Functions 11

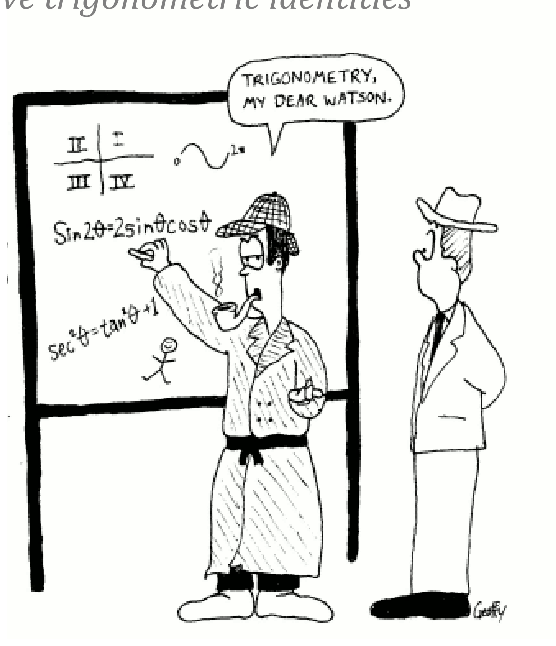
Course Notes

Chapter 5 – Trigonometric Ratios

TRIGONOMETRY IS MORE THAN TRIANGLES

We will learn

- *The three reciprocal trigonometric ratios*
- *To relate the six trigonometric ratios to the unit circle*
- *To solve problems using trig ratios, properties of triangles, and the sine/cosine laws*
- *How to prove trigonometric identities*



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MATH@TD

Chapter 5 – Trigonometric Ratios Name____________________.

5.1 – Trigonometric Ratios of Acute Angles

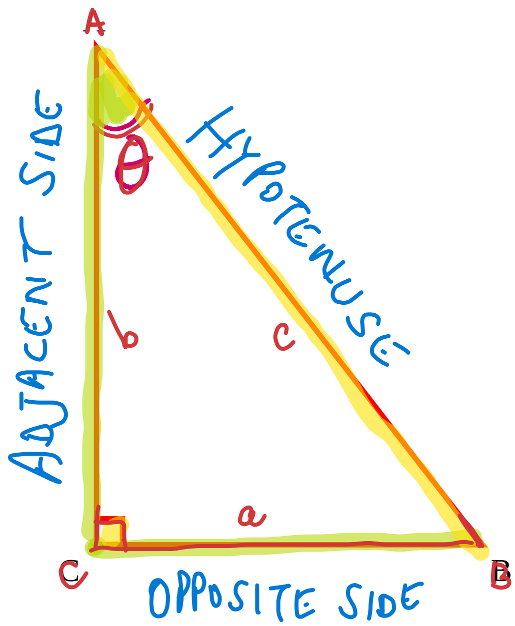
Learning Goal: We are learning to evaluate reciprocal trigonometric ratios.

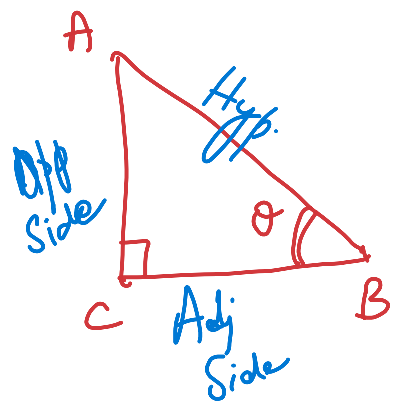
Recall from Grade 10 the mnemonic

SOH CAH TOA

We use SOH CAH TOA to calculate the so-called "trig ratios" for **right angled triangles**.

Consider the triangle:





The Trigonometric Ratios

Primary Trig Ratios

Sine $\theta \equiv \sin\theta = \frac{O}{H} = \frac{a}{c}$

Cosine $\theta \equiv \cos\theta = \frac{A}{H} = \frac{b}{c}$

Tangent $\theta \equiv \tan\theta = \frac{O}{A} = \frac{a}{b}$

Reciprocal Trig Ratios

$\frac{1}{\sin\theta} \equiv$ Cosecant $\theta = \csc\theta = \frac{H}{O} = \frac{c}{a}$

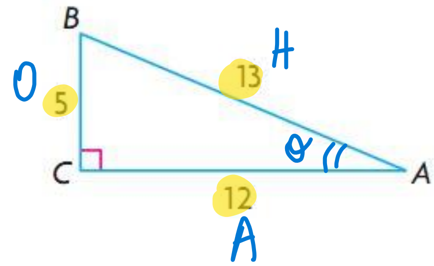
$\frac{1}{\cos\theta} \equiv$ Secant $\theta = \sec\theta = \frac{H}{A} = \frac{c}{b}$

$\frac{1}{\tan\theta} \equiv$ Cotangent $\theta = \cot\theta = \frac{A}{O} = \frac{b}{a}$

Example 5.1.1

From your text, Pg. 280 #1

Given $\triangle ABC$, state the six trigonometric ratios for $\angle A$.



