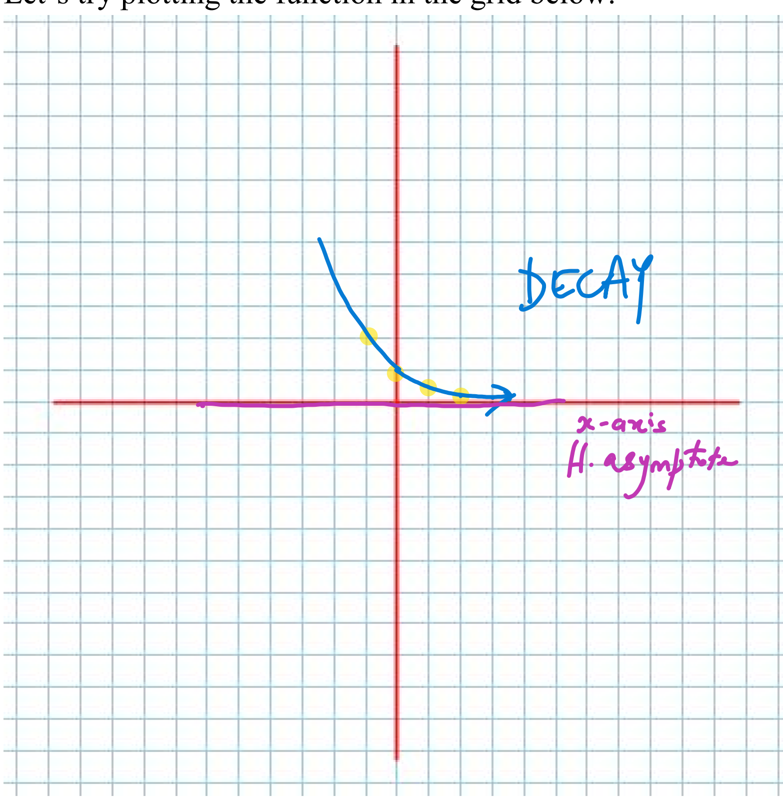
x # of bounces	y Peak Height (in metres) *Use fractions*	Peak Height in exponential form
0	1	$\left(\frac{1}{2}\right)^0$
1	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^1$
2	$\frac{1}{2}\left(\frac{1}{2}\right)=\frac{1}{4}$	$\left(\frac{1}{2}\right)^2$
3	$\frac{1}{2}\left(\frac{1}{4}\right)=\frac{1}{8}$	$\left(\frac{1}{2}\right)^3$
4	$\frac{1}{2}\left(\frac{1}{8}\right)=\frac{1}{16}$	$\left(\frac{1}{2}\right)^4$
5	$\frac{1}{2}\left(\frac{1}{16}\right)=\frac{1}{32}$	$\left(\frac{1}{2}\right)^5$
6	$\frac{1}{2}\left(\frac{1}{32}\right)=\frac{1}{64}$	$\left(\frac{1}{2}\right)^6$

If you could represent the situation as a function, what would it be?

(Hint: *Take the independent variable as x and try defining f(x)* 😉***)***

$$f(x)=\left(\frac{1}{2}\right)^x$$

Let's try plotting the function in the grid below:



From the above activities, it is pretty clear that the base of the exponential function i.e. the constant **b** determines

the RATE of CHANGE of the exponential function.

i. If the base **b** > 1, then the Exponential function defines GROWTH.

For example, the growth of population, bacteria, money in some investment, etc.

ii. If the base 0 < **b** < 1, then the Exponential function defines DECAY.

fractional

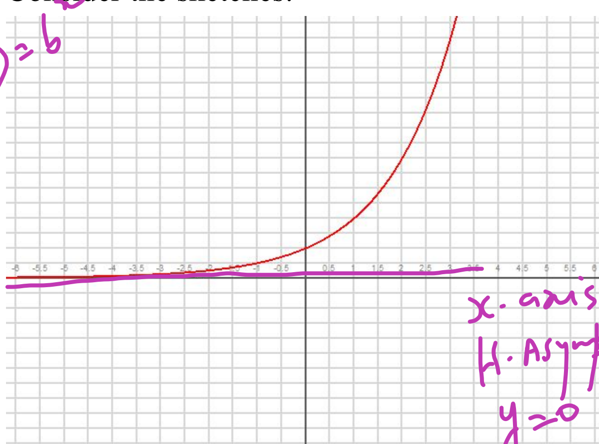
For example, the death of populations, radioactive decay, the temperature of a hot cup of coffee left on the desk, depreciation of a car, etc.

