CALCULUS

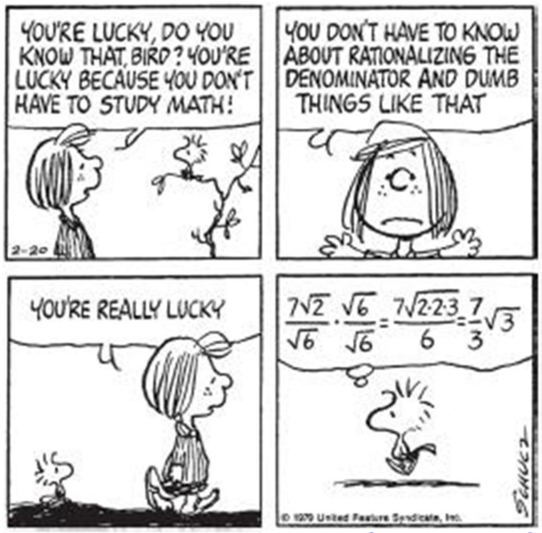
Chapter 1 –Introduction to the Calculus

(*Material adapted from Chapter 1 of your text*)

A∞Ω

MATH@TD

1.1 Radical Expressions: Rationalizing



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In the above cartoon Peppermint Patty calls the bird lucky for not having to know how to rationalize radicals (square roots, really). As it turns out, the Woodstock is actually lucky because he *can* rationalize. In Calculus being able to rationalize a denominator (or a numerator) is a necessary skill, and so we'll spend a little time honing that skill.

Definition 1.1.1

The **conjugate** of a binomial expression $a+b$ is $a-b$.

Example 1.1.1

$c - d \Rightarrow c + d$

Determine the conjugate of

a) $5-\sqrt{3x}$ b) $\sqrt{2x^2}+\sqrt{10}$

$5 + \sqrt{3x}$ $\sqrt{2x^2} - \sqrt{10}$

We can use the conjugate to **rationalize** a binomial expression which contains square roots. That is to say, we can **eliminate the square roots** (the irrational part) of a binomial expression (sort of).

$(1+\sqrt{2})(1-\sqrt{2})$

$= 1^2 - (\sqrt{2})^2 = 1 - 2$

$= -1$ radical is 'gone'

$(a+b)(a-b)$

$= a^2 - b^2$

Example 1.1.2

Rationalize the denominator of $\dfrac{3+2x}{\sqrt{8+h}-\sqrt{h}}$

Note: Only "conjugate" the part of the expression indicated. So, in this example, **leave the numerator alone**. Also remember **FACTORED FORM IS YOUR FRIEND**.

(if there is a variable)

• if we have only numbers, simplify everything

$$= \frac{3+2x}{\sqrt{8+h}-\sqrt{h}} \cdot \frac{\sqrt{8+h}+\sqrt{h}}{\sqrt{8+h}+\sqrt{h}}$$

$$= \frac{(3+2x)(\sqrt{8+h}+\sqrt{h})}{(8+h)-(h)} = \frac{(3+2x)(\sqrt{8+h}+\sqrt{h})}{8}$$

Example 1.1.3

Rationalize the numerator of $\dfrac{\sqrt{7}+5}{3\sqrt{7}-6}$.

$$= \frac{\sqrt{7}+5}{3\sqrt{7}-6} \cdot \frac{\sqrt{7}-5}{\sqrt{7}-5}$$

don't leave the denominator in factored form since we have no variables. FOIL

$(3\sqrt{7}-6)(\sqrt{7}-5)$

$$= \frac{7-25}{3(7)-15\sqrt{7}-6\sqrt{7}+30}$$

$$= \frac{-18}{51-21\sqrt{7}}$$

Note: "-3" is a common

$$= \frac{-\cancel{18}\,6}{\cancel{-3}(-17+7\sqrt{7})} = \frac{6}{7\sqrt{7}-17}$$