$\sin A = \frac{5}{13}$

$\cos A = \frac{12}{13}$

$\tan A = \frac{5}{12}$

$\csc A = \frac{13}{5}$

$\sec A = \frac{13}{12}$

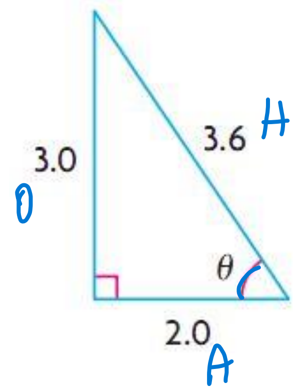
$\cot A = \frac{12}{5}$

Example 5.1.2

For the given right triangle determine:

a) $\csc(\theta)$, $\sec(\theta)$, and $\cot(\theta)$.

b) the angle θ to the nearest degree.



a) $\csc\theta = \frac{H}{O} = \frac{3.6}{3} = 1.2$

$\sec\theta = \frac{H}{A} = \frac{3.6}{2} = 1.8$

$\cot\theta = \frac{A}{O} = \frac{2}{3}$

b) $\tan\theta = \frac{3}{2}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56^\circ$ (approx)

Example 5.1.3

a) Determine the corresponding reciprocal ratio:

i) $\sin(\theta) = \frac{2}{5}$ ii) $\tan(\theta) = \frac{-3}{1}$

$\csc\theta = \frac{5}{2}$ $\cot\theta = \frac{-1}{3}$

b) Calculate to the nearest hundredth: $\sec(34^\circ)$ *(There is no sec button in your calculator!!! What would you do?)*

$\sec 34 = \frac{1}{\cos 34} = 1 \div (\cos 34) \approx 1.206 \approx 1.21$

c) Determine the value of θ to the nearest degree: $\csc(\theta) = 2.46$ *(There is no csc button either!!!)*

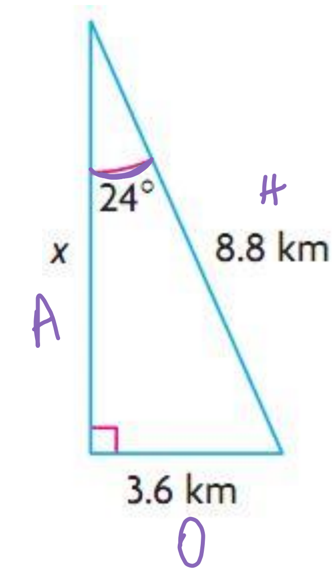
$\csc\theta = 2.46$

$\Rightarrow \sin\theta = \frac{1}{2.46}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2.46}\right) = 24^\circ$ (approx)

Example 5.1.4 SOH CAH TOA

Given the right triangle, determine the unknown side using two different trig ratios:



$\cos 24 = \frac{x}{8.8}$

$\Rightarrow 8.8 \cos 24 = x$

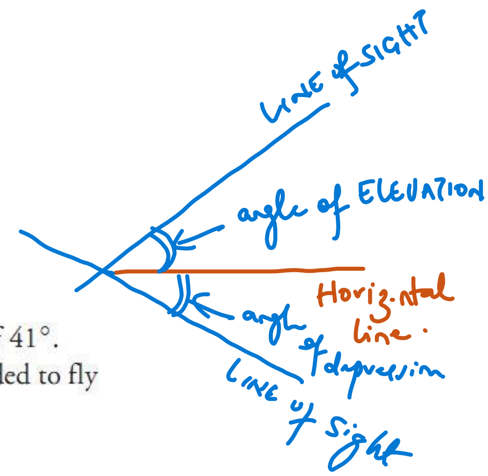
$\Rightarrow 8.04 \approx x$

$\tan 24 = \frac{3.6}{x}$

$\Rightarrow x \tan 24 = 3.6$

$\Rightarrow x = \frac{3.6}{\tan 24}$

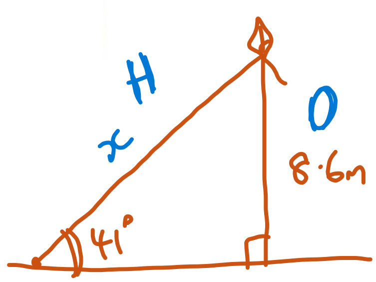
$\Rightarrow x \approx 8.09$



Example 5.1.5

From your text, Pg. 282 #11

A kite is flying 8.6 m above the ground at an angle of elevation of 41°. Calculate the length of string, to the nearest tenth of a metre, needed to fly the kite



SOH

$\sin 41 = \frac{8.6}{x}$

$x = \frac{8.6}{\sin 41} = 13.1\text{ m.}$

∴ The length of the string is 13.1 m.

HW: Section 5.1

Pg. 280 – 282 # 3, 4, 5i,ii,iv, 6, 7, 8a, 12, 14, 15

Success Criteria:

- I can use SOH CAH TOA to determine the primary and reciprocal trigonometric ratios
- I can evaluate problems using the reciprocal trigonometric ratios
- I cannot use my calculator to directly solve a reciprocal trigonometric ratio

Chapter 5 – Trigonometric Ratios Name____________________.

5.6: The Sine Law

Learning Goal: We are learning to use the Sine law to solve non-right-angle triangles.

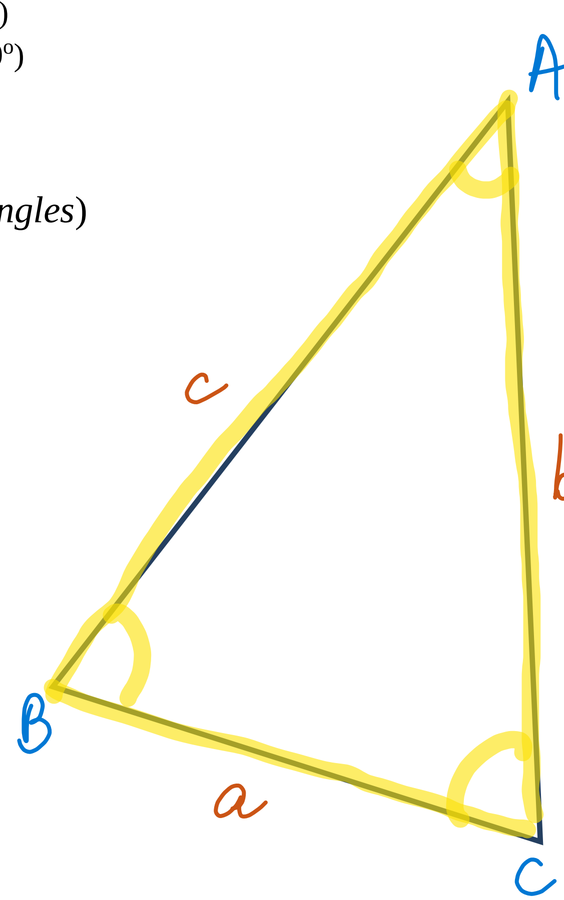
Last year you learned the Sine Law. It is a "formula" we can use to **solve triangles which are not right-angled triangles**. There is one requirement to be able to use the Sine Law.