Note that if exponential functions are some kind of Growth or Decay, then we need to have something to begin with. Hence, we will not consider the **b** to be negative and choose to consider only **positive bases 'b'.**

Domain and Range of the Basic Exponential Function

Consider the sketches:

$f(x) = b^x$



x-axis
H. Asymptote
$y = 0$

Growth Function

D = $\{x \in \mathbb{R}\}$.

R = $\{f(x) \in \mathbb{R} \mid f(x) \geq 0\}$.

Decay Function

D = $\{x \in \mathbb{R}\}$.

R = $\{f(x) \in \mathbb{R} \mid f(x) \geq 0\}$.

ALL Exponential Functions have a Horizontal **ASYMPTOTE** (Basic Exponential Functions have $y = 0$ i.e. x-axis as their Horizontal Asymptote.

The Horizontal Asymptote of a Transformed Exponential Function, however, depends on the "c" ________________.)

$g(x) = a f(k(x-d)) + c$

$\boxed{\text{H. Asymptote} \to y = c}$

Example. $f(x) = 2^x + 1$ ← c=1 ∴ H. Asymptote: y = 1

↑ b = 2 ∴ GROWTH FUNCTION.

ALL BASIC Exponential Functions pass through the point $(0,1)$ which is the y-intercept. Transformed Exponential Functions will have a *y*-intercept too, but depends on the vertical shift/stretch and horizontal shift/stretch. So, we can find the y-intercept by taking c (vertical shift).

The Transformed Exponential Function

The general form of an exponential function is:

$$f(x) = a \cdot b^{k(x-d)} + c$$

Where:

$a \to$ V. stretch

$a < 0 \to$ V. Flip.

$k \to$ H. stretch by $1/k$

$d \to$ H. shift

d: Left or Right

$c \to$ V. shift

c: up when positive, down when negative

∴ H. Asymptote ⇒ $y = c$

Example 4.6.1

State the transformations applied to the parent function $\boxed{f(x)=3^x}$. Also state the y-intercept, and the equation of the horizontal asymptote of the transformed function.

$$g(x)=-2\cdot 3^{3x+3}+4$$

$g(x) = af(k(x-d))+c$

$g(x) = -2\, 3^{\frac{(3x+3)}{3 \quad 3}} + 4$

$g(x) = -2\, 3^{3(x+1)} + 4$

$a = -2$

V. stretch & V. flip. by a

$c = 4$

H. shift 4 units up

$1/k = \frac{1}{3}$

H. stretch by $\frac{1}{3}$

$d = -1$

V. shift 1 unit down.

$y_{int} = y_{x=0} = g(0) = -2 \cdot 3^{3(0)+3} + 4$

$= -2 \cdot 3^3 + 4$

$y_{int} = -50$

$\therefore$ Equation of H. Asymptote : $y = 4$

Example 4.6.2

From your Text: Page 252 #7

7. A cup of hot liquid was left to cool in a room whose temperature was 20 °C. The temperature changes with time according to the function $\boxed{T(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{30}} + 20.}$ Use your knowledge of transformations to sketch this function. Explain the meaning of the y-intercept and the asymptote in the context of this problem.

(i) $b = \frac{1}{2}$ $\therefore$ DECAY PROBLEM!!

(ii) time has been chunked into intervals of 30.

