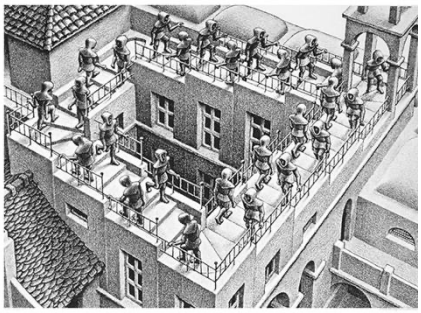
To Infinity & Beyond

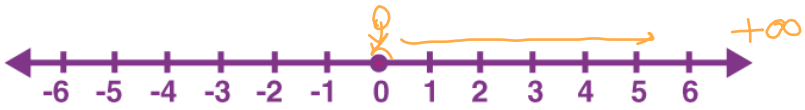
Learning Goals:

- To understand countable infinities
- To improve problem-solving skills

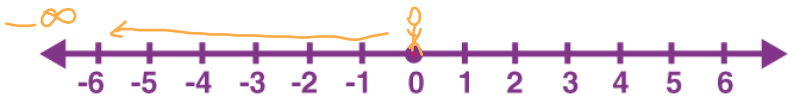
What is infinity? It is not a number.
It is a concept or an idea of going on forever.



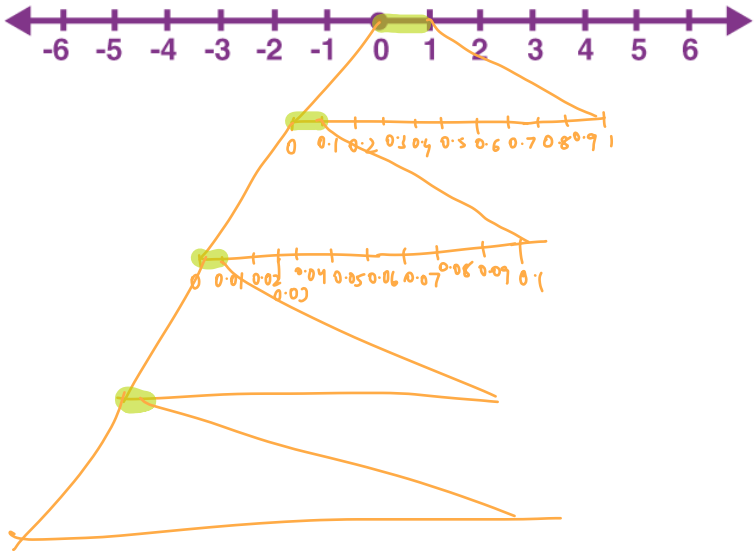
On a number line, you can go towards positive infinity…



Or you can go towards negative infinity…

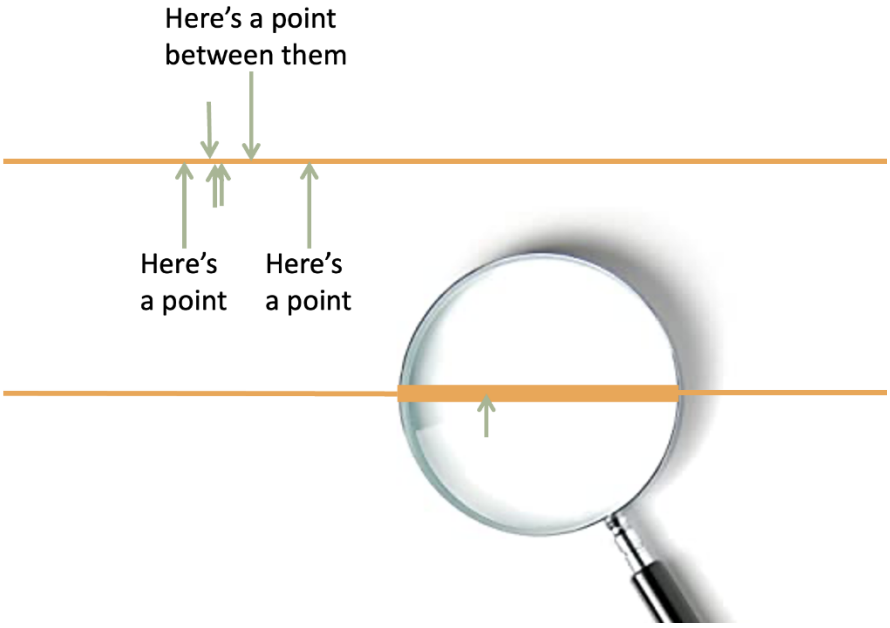


How dense is our Real Number Line?



Our Real Number line is extremely dense!!

Clearly, squeezed between *every* 2 points, there is an infinite number of points!
Even if those 2 points are very close together! And each of these points represents a number on our Real Number Line.



So, a quick question to wonder:
Is there only one infinity or are there different kinds of infinity?

a) Yes
b) No

There are an infinite *of types of infinities!!!*

Countable Infinity

What is a **countable infinity**?