You Must Have an Angle with Its Corresponding Side!

So far, we have been using Right Angle Triangles along with SOH CAH TOA to "solve" triangles. BUT right-angle triangles aren't always the best triangles to use;

Sometime using a right-angle triangle just can't be done. We then need to use so-called "**OBLIQUE TRIANGLES**". Oblique triangles come in two forms:

1) Acute (all angles are less than 90°)
2) Obtuse (one angle is more than 90°)



The Sine Law (*for oblique triangles*)

(There are **TWO FORMS** you should know!!)

Given the non-right triangle, ΔABC, then:

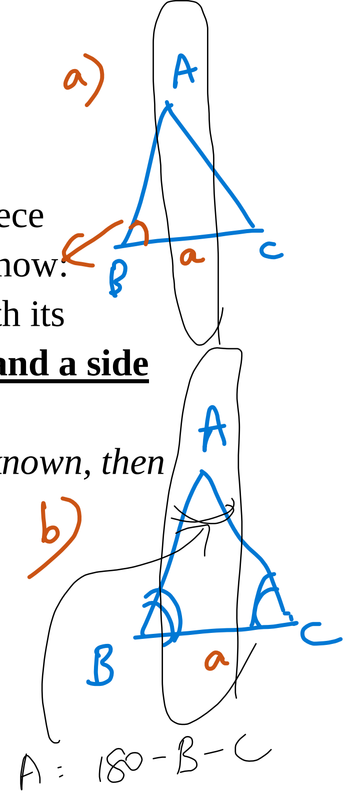
$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$

or

$$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$$

Notes:

1) Memorize the SINE LAW!
2) If we are trying to **find an angle**, use the first form of the Sine Law (**angles on top**)
3) If we are trying to **find the length** of a side, use the second form of the law (**with sides on top**)



4) In order to use the Sine Law, you must have the correct three piece information in the triangle. You can apply the Sine law if you know:

a) **<u>any one "CORRESPONDING PAIR"</u>** – an angle with its opposite side (for example you might have side *a* and angle *A*) **<u>and a side</u>** **or** else,

b) **<u>two angles and any side</u>** *(Note that if two angles are known, then the third angle is obvious by using Angle Sum Property)*

Note: **Sin θ = Sin (180 - θ)**

Let's put our calculators to good use and verify this.

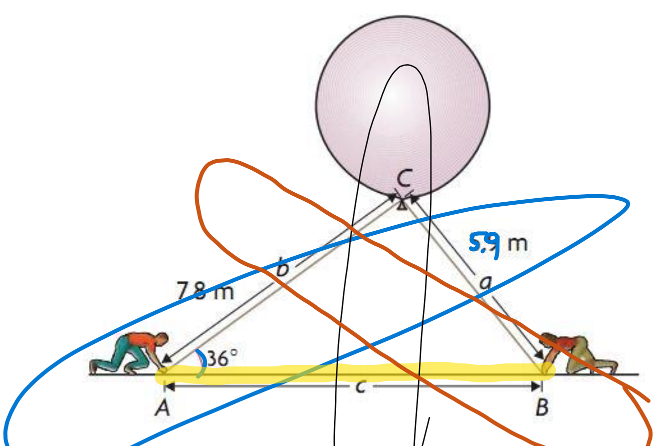
e.g. $\sin(51^\circ) =$ 0.77.... $\sin(129^\circ) =$ 0.77....
sin(180−51)=

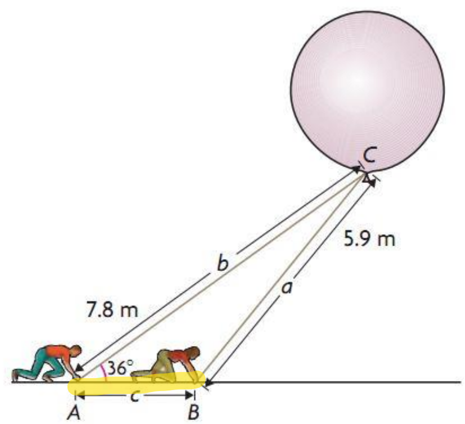
Let's consider Example 1 in your text: Pg. 312 – 314 .

Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long.

The question we are asked is: **How far is Albert from Belle, to the nearest meter?**

<u>Possible Pictures:</u>





Both Pictures describe the problem completely. So which is correct?

Well…BOTH ARE POSSIBLE solutions. This is known as the "AMBIGUOUS CASE". Because both are possible solutions, you must find both. <u>Such a case occurs when you know or are given two side length and an angle opposite one of the sides (ASS triangle)</u>

Note that to find angle B using the Sine law, your calculator will give you an answer for only the Acute possibility. So, you must apply the supplementary angle theorem to find the other possibility for angle B.

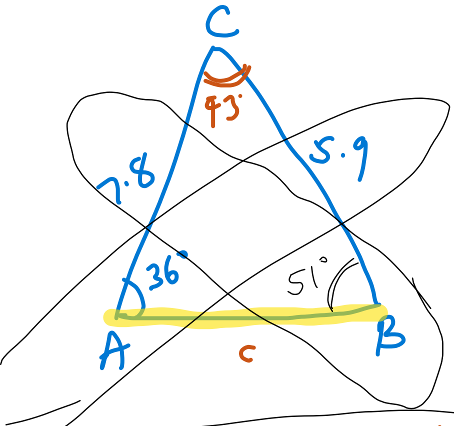
Note: If the **GIVEN ANGLE** is **ACUTE**, then this so-called Ambiguous Case MAY APPLY. But, if the Given Angle is Obtuse, then the Ambiguous Case CANNOT APPLY. (*And* ***Sometimes****, there is no triangle which solves the problem.*)

Why? This is because if the given angle is Obtuse, the other two angles MUST BE Acute to satisfy the Angle Sum Property of a Δ (add up to 180°)

Example 5.6.1

Solve the triangles above.

