Unit 5 – Trigonometric Ratios

5.5 – Trigonometric Identities

Learning Goal: We are learning to prove trigonometric identities.

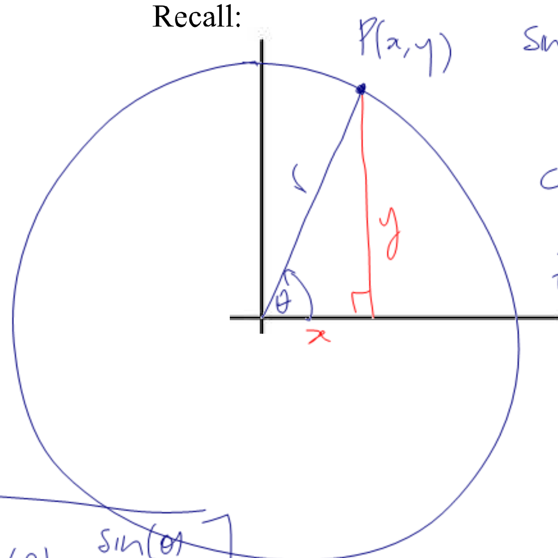
Proving Trigonometric Identities is so much fun it's **ridiculous**!

SOH CAH TOA

Let's start with a simple identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Recall:



$\sin(\theta) = \dfrac{y}{r}$ (*)

$\cos(\theta) = \dfrac{x}{r}$ (**)

$\tan(\theta) = \dfrac{y}{x}$ (~)

(*) $\Rightarrow y = r \cdot \sin(\theta)$

(**) $\Rightarrow x = r \cdot \cos(\theta)$

sub into (~)

$\Rightarrow \tan(\theta) = \dfrac{\cancel{r}\sin(\theta)}{\cancel{r}\cos(\theta)} \Rightarrow \boxed{\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}}$

Our second identity:

$$\sin^2\theta + \cos^2\theta = 1$$

(A pythagorean trigonometric identity)

By Pythagorus, the triangle above shows

$$x^2 + y^2 = r^2$$

by (*) & (**)

$$(r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2$$

$$\Rightarrow r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$$

$$\Rightarrow r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2$$

$\div r^2$

$$\Rightarrow \boxed{\cos^2(\theta) + \sin^2(\theta) = 1}$$

$\Rightarrow \sin^2(\theta) = 1 - \cos^2(\theta)$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

When proving trig identities, it's helpful to keep a few things in your mind. Things such as:

- The Reciprocal Trig Identities
- Converting everything to sin and cos can be helpful
- Start with the side which has the most "stuff" to work with, and work toward the other side
- A few special formulas, which we need to find…
- helpful to factor if you can

Example 5.5.1

Prove $\cos(x)\tan(x)=\sin(x)$

$$LS = \cos(x)\cdot\tan(x)$$

$$= \cancel{\cos(x)}\cdot\frac{\sin(x)}{\cancel{\cos(x)}}$$

$$= \sin(x) = RS \square$$

since $\cot(\theta)=\frac{1}{\tan\theta}$

$\Rightarrow \cot(\theta)=\frac{\cos\theta}{\sin\theta}$

Example 5.5.2

Prove $1+\cot^2(x)=\csc^2(x)$

$$LHS = 1+\cot^2(x)$$

$$= \frac{1}{1}+\frac{\cos^2(x)}{\sin^2(x)}$$

(get a common denom → promote world peace

$$= \frac{\sin^2(x)}{\sin^2(x)}+\frac{\cos^2(x)}{\sin^2(x)}$$

$$= \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}$$

$$= \frac{1}{\sin^2(x)} = \csc^2(x) = RS \square$$

The Other Pythagorean Trig Identities

Prove $1+\tan^2(\theta)=\sec^2(\theta)$

$$LS = 1+\tan^2(\theta)$$

$$= 1+\frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2(\theta)+\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

$$= \sec^2(\theta) = RS \square$$

Example 5.5.3

From your text: Pg. 310 #8b

Prove $\frac{\tan^2\alpha}{1+\tan^2\alpha} = \sin^2\alpha$

$$LS = \frac{\tan^2\alpha}{1+\tan^2\alpha}$$

$$= \frac{\frac{\sin^2\alpha}{\cos^2\alpha}}{1 + \frac{\sin^2\alpha}{\cos^2\alpha}}$$

$$= \frac{\frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha}{\cos^2\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha}}$$

$$= \frac{\frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}}$$

$$= \frac{\frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha}}$$

$$= \frac{\sin^2\alpha}{\cancel{\cos^2\alpha}} \cdot \frac{\cancel{\cos^2\alpha}}{1}$$

$$= \sin^2\alpha = RS \;\square$$

Another path

$$LS = \frac{\tan^2\alpha}{1+\tan^2\alpha}$$

$$= \frac{\tan^2\alpha}{\sec^2\alpha}$$

$$= \frac{\frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha}}$$

$$= \sin^2\alpha. = RS \;\square$$

Example 5.5.4

Prove $1-2\cos^2\phi = \sin^4\phi - \cos^4\phi$

$$RS = \sin^4(\phi) - \cos^4(\phi)$$

difference of squares

$$a^2 - b^2 = (a-b)(a+b) \Rightarrow a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$$

$$= \left(\sin^2(\phi) - \cos^2(\phi)\right)\left(\sin^2(\phi) + \cos^2(\phi)\right)$$

1

$$= \sin^2(\phi) - \cos^2(\phi)$$

$$LS = 1 - 2\cos^2(\phi)$$

collect like terms

$$= \underbrace{\sin^2(\phi) + \cos^2(\phi)} - 2\cos^2(\phi)$$

$1x - 2x = -x$

$$= \sin^2(\phi) - \cos^2(\phi)$$

$$= RS \;\square$$

Example 5.5.5

Prove $\sin\theta + \sin\theta \cot^2\theta = \csc\theta$

$\cancel{a} \cdot \frac{b}{a^4}$

$LS = \sin\theta + \sin\theta \cdot \cot^2(\theta)$

$= \sin(\theta) + \cancel{\sin(\theta)} \cdot \frac{\cos^2(\theta)}{\sin^{\cancel{2}}(\theta)}$

$= \frac{\sin(\theta)}{1} + \frac{\cos^2\theta}{\sin\theta}$ get a common denom $(\sin\theta)$

$= \frac{\sin\theta}{1} \cdot \frac{\sin\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin(\theta)}$

$= \frac{1}{\sin(\theta)} = \csc(\theta) = RS$ $\square$

Example 5.5.6

Prove $\sec^2\theta + \csc^2\theta = \sec^2\theta \csc^2\theta$

$LS = \sec^2\theta + \csc^2\theta$

$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$ common denominator : $\cos^2(\theta) \cdot \sin^2\theta$

$= \frac{\sin^2\theta}{\cos^2\theta \cdot \sin^2\theta} + \frac{\cos^2\theta}{\sin^2(\theta)\cos^2\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \cdot \sin^2\theta}$

$= \frac{1}{\cos^2\theta \cdot \sin^2\theta}$

$= \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta}$

$= \sec^2\theta \cdot \csc^2\theta = RS$ $\square$

Success Criteria:

- I can prove trig identities using a variety of strategies:
  - Using the reciprocal, quotient, and Pythagorean identities
  - Factoring
  - Converting to sin and cos
  - Common denominators
- I can recognize the proper form to proving trigonometric identities (begin with only one side)