- Also called a "LISTABLE ____" infinity
- An infinite set of numbers that we can list.
- A set is countable if you can count its elements. Of course, if the set is finite, you can easily count its elements. If the set is infinite, being countable means that you are able to put the elements(numbers) of the set, in order, just like natural numbers are in order.

Examples:

N = 1, 2, 3, 4, 5, ….

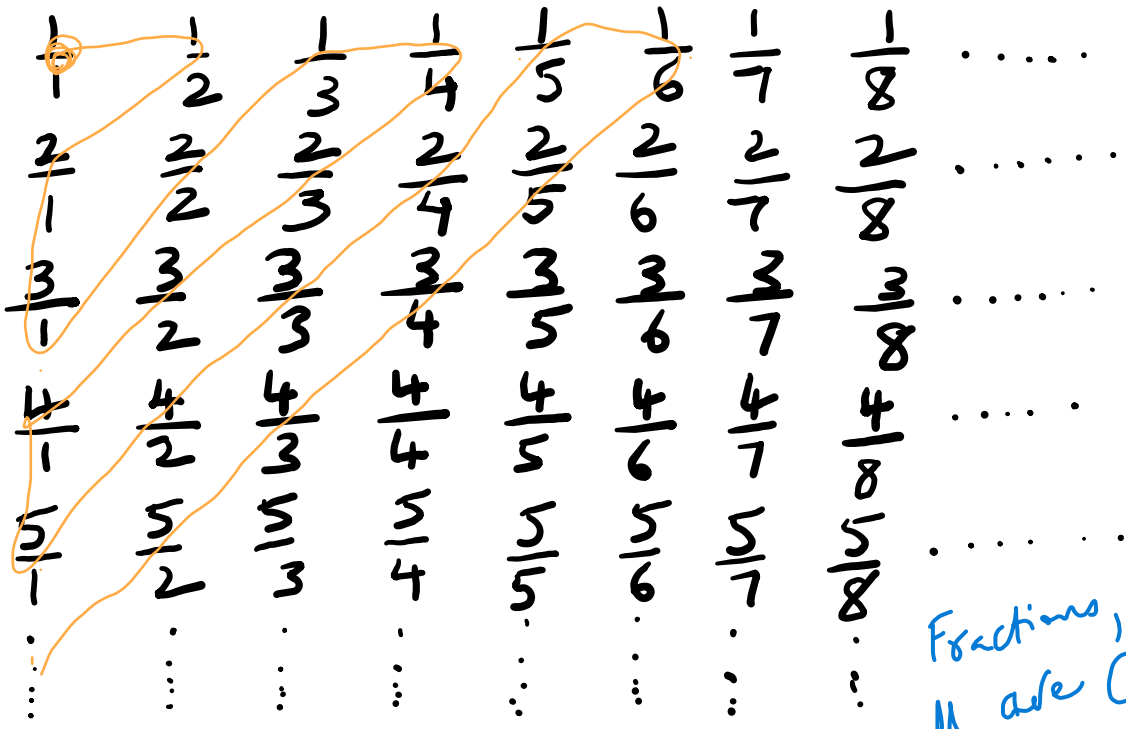
W = 0, 1, –1, 2, –2, 3, –3, 4, –4, 5, ….

: ½, ¼, ⅛, …

3.14159265… (digits of pi)

The number of hours in forever

Is the set of all fractions also a countable infinity? Can we list all the possible fractions?



Fractions, Rational #s all are COUNTABLE also.

Which of these countable infinities is the largest?

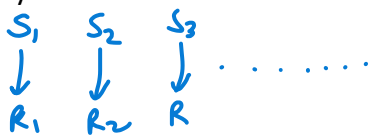
a) 2, 4, 6, 8, 10,….
b) 1, 2, 3, 4, 5, ….
c) 1, –1, 2, –2, 3, –3, 4, –4,…
d) ✓ They are all the same size

All of these countable infinities are the SAME. If the infinity is listable, it is all the same!!! We call the size of a countable infinity: $\aleph_0$ *"Aleph null"*

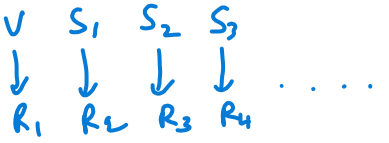
The Infinite Hospitality Problem:

You are the owner of an infinite hotel, which has an infinite number of rooms, labelled room 1, room 2, room 3, room 4, etc…

a) One day, an infinite bus with an infinite number of passengers pulls up to your infinite hotel. Each seat on the bus is labeled seat 1, seat 2, seat 3, seat 4, etc… The driver of the bus stops by your front desk and asks, "Can all my passengers find rooms in your hotel?" What instruction can you give the passengers to assign every person on the bus a room in your hotel?