CASE 1: Albert & Bella are on opposite sides of the weather balloon.



$$\frac{\sin B}{7.8} = \frac{\sin 36}{5.9}$$

$$\sin B = \frac{7.8 \sin 36}{5.9}$$

$$B = \sin^{-1}(\ldots)$$

$$B = 51°$$

$$\therefore C = 180 - 36 - 51 = 93°$$

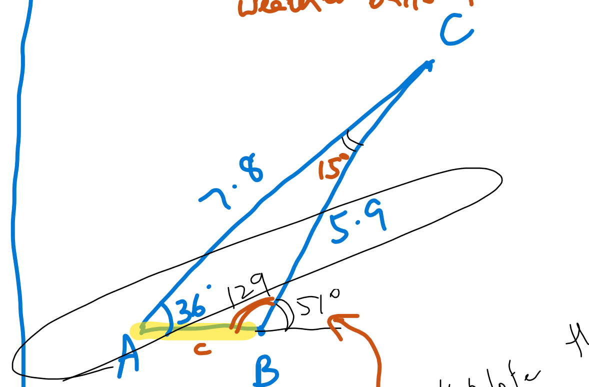
$$\frac{c}{\sin 93} = \frac{5.9}{\sin 36}$$

$$c = \frac{5.9 \sin 93}{\sin 36}$$

$c = 10$ m. (approx)

$\therefore$ Albert & Bella could be 10 m apart or 2.6 m apart.

CASE 2: Albert & Bella are on the side of the weather balloon.



$$\frac{\sin B}{7.8} = \frac{\sin 36}{5.9}$$

$\vdots$

$B = 51°$

$\therefore 180 - B = 180 - 51 = 129°$

$\sin(B) = \sin(180 - B)$

$\sin(51) = \sin(129)$

* Note that the problem is bcz the calculator only gives acute angles and does not cover the obtuse case. So, we'll manually find the obtuse angle

$\therefore$ Obtuse $\angle B = 129°$

$\therefore \angle C = 180 - 36 - 129 = 15°$

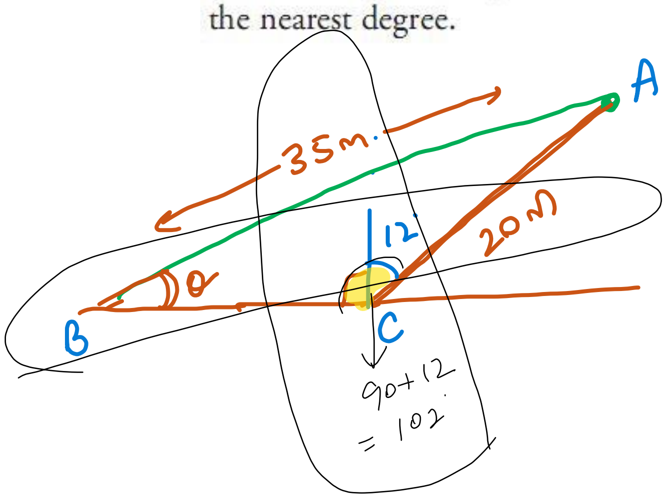
$$\therefore \frac{c}{\sin 15} = \frac{5.9}{\sin 36}$$

$$c = \frac{5.9 \sin 15}{\sin 36} = 2.6\text{ m (approx)}$$

Example 5.6.2

From your text: Pg. 319 #6

The trunk of a leaning tree makes an angle of 12° with the vertical. To prevent the tree from falling over, a 35.0 m rope is attached to the top of the tree and is pegged into level ground some distance away. If the tree is 20.0 m from its base to its top, calculate the angle the rope makes with the ground to the nearest degree.



$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin B}{20} = \frac{\sin 102}{35}$$

$$\sin B = \frac{20 \cdot \sin 102}{35}$$

$$B = \sin^{-1}(\text{Ans.})$$

$$B = 34^\circ$$

∴ The rope makes an angle of 34° with the ground.

HW: Section 5.6

Pg. 318 – 320 #4, 5bd, 7 (If only you had a side of the right-angle triangle…), #9 (recall the meaning of angle of depression??), 10, 13

Success Criteria

- I can recognize when the sine law applies and use it to solve for an unknown value
- I can identify, given S-S-A, that there will be two solutions (the ambiguous case)

5.7: The Cosine Law

Learning Goal: We are learning to use the cosine law to solve non-right-angle triangles.

The Cosine Law is another "formula" for solving Oblique Triangles.

*In order to use the Cosine Law, you must again have three pieces of information about the triangle. You can apply the Cosine law if you know:

a) **all three sides** (SSS)

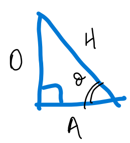
or else,

b) **two sides and included angle** (SAS)

So, the main question you will have to be able to answer while trying to solve a triangle is this:

When will I use

1) SOH CAH TOA ?

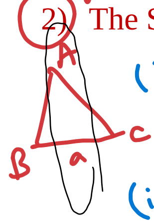


Right Δ → SOH CAH TOA

If Not Right Δ

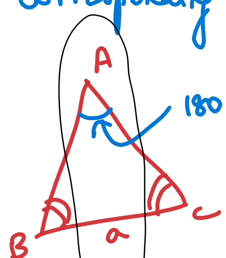
2) The SINE LAW ?