Class/Homework for Section 1.1

Pg. 9 #2, 3, 5 – 7

1.2 The Slope of a Tangent

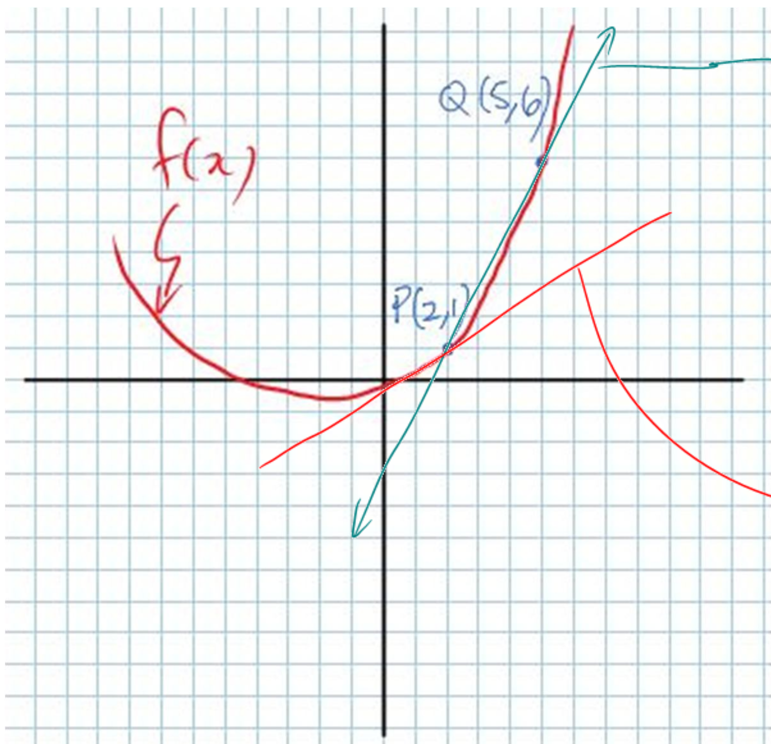
Advanced Functions, con't.

This concept is a key to unlocking the tool box of Differential Calculus.

We'll begin by looking at a couple of examples.

Example 1.2.1

Consider the diagram:



secant line will slope represents the AROC of f(x) over x ∈ [2,5]

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$$

$$= \frac{f(5) - f(2)}{5 - 2}$$

tangent line will slope m_{tan} ⇒ represents the IROC of f(x) at P(2,1)

Problem (As seen in Advanced fns)

We $\boxed{\text{CANNOT}}$ calculate the slope of the tangent using the traditional slope formula

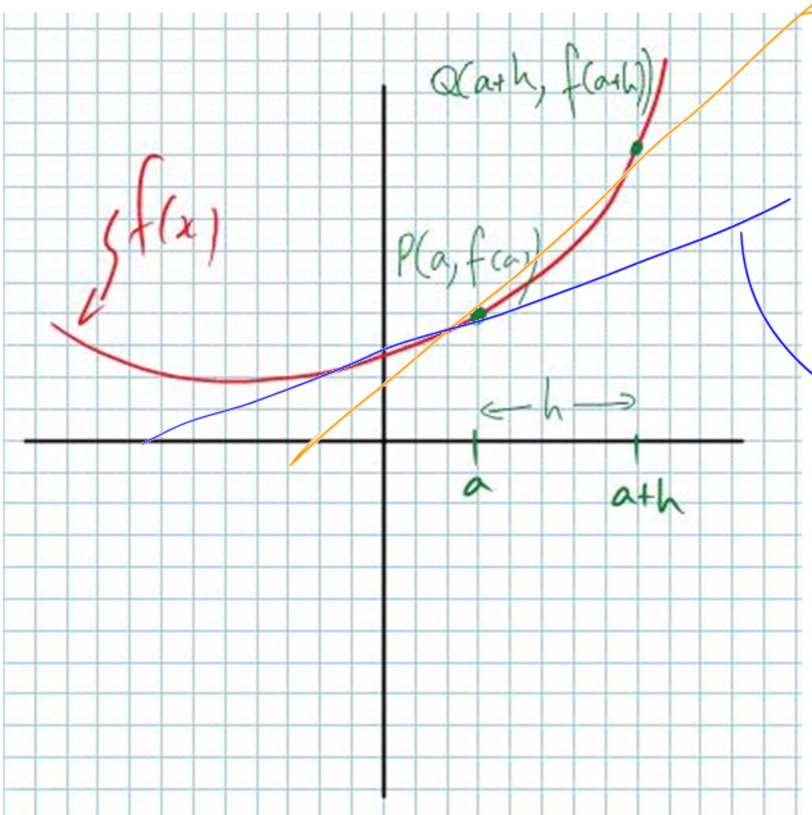
Question: Why can we **always** calculate the slope of a secant?