(iii) $c = 20$

$\therefore$ H. asymptote : $y = 20$

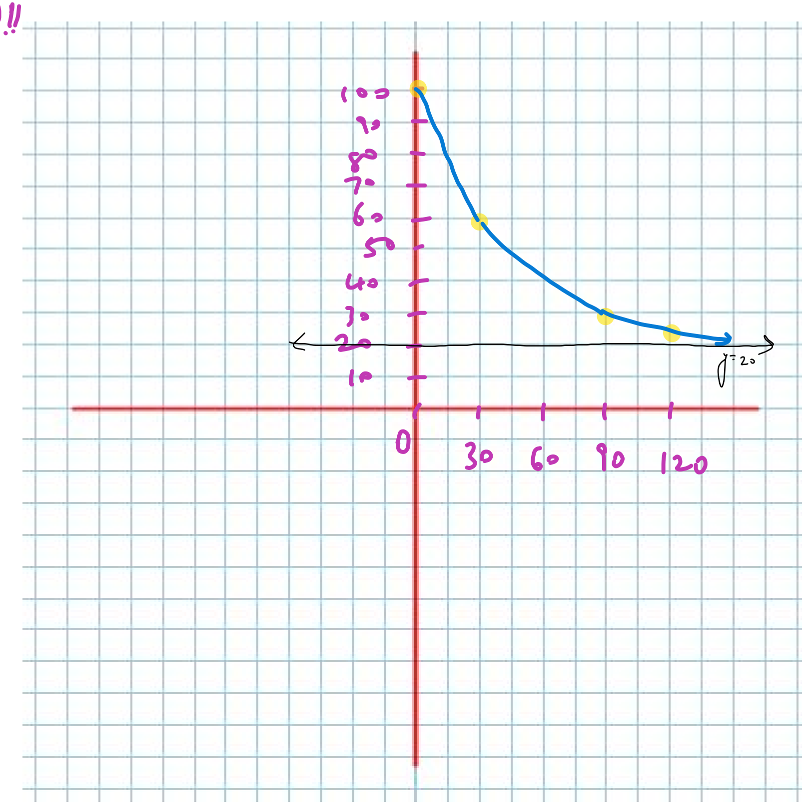
(iv) $y_{int} = T(t=0)$

$= 80(0.5)^{\frac{0}{30}} + 20$

$= 80(0.5)^{0} + 20$

$= 80(1) + 20$

$= 100°$



(iv) Temp of Coffee

(iii) Temp. of Room

t	$T = 80(0.5)^{\frac{t}{30}} + 20$
0	100°
30	60
60	40
90	30
120	25

Example 4.6.3

From your Text: Page 252 #5a

Let $f(x)=4^x$. For the function which follows,

- State the transformations applied to $f(x)$
- State the y-intercept, and the horizontal asymptote
- Sketch the transformed function, and write the function "properly"
- State the domain and range of the transformed function

$$g(x)=0.5f(-x)+2$$

$$g(x) = \overset{a=0.5}{0.5} \cdot 4^{\overset{k=-1}{-}(x-\overset{d=0}{0})} + \overset{c=2}{2}$$

OBSERVATION:

$b=4$ has a negative exponent which flips the base to $\frac{1}{4}$

$\therefore$ DECAY PROBLEM!!

• Transformation:

$a=0.5$ V. stretch

$c = 2$ V. shift 2 units up

$k = -1$ H. flip.