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \iff \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



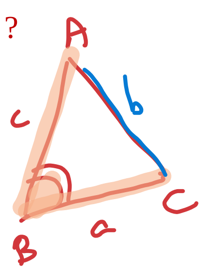
(i) Corresponding pair of side and angle given (EXPLICIT)

(ii) $180 - \angle B - \angle C = \angle A$



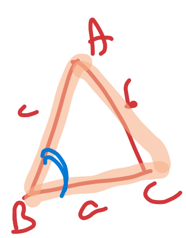
3) The COSINE LAW ?

(i) SAS



$$b^2 = a^2 + c^2 - 2ac\cos B$$

(ii) SSS



$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

The Cosine Law *(for oblique triangles)*

There are **THREE SIDE FORMS** you should know!!

Given the non-right triangle, ΔABC, then:

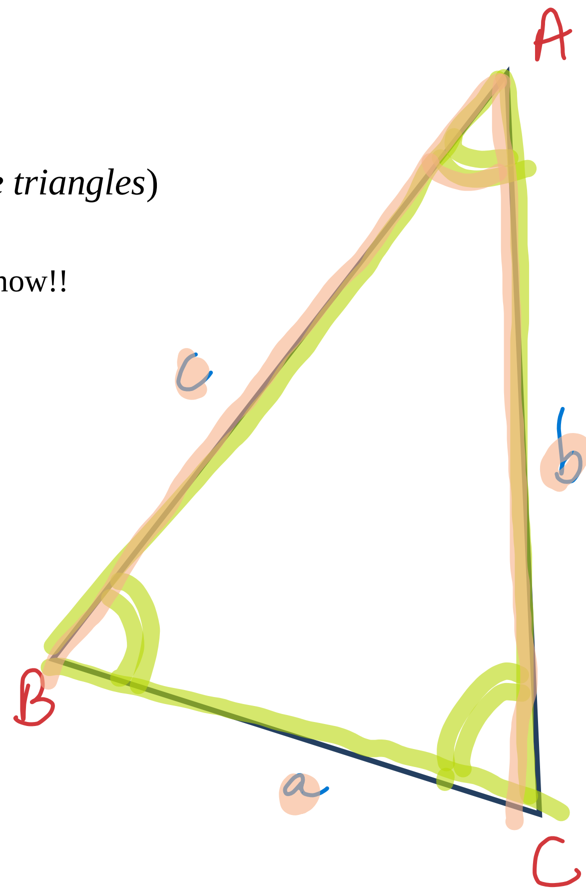
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$ SAS

or

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

or

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



Also, there are **THREE ANGLE FORMS** you should know!!

$$\cos(A) = \frac{a^2 - b^2 - c^2}{-2bc}$$ SSS

or

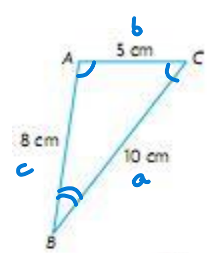
$$\cos(B) = \frac{b^2 - a^2 - c^2}{-2ac}$$

or

$$\cos(C) = \frac{c^2 - a^2 - b^2}{-2ab}$$

The formula you use depends on which side or angle you are looking for!!!

e.g. Determine angle B



$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

$$\cos B = \frac{5^2 - 10^2 - 8^2}{-2(10)(8)}$$

$$\cos B = \frac{25 - 100 - 64}{-160}$$

$$\cos B = \frac{+139}{+160}$$

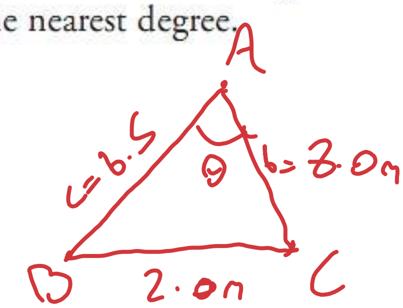
$$B = \cos^{-1}\left(\frac{139}{160}\right)$$

$$B \approx 30^\circ$$

Example 5.7.1

From your text: Pg. 326 #5

The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.



$$\cos A = \frac{a^2 - b^2 - c^2}{-2(b)(c)}$$

$$\cos A = \frac{2^2 - 8^2 - 6.5^2}{-2(8)(6.5)}$$

$$\cos A = \frac{4 - 64 - 42.25}{-104} = \frac{+102.25}{+104}$$

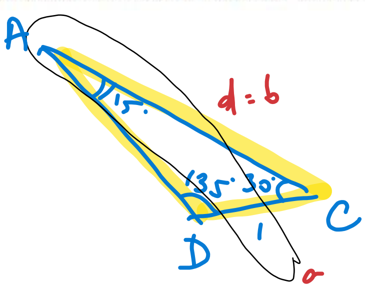
$$A = \cos^{-1}\left(\frac{102.25}{104}\right) \approx 11^\circ$$

∴ The shot must be made within an angle of 11°

Example 5.7.2

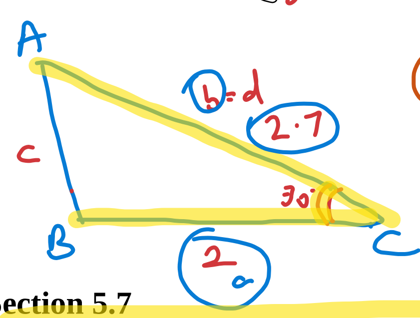
From your text: Pg. 327 #7

Given $\triangle ABC$ at the right, $BC = 2.0$ and D is the midpoint of BC. Determine AB, to the nearest tenth, if $\angle ADB = 45^\circ$ and $\angle ACB = 30^\circ$.



$$\frac{d}{\sin 135} = \frac{1}{\sin 15}$$

$$d = \frac{\sin 135}{\sin 15} = 2.7 \text{ (approx)}$$



SAS

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 2^2 + 2.7^2 - 2(2)(2.7)\cos 30^\circ$$

$$c^2 = 4 + 7.29 - 10.8\cos 30$$

$$c = \sqrt{(\quad)} \approx 1.4 = AB$$

∴ AB = 1.4 units (approx.)