↳ we always have two points

But we can approximate $m_{tan} \sim m_{sec}$ IF P and Q are close together

Example 1.2.2

Consider the diagram:



$$m_{sec} = \frac{f(a+h) - f(a)}{h} \leftarrow (a+h) - a.$$

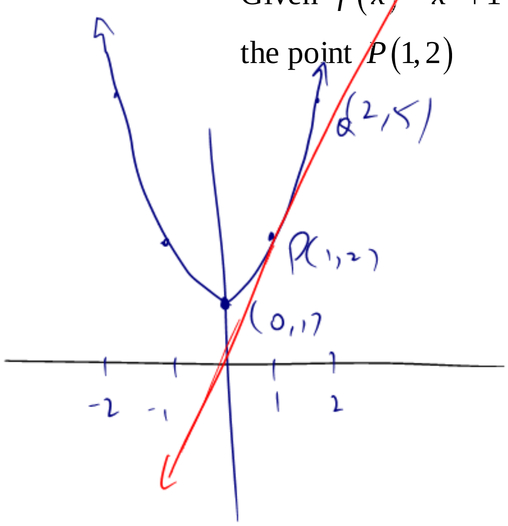
This ↑ is the awesome Difference Quotient.

$m_{tan} \sim m_{sec}$ if h is small

Example 1.2.3

Given $f(x) = x^2 + 1$ ***numerically*** approximate the slope of the tangent to the function at the point $P(1,2)$

$m_{tan} = ?$



Pictures are as much your **friends** as Factors are

$$m_{tan} \sim m_{sec} = \frac{f(a+h) - f(a)}{h}, \text{ if } h \text{ is small}, h \neq 0$$

$a = 1$

$Q(a+h, f(a+h))$

varies w/ h $\rightarrow$ $h = 1 \Rightarrow Q(1+1, f(1+1)) = (2, 5)$

h	P	Q	m_{sec}
1	(1,2)	(2,5)	$3 = \frac{5-2}{2-1}$ $\left(\frac{\text{rise}}{\text{run}}\right)$
0.5	(1,2)	(1.5, 3.25)	2.5
0.25	(1,2)	(1.25, f(1.25))	2.25
0.1	(1,2)	(1.1, 2.21)	2.1
0.01	(1,2)	(1.01, 2.02)	2.01

$f(1.01) = (1.01)^2 + 1$

$f(x) = x^2 + 1$

$f(☺) = ☺^2 + 1$

$f(\square + \triangle) = (\square + \triangle)^2 + 1$

Consider the following simplification

$$m_{tan} \sim \frac{f(1+h) - f(1)}{h}$$

$$= \frac{[(1+h)^2 + 1] - 2}{h}$$

$$= \frac{(1 + 2h + h^2 + 1) - 2}{h}$$

$$= \frac{2h + h^2}{h} = \frac{\cancel{h}(2+h)}{\cancel{h}} = 2 + h$$

$\Rightarrow m_{tan} \sim 2$

Algebraic Technique

"tends to, but never gets there

In making h smaller and smaller (that is, as we let $h \to 0$), we are actually using what we call a **limit technique**.

If we write for the slope of a secant to a function

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

then, $$m_{\tan} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

Siyi needs to be happy

Example 1.2.4

Determine the slope of the tangent to $f(x) = 3x^2 + 1$ at $x = 2$. $a = 2$

$$m_{\tan} = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(3(2+h)^2 + 1) - (3(2)^2 + 1)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{3(4 + 4h + h^2) + 1 - 13}{h} \right)$$

"Take the limit"

$$= \lim_{h \to 0} \left(\frac{12h + 3h^2}{h} \right) = \lim_{h \to 0} \left(\frac{\cancel{h}(12 + 3h)}{\cancel{h}} \right) = 12$$

Example 1.2.5

Calculate the slope of the tangent to $g(x)=\sqrt{x+1}$ at $x=3$. $a=3$

$$m_{tan}=\lim_{h\to 0}\left(\frac{g(3+h)-g(3)}{h}\right)$$

$$m_{tan}=\lim_{h\to 0}\left(\frac{g(a+h)-g(a)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sqrt{(3+h)+1}-\sqrt{3+1}}{h}\right)$$

Conjugate(!!!!!!)

$$=\lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\right)\cdot\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$

leave denom. in FACTORED FORM

$$=\lim_{h\to 0}\left(\frac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)}\right)=\lim_{h\to 0}\left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{4+h}+2\right)}\right)$$

$$=\frac{1}{4}$$

Class/Homework for Section 1.2

Pg. 18 – 21 #4,6 – 9, 11, 16, 20 – 22.