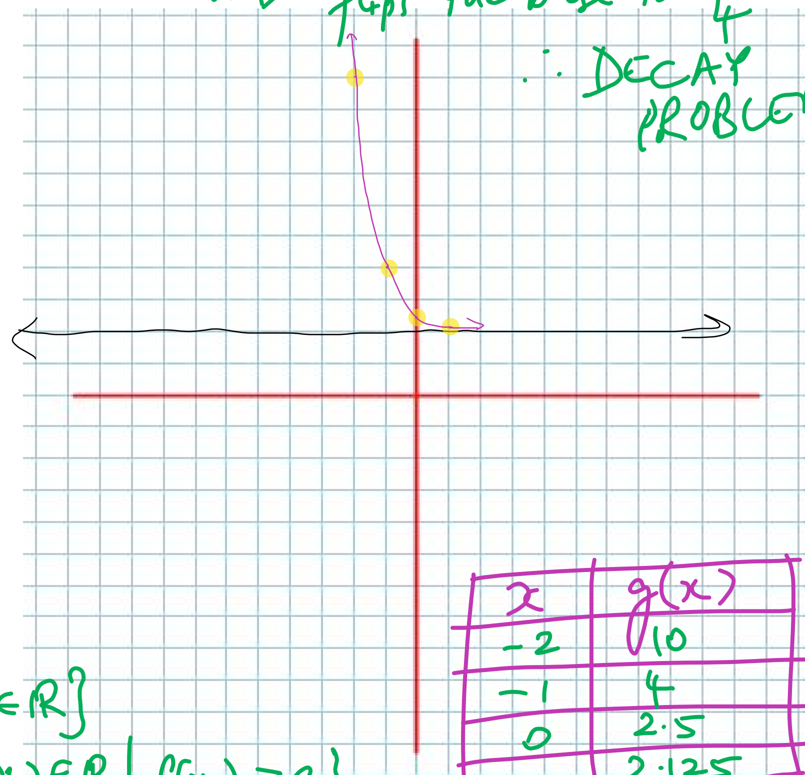
$d = 0$ No. H. shift

• $y.\text{int} = g(x=0) = 0.5 \cdot 4^{-0}+2$

$= 0.5(1)+2$

$= 0.5+2 = 2.5$

H. asymptote: $y=2$



x	$g(x)$
-2	10
-1	4
0	2.5
1	2.125

• $D = \{x \in \mathbb{R}\}$

$R = \{f(x) \in \mathbb{R} \mid f(x) > 2\}$

RECOMMENDED

not mandatory

HW- Section 4.5/4.6

Pg. 251 – 253 #1 – 3, 5bcd (write each transformed function "properly), 8 – 10 (*for #10 please see example 4 on page 250*)

Success Criteria:

- I can identify the graph of an exponential function
- I can identify and apply the four transformations (a, k, d, c) to the equation of an exponential function

Chapter 4 – Exponential Functions Name____________________.

4.7 Applications of Exponential Functions

Learning Goal: We are learning to use exponential functions to solve problems involving exponential growth and decay.

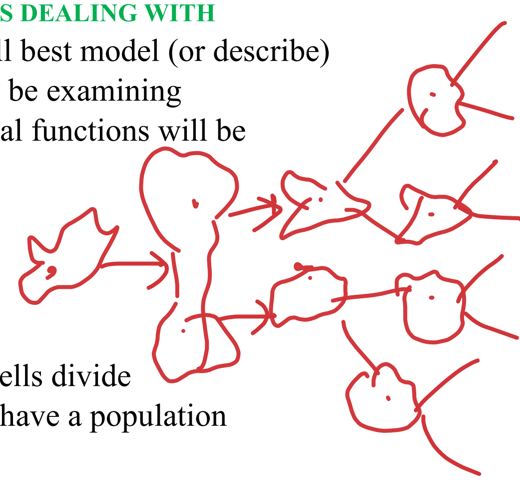
Anything in the real world which grows, or decays can be "**MODELED**" (or in some sense "***DESCRIBED***") with words, or pictures or mathematics. **Mathematical models** are useful for getting solutions to problems, and making predictions.

So far in Mathematics 11U we have studied the basics of functions in general (chapter 1), we've done some algebra (chapter 2), and we've examined Quadratic functions (chapter 3). Part of our study of Quadratics was learning how to use the vertex of a parabola to answer questions about maxima and minima for some real word problems.

We now know that Quadratic **MODELS** are very useful for solving max/min problems.

In this lesson we want to work on **LEARNING HOW TO SOLVE PROBLEMS DEALING WITH GROWTH AND DECAY.** We have to decide what type of function will best model (or describe) the type of growth/decay seen in the problem (hint: for this lesson we'll be examining Exponential Growth and Decay, and therefore we expect that exponential functions will be used…shocking, I know :P)

Q. What is Exponential Growth or Decay again?



Consider the following:

A single cell divides into two "daughter" cells. Both daughter cells divide resulting in four cells. Those four cells each divide and we now have a population of ____8____ cells.