HW: Section 5.7

Page 326 - 327 #4ad (do you "need" to use the cosine law?), 6, 8 – 10

Success Criteria:

- I can use the cosine law, given S-A-S or S-S-S
- I can rearrange the cosine law to solve for an unknown angle

5.8: 3D Problems

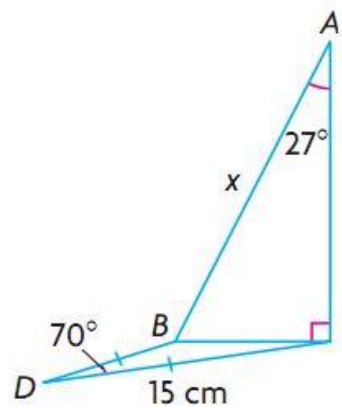
Learning Goal: We are learning to use trigonometry to solve 3-dimensional problems.

We will be using SOH CAH TOA, the Sine Law, and the Cosine Law for these problems. We'll jump right in by solving some problems since we already know how to use the various techniques! **One thing to keep in mind, though, is that these sorts of problems can be difficult to draw, or even simply visualize because we are working in 3D! Art specialists – REJOICE!** 😉

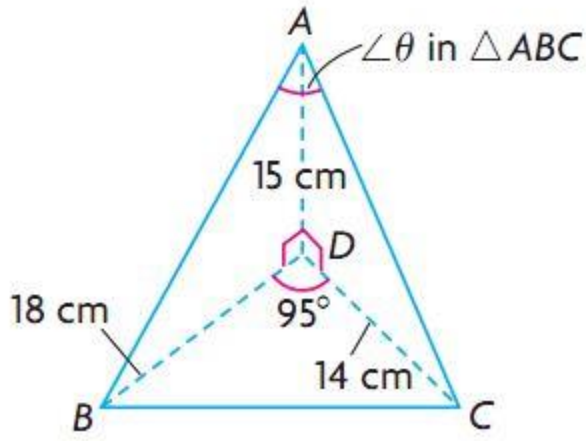
Example 5.8.1

From your text: Pg. 332 #4b

Solve for *x*



d) Solve for θ



Example 5.8.2

From your text: Pg. 333 #5

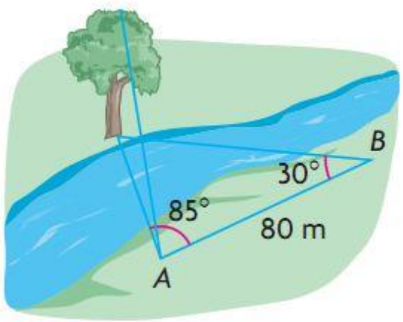
While Travis and Bob were flying a hot-air balloon from Beamsville to Vineland in southwestern Ontario, they decided to calculate the straight-line distance, to the nearest metre, between the two towns.

- From an altitude of 226 m, they simultaneously measured the angle of depression to Beamsville as $2°$ and to Vineland as $3°$.
- They measured the angle between the lines of sight to the two towns as $80°$.

Is there enough information to calculate the distance between the two towns? Justify your reasoning with calculations.

Example 5.8.3

From your text: Pg. 334 #11



Bert wants to calculate the height of a tree on the opposite bank of a river. To do this, he lays out a baseline 80 m long and measures the angles as shown at the left. The angle of elevation from *A* to the top of the tree is 28°. Explain if this information helps Bert to calculate the height of the tree to the nearest metre. Justify your reasoning with calculations.

HW: Section 5.8

Pg. 332 – 334 #3ac, 4a, 6, 9, Bonus: 7 (this one is tricky!!!)

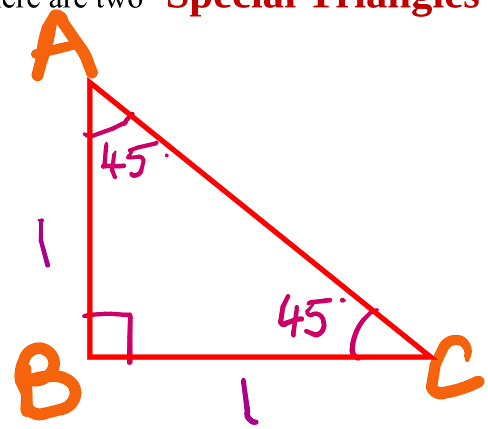
Success Criteria:

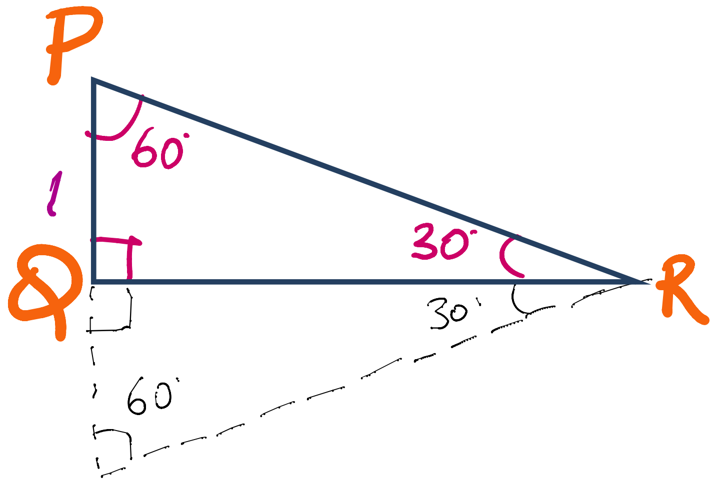
- I can sketch, to the best of my ability, a representation of the question
- I can identify the correct method to solve the unknown(s) in a given problem

5.2 – Trigonometric Ratios and Special Triangles

Learning Goal: We are learning to find the EXACT values of sin, cos, and tan for specific angles.