b) What if **two vehicles** are looking for shelter: a single passenger in a Volkswagon + an infinite bus full of infinite passengers, each with seats numbered seat 1, seat 2, seat 3, seat 4, … No one wants to share a room. What instruction will you give the passenger in the Volkswagon and the bus riders now?



passenger in Volkswagon → R_1

All in the infinite bus go to room # = Seat # + 1

Seat # = x ⇒ Room # = $x+1$

c) What if **two infinite buses** arrive? Both buses are filled with infinite passengers sitting on seats numbered seat 1, seat 2, seat 3,… No one wants to share a room. What instruction will you give the passengers on bus A? What instruction will you give the passengers on bus B?

one bus passengers go to odd rooms

other go to the even rooms

Bus 1: Seat # = x ⇒ Room # $2x$

Bus 2: Seat # x ⇒ Room # $2x-1$

Uncountable Infinities

Learning Goals:

- To become familiar with examples of uncountable infinities
- To develop an intuition for the different sizes of countable and uncountable infinities

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Countable infinities can be counted or be listed.

But what is an **un**countable infinity?

Example: How many infinitely tiny points are there on our number line? Each point on our number line can be represented by a decimal number. Some decimals are terminating, some are non-terminating but repeating, while the rest are non-terminating and non-repeating.

As observed earlier, squeezed between *every* 2 points, there is an infinite number of points! Even if those 2 points are very close together!

The number of points on a line is an **uncountable infinity**. ∴ Real #s Set is Uncountable

The size of uncountable infinities is always greater than $\aleph_0$

Is it possible to list out every decimal number on our Number line?

No.

But even if we assumed it was possible, the German mathematician George Cantor found a way to show that not all decimals can be listed.

Cantor's Diagonal Proof:

Imagine trying to write down all the real numbers between 0 and 1.

N1. 0.000012211173…
N2. 0.11122000044…
N3. 0.22232111110… 0.0123456....
N4. 0.31334210011…
N5. 0.42424242424…
…

Overall idea: No matter what strategy you use for writing down these real numbers, you'll always be able to find a number you missed squeezed in somewhere

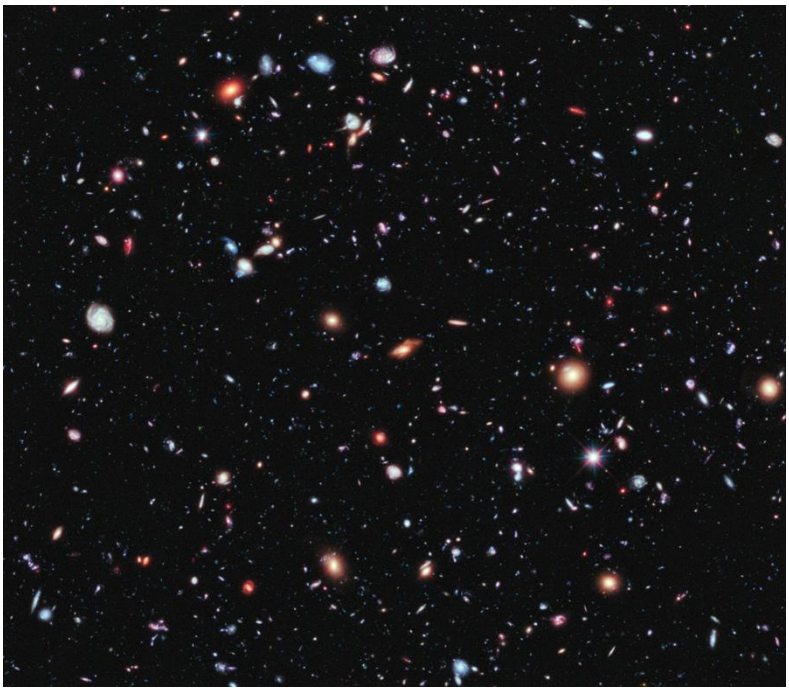
Here's a way to find a number you definitely missed (even though you've made an infinitely long list):

look above
Cantor's Diagonal.

Conclusion: Our Real numbers are a set of uncountably infinite numbers.

Is the Universe Infinite?

- Physicists pointed Hubble telescope at a tiny uninteresting pinprick of sky in the middle of nowhere and took long exposure photos for 10 days.
- And in that tiny pinprick, they discovered hundreds of distant galaxies – whole galaxies, each filled with billions of stars!



So is the universe infinite?

I don't know

Practice with Infinities:

Which is larger?

a) The number of minutes in forever
b) The number of seconds in forever
c) Both are the same size

Which number is closer to infinity?

a) 5
b) 5,000,000,000,000
c) Both are the same distance to infinity

Which is larger?

a) The number of real numbers
b) The number of integers
c) Both are the same size

Which is larger?

a) The number of moments in the life of a baby
b) The number of moments in the life of an elderly person
c) Both are the same size

What is $\aleph_0 - 1$? $= \aleph_0$

What is $2 \times \aleph_0$? $= \aleph_0$