Describe, using mathematics, how the cell population changes from generation to generation.

Generation	G_0	G_1	G_2	G_3	. . . .
Population	1	2	4	8	. . .
	2^0	2^1	2^2	2^3	. . .

$\therefore$ Population in Generation $x = 2^x$

Example 4.7.1

Being a financial wizard, you deposit $1,000 into an account which pays 3.5% interest, compounded annually.

P

$\frac{3.5}{100} = 0.035 = r$

a) Determine how much money is in your account after t = 1,2,3, and 4 years. A

b) Determine a mathematical model which can describe how the value of the account is changing from year to year.

$A = P + I$

Time (in years)	$t = 0$	$t = 1$	$t = 2$
Amount in bank ($)	$A_0 = 1000$	$A_1 = 1000 + 0.035(1000)$ $\boxed{A_1 = 1000(1+0.035)}$ $A_1 = 1035$	$A_2 = 1035 + 0.035(1035)$ $A_2 = 1035(1+0.035)$ $A_2 = 1000(1+0.035)(1+0.035)$

$A_2 = \boxed{1000(1+0.035)^2}$

$1 \cdot P + 0.035P$

$= P(1 + 0.035)$

$t = 3 \Rightarrow A_3 = 1000(1+0.035)^3$

$t = n \text{ years} \Rightarrow A_n = 1000(1+0.035)^n$

Definition 4.7.1

A function describing Exponential Growth is of the form:

$$A(x) = A_0(1+R)^x$$

x is # GROWTH PERIODS

A_0 is the INITIAL AMOUNT.

GROWTH RATE and $0 < R < 1$

A function describing Exponential Decay is of the form:

$$A(x) = A_0(1-R)^x$$

x is # DECAY PERIODS

R is the DECAY RATE

Example 4.7.2

From your text, Pg. 263

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents. 0.01

a) the percent of colour left if blue jeans lose 1% of their colour every time they are washed

b) the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for *t* years

c) the population of a colony if a single bacterium takes 1 day to divide into two; the population is P after t days

$r = 100\% = 1$

a) $C(w) = C_0(1-0.01)^w = C_0(0.99)^w$

DECAY PROBLEM

$C(w) \rightarrow$ color of jeans after w washes.

initial colour of jeans

we can take the original colour as 1.

$\therefore C(w) = 1 \cdot (0.99)^w$

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b) $P(t) = 2500(1+0.005)^t = 2500(1.005)^t$

GROWTH PROBLEM

$P(t) \rightarrow$ population after t years

initial pop.

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c) $P(t) = 1(1+1)^t = 2^t$

doubling problem

Example 4.7.3
From your text, Pg. 263

11. A population of yeast cells can double in as little as 1 h. Assume an initial population of 80 cells.
   a) What is the growth rate, in percent per hour, of this colony of yeast cells?
   b) Write an equation that can be used to determine the population of cells at *t* hours.
   c) Use your equation to determine the population after 6 h.
   d) Use your equation to determine the population after 90 min. = 1.5h.
   e) Approximately how many hours would it take for the population to reach 1 million cells? $t = ?$ $P = 1{,}000{,}000$
   f) What are the domain and range for this situation?

a) 100% $\therefore R = \frac{100}{100} = 1$

b) $P(t) = P_0(1+R)^t$

$P(t) = 80(1+1)^t = 80(2)^t$

c) $t = 6$

$\therefore P(t=6) = 80(2)^6 = 5120$ yeast cells.

d) $t = 1.5$

$\therefore P(t=1.5) = 80(2)^{1.5} = 226.27$

≈ 226 yeast cells.

e) $P(t) = 80(2)^t$

$\frac{1000000}{80} = \frac{\cancel{80}}{\cancel{80}}(2)^t \rightarrow 12{,}500 = (2)^t$