There are two "**Special Triangles**"





MEMORIZE THESE!

The Primary Trigonometric Ratios of the Special Angles

$\sin(30^\circ)$ $\sin(45^\circ)$ $\sin(60^\circ)$

$\cos(30^\circ)$ $\cos(45^\circ)$ $\cos(60^\circ)$

$\tan(30^\circ)$ $\tan(45^\circ)$ $\tan(60^\circ)$

Example 5.2.1

Evaluate <u>exactly</u> (ie without the use of calculators)

a) $\sin(45)\cdot\cos(60)$

b) $\cos^2(30)+\sin^2(30)$

c) $\tan(60)\cdot\cos(60)-\sin(60)$

d) $\tan(30)\cdot\frac{\sin(60)}{\cos(45)}$

Example 5.2.2

Determine the angle θ (where $0\le\theta\le 90^\circ$) given:

a) $\sec(\theta)=\frac{2}{\sqrt{3}}$

b) $\tan(\theta)=\frac{\sqrt{3}}{3}$

HW: Section 5.2

Pg 286 – 288 #3 – 9, 11, 13

Success Criteria:

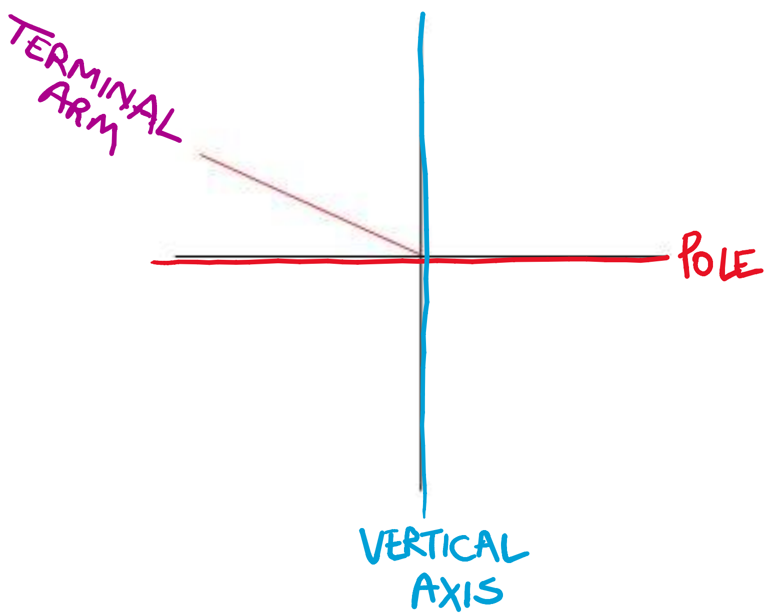
- I can draw the two special triangles
- I can identify the EXACT values for 30º, 45º, 60º, using the special triangles
- I can evaluate EXACTLY (no calculators…OR capes!!!) problems involving the special triangles

5.3 – 5.4 – Trigonometric Ratios for Angles Larger than 90º

Learning Goal: We are learning to use a Cartesian plane to evaluate trig ratios for angles between 0º and 360º.

Angles Larger than 90º

Consider the following sketch of the angle $\theta = 150^\circ$



e.g. Calculate (using calculator!)

$\sin(150)$

$\sin(30)$

$\cos(150)$

$\cos(30)$

$\tan(150)$

$\tan(30)$

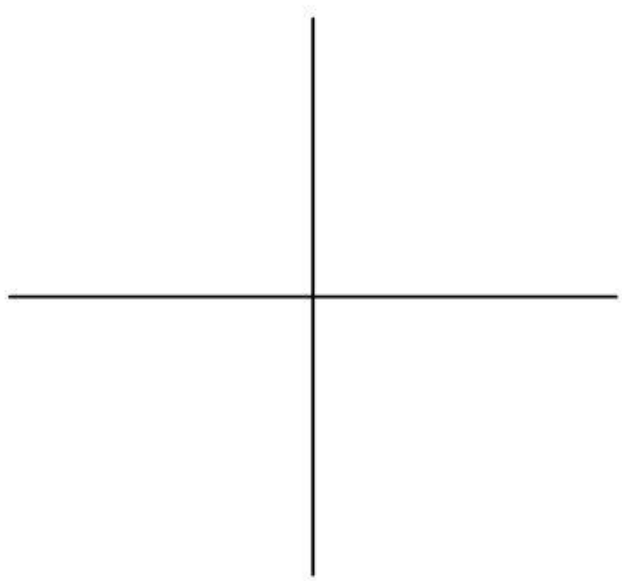
There clearly is a connection, BUT WHAT THE RIP IS GOING ON ?!!!!

In this example, we call $\theta = 150^\circ$ the **PRINCIPAL ANGLE**, or the **angle in Standard Position**

Note: The angle $\beta = 30^\circ$ is called the **RELATED ACUTE ANGLE**

Example 5.3.1

Sketch the angle of rotation $\theta = 225^\circ$ and determine the related acute angle.



