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MATH@TD

Chapter 4 – Exponential Functions

4.7 – Applications of Exponential Functions

Learning Goal: We are learning to use exponential functions to solve problems involving exponential growth and decay.

Anything in the real world which grows, or decays can be "**MODELED**" (or in some sense "***DESCRIBED***") with words, or pictures or mathematics. **Mathematical models** are useful for getting solutions to problems, and making predictions.

So far in Mathematics 11U we have studied the basics of functions in general (chapter 1), we've done some algebra (chapter 2), and we've examined Quadratic functions (chapter 3). Part of our study of Quadratics was learning how to use the vertex of a parabola to answer questions about maxima and minima for some real word problems. For example we saw a question where we tried to maximize revenue for a school store. Quadratic **MODELS** are very useful for solving max/min problems.

In this lesson we want to work on **LEARNING HOW TO SOLVE PROBLEMS DEALING WITH GROWTH AND DECAY.** We have to decide what type of function will best model (or describe) the type of growth/decay seen in the problem (hint: for this lesson we'll be examining Exponential Growth and Decay, and therefore we expect that exponential functions will be used…shocking, I know)

Q. What is Exponential Growth or Decay?

Consider the following:

A single cell divides into two "daughter" cells. Both daughter cells divide resulting in four cells. Those four cells each divide and we now have a population of 8

Describe, using mathematics, how the cell population changes from generation to generation.

generation	G_0	G_1	G_2	G_3	G_4	G_5
# of cells	1	2	4	8	16	32
powers	2^0	2^1	2^2	2^3	2^4	

n is a whole number

Model : $G(n) = 2^n$, $n \in \mathbb{N} + \{0\}$

Example 4.7.1

%'s are NOT numbers convert to decimal

Being a financial wizard, you deposit \$1,000 into an account which pays 3.5% interest, annually.

$\frac{3.5}{100} = 0.035$

a) Determine who much money is in your account after t = 1,2,3, and 4 years.

b) Determine a mathematical model which can describe how the value of the account is changing from year to year.

a) year 0: 1000

year 1: \$1000 + interest on \$1000

$= \boxed{1000 + (0.035)(1000)}$

$= 1035$

$\boxed{1000(1+0.035)}$

year 2: $\boxed{1035 + (0.035)(1035)}$

$= 1071.23$

$\boxed{1035}(1+0.035)$

$[1000(1.035)](1.035)$

$= 1000(1.035)^{2}$

year 3: $1000(1.035)^3$

year 4

starting amount

length of time

b) $\boxed{A(t) = 1000(1.\boxed{035})^{t}}$, t is the # of years

growth rate

Definition 4.7.1

A function describing Exponential Growth is of the form:

$$A(t) = \underbrace{A_0}_{\text{A naught}}(1 \boxed{+} r)^{t}$$

growth — amount of time — growth rate

A function describing Exponential Decay is of the form:

decay

$$P(t) = P_0(1 - r)^{t}$$

Both formulas in one line

$$A(t) = A_0(1 \pm r)^{t}$$

growth (+), decay (−)

$$\boxed{A(t) = A_0(1 \pm r)^t}$$

Example 4.7.2

From your text, Pg. 263

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.

$C(w)$

a) the percent of colour left if blue jeans lose 1% of their colour every time they are washed

decay

b) the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for *t* years

a) $C(w) = C_0(1-r)^w$

$C(w) = 1(1-0.01)^w$

$C(w) = (0.99)^w$

$$\boxed{\begin{aligned} C_0 &= 100\% = 1 \\ r &= 1\% = 0.01 \end{aligned}}$$

b) $P(t) = P_0(1+r)^t$, $P_0 = 2500$, $r = 0.005$

$P(t) = 2500(1.005^t)$

Example 4.7.3

decay

A new car depreciates at a rate of 20% per year. Steve bought a new car for \$26,000. , $A_0 = 26\,000$

$r = 0.2$

a) Write the equation that models this scenario.

$A(t) = A_0(1-r)^t \Rightarrow A(t) = 26000(0.8^t)$

b) How much will Steve's car be worth in 3 years?

$A(3) = 26\,000(0.8^3)$

$= \$13\,312$

A logarithm "undoes" exponent (like ÷ undoes ×)

$\log(a^b) = b \cdot \log(a)$

power is isolated!

c) When will Steve's car be worth \$4000? $A(t)$

t is unknown

$A(t) = A_0(1-r)^t$

$4000 = 26\,000(0.8^t)$

isolate the power ÷ 26000

$\Rightarrow \dfrac{4000}{26\,000} = 0.8^t$

$\dfrac{2}{13} = 0.8^t$

$\log\left(\frac{2}{13}\right) = \log(0.8^t)$

$\Rightarrow \log\left(\frac{2}{13}\right) = t \cdot \log(0.8)$

$\Rightarrow \quad t = \dfrac{\log\left(\frac{2}{13}\right)}{\log(0.8)} \doteq 8.4$ years

Additional Applications – *Doubling and Half-Life*

Thus far, we have only seen examples with single period rates: "yearly" "monthly" "daily"

Unfortunately, it's not always that simple…Our rates could be…

Every 3 years   Every 6 hours   Every 4 days   (chunked time)

How do we deal with the exponent in these cases?

we chunk the time by $\frac{t}{D}$. D is the doubling period

Example (Doubling)

$P(t) = P_0(1+r)^{\frac{t}{D}}$

A species of bacteria has a population of 300 at 9 am. It doubles every 3 hours. $P_0 = 300 \quad r = 100\% = 1$

a) Write the function that models the growth of the population, P, at any hour, t

$P(t) = 300(2)^{t/3}$

base 2 ⇒ doubling problem.

b) How many will there be at 6 pm?

9am → 6pm = 9 hours $t = 9$

$P(9) = 300\left(2^{9/3}\right) = 300\left(2^3\right) = 2400$ bacteria.

c) How many will there be at 11 pm? $t = 14$

$P(14) = 300\left(2^{14/3}\right) = 7169$ bacteria.

7619.5 – round down always (since 0.5 bacteria don't exist)

d) Determine the time at which the population first exceeds 3000. $= P(t)$

want t

$P(t) = 300\left(2^{t/3}\right)$

$\Rightarrow \quad 3000 = 300 \cdot 2^{t/3}$ $\div 3000$

$10 = 2^{t/3}$

$\log(10) = \frac{t}{3} \cdot \log(2)$ $\div \log(2)$

$\Rightarrow \quad \dfrac{\log(10)}{\log(2)} = \dfrac{t}{3}$ $\times 3$

$\Rightarrow \quad t = \dfrac{3 \cdot \log(10)}{\log(2)} \doteq 10$ hours decay

$A(t) = A_0 (1 - r)^{t/h}$, h = half-life. $r = \frac{1}{2}$

∴ At about 7pm the population is ~3000 bac.

Example (Half-Life)

A 200g sample of radioactive material has a half-life of 138 days. How much will be left in 5 years?

$$A(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

Note: h is in days ⟹ t is in days also

5 years = $5 \times 365.25 = 1826.25$

$$A(1826.25) = 200\left(\frac{1}{2}^{\frac{1826.25}{138}}\right)$$

$$= 0.021 \text{ g.}$$

Success Criteria:

- I can differentiate between exponential growth and exponential decay
- I can use the exponential function $f(x) = ab^x$ to model and solve problems involving exponential growth and decay
    - Growth rate is $b = 1 + r$. Decay rate is $b = 1 - r$.
    - r is a DECIMAL, not a percent!!!!!