ROUGH TRIAL & ERROR

$2^{20} = 1{,}048{,}576$

$2^{18} = 262{,}144$

$2^{15} = 32{,}768$

$t =$

$2^{13} = 8192$

$2^{14} = 16384$

$\therefore$ It will take about 14h. for the yeast cells to cross 1 million cells population.

f) $D = \{ t \in \mathbb{R} \mid t \geq 0 \}$

$R = \{ P(t) \in \mathbb{R} \mid P(t) \geq 80 \}$

Example 4.7.4

$r = 0.2$ $\quad$ C_0

A new car depreciates at a rate of 20% per year. Steve bought a new car for $26,000.

a) Write the equation that models this scenario.

$$C(t) = 26000(1-0.2)^t = 26000(0.8)^t$$

b) How much will Steve's car be worth in 3 years?

$t = 3$ years.

$$\therefore C(t=3) = 26000(0.8)^3 = 13312$$

c) When will Steve's car be worth $4000?

$C(t) = 4000 \qquad t = ?$

$$\therefore 4000 = 26000(0.8)^t$$

$$\frac{4000}{26000} = (0.8)^t$$

$$\approx 0.153846 = (0.8)^t$$

$0.8^7 = 0.209715$

$0.8^8 = 0.1677\ldots$

$0.8^9 = 0.1342\ldots$

$\therefore$ After 8 years, Steve's car will be worth $4000.

$$A(x) = A_0(2)^{\frac{x}{D}}$$

doubling period.

Additional Applications – ***DOUBLING AND HALF-LIFE***

Thus far, we have only seen examples with single period rates: "yearly" "monthly" "daily"

Unfortunately, it's not always that simple…Our rates could be…

Every 3 years Every 6 hours Every 4 days

How do we deal with the exponent in these cases?

Example 1 (Doubling)

P_0 $r = 100\%$ D

A species of bacteria has a population of 300 at 9 am. It doubles every 3 hours.

a) Write the function that models the growth of the population, P, at any hour, t

$$P(t) = 300(2)^{\frac{t}{3}}$$

b) How many will there be at 6 pm?

$9am \longrightarrow 6pm \Rightarrow t = 9$ hrs.

$$P(t=9) = 300(2)^{\frac{9}{3}} = 300(2)^3 = 2400$$

c) How many will there be at 11 pm?

$9am \longrightarrow 11pm \Rightarrow t = 14$ hrs.

$$P(t=14) = 300(2)^{\frac{14}{3}} = 7619$$

d) Determine the time at which the population first exceeds 3000.

$$3000 = 300(2)^{\frac{t}{3}}$$

$$\frac{3000}{300} = (2)^{\frac{t}{3}}$$

$$10 = (2)^{\frac{t}{3}}$$

$$2^3 = 2^{\frac{t}{3}} \longrightarrow 3 = \frac{t}{3} \Rightarrow 9 = t$$

$2^3 = 8$

$2^4 = 16$

∴ The pop. first exceeds 3000 in about 10 hrs.

$$A(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{D}}$$

half life period.

Example (Half-Life)

A_0

A 200g sample of radioactive material has a half-life of 138 days. How much will be left in 5 years?

$A(x)$

$t = 5$ years $= 5 \times 365$

$= 1825$ days.

$D = 138$ days

$$A(t) = 200\left(\frac{1}{2}\right)^{\frac{1825}{138}} = 0.021\text{ g (approx)}$$

HW- Section 4.7

READ Example 2 on pages 256 – 257 (which method do you prefer: Guess and Check, or Graphing Calculator?)

READ Example 4 on pages 259 – 260.

Pg. 261 – 263 #1, 3 – 9, 11 – 16

Success Criteria:

- I can differentiate between exponential growth and exponential decay
- I can use the exponential function $f(x) = ab^x$ to model and solve problems involving exponential growth and decay
  - Growth rate is $b = 1 + r$. Decay rate is $b = 1 – r$.
  - r is a DECIMAL, not a percent!!!!!