e.g. Calculate

$\sin(225)$

$\sin(\quad)$

$\cos(225)$

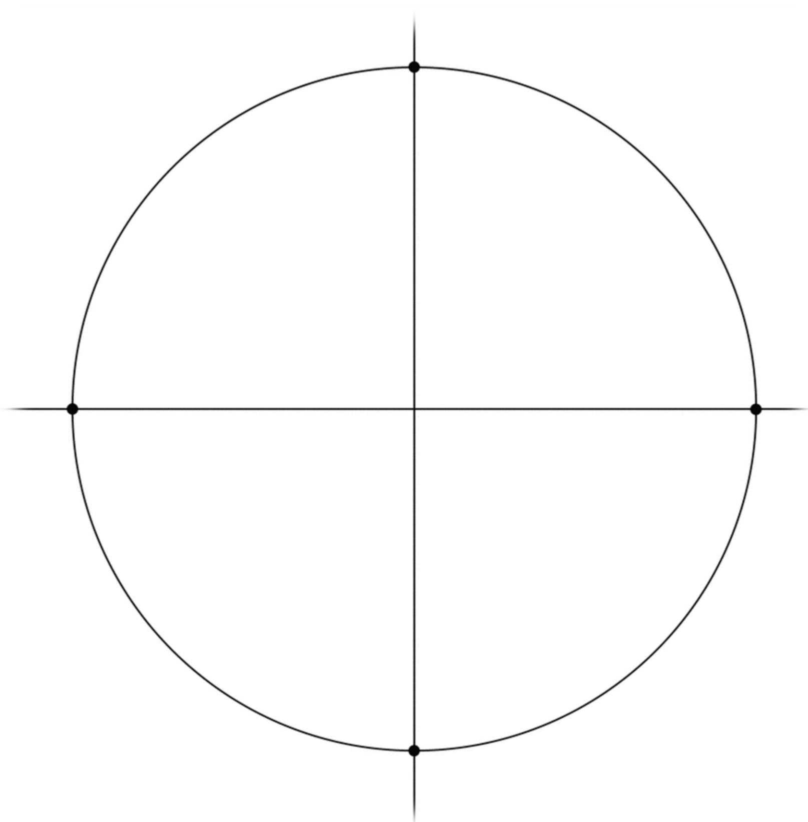
$\cos(\quad)$

$\tan(225)$

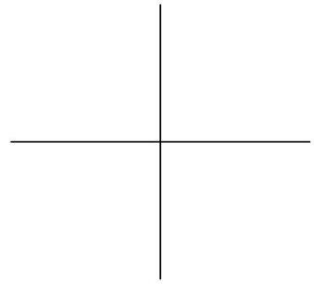
$\tan(\quad)$

What is up with these signs??? (**BE CAREFUL WITH YOUR SIGNS!!!!!!!!!**)

Looking at the TRIG ratios on a Cartesian Plane



The **CAST RULE** determines the sign (+ or -) of the trig ratio



We now have enough tools to calculate the trigonometric ratios of any angle!

For any given angle θ we will:

1) Draw θ in **STANDARD POSITION** (i.e. draw the principal angle for θ)
2) Determine the **RELATED ACUTE ANGLE** (β) (between the terminal arm and the x-axis (also called the polar axis))
3) Use the related acute angle and the **CAST RULE** (and SOH CAH TOA) to determine the trig ratio (along with its sign…BE CAREFUL WITH YOUR SIGNS) in question

Example 5.3.2

Determine the trig ratio $\sin(135)$

Determine the trig ratio tan (240)

Example 5.3.3

The point $P(x, y) = (6, 8)$ lies on the terminal arm (of length *r*) of an angle of rotation. Sketch the angle of rotation.

Determine: a) the value of *r*

b) the primary trig ratios for the angle

c) the value of the angle of rotation in degrees, to two decimal places

Example 5.3.4

The point $(-3, 5)$ lies on the terminal arm (of length *r*) of an angle of rotation. Sketch the angle of rotation.

Determine: a) the value of *r*

b) the primary trig ratios for the angle

c) the value of the angle of rotation in degrees, to two decimal places

Example 5.3.5 (going backwards!)

a) Given $\sin(\theta) = +\frac{1}{2}$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

b) Given $\cos(\theta) = -0.5372$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

c) Given $\sin(\theta) = -0.4567$ determine BOTH values of θ for $0^\circ \le \theta \le 360^\circ$

Section 5.3/5.4

Pg. 299 – 301 #1 – 3 (*For #3, **READ** example 3, pg. 296*), 5, 6 (*see example 5.3.4 above*), #8 – 10, 12

If you struggle with this stuff…ASK QUESTIONS in EDSBY!!! (and in class too!)

Success Criteria:

- I can identify a positive or negative angle based on the direction of rotation
- If the principal angle (Θ) lies in quadrants 2, 3, or 4 there is a related acute angle, β
- I can identify where a trigonometric ratio is + or – using the CAST Rule
- Every trigonometric ratio has two principal angles between 0º and 360º

 Name____________________.

5.5 – Trigonometric Identities

Learning Goal: We are learning to prove trigonometric identities.

Proving Trigonometric Identities is so much fun it's **ridiculous**!

Let's start with a simple identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Recall:

Our second identity:

$$\sin^2\theta + \cos^2\theta = 1$$

When proving trig identities, it's helpful to keep a few things in your mind. Things such as:

- The Reciprocal Trig Identities
- Converting everything to sin and cos can be helpful
- Start with the side which has the most "stuff" to work with, and work toward the other side
- A few special formulas, which we need to find…

Example 5.5.1

Prove $\cos(x)\tan(x) = \sin(x)$

Example 5.5.2

Prove $1 + \cot^2(x) = \csc^2(x)$

Example 5.5.3

From your text: Pg. 310 #8b

Prove $\frac{\tan^2 \alpha}{1+\tan^2 \alpha} = \sin^2 \alpha$

Example 5.5.4

Prove $1-2\cos^2 \phi = \sin^4 \phi - \cos^4 \phi$

Example 5.5.5

Prove $\sin\theta + \sin\theta\cot^2\theta = \csc\theta$

Example 5.5.6

Prove $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$

HW: Section 5.5

Handout

Success Criteria:

- I can prove trig identities using a variety of strategies:
  - Using the reciprocal, quotient, and Pythagorean identities
  - Factoring
  - Converting to sin and cos
  - Common denominators
- I can recognize the proper form to proving trigonometric identities