1.4 The Limit of a Function (*Skipping 1.3*)

(*Geometric Point of View*)

Recall the definition of a function: → A number generator making points: $P(x, f(x))$

An algebraic rule which assigns exactly one element in a set called the range to each element in a set called the domain

e.g. Given $f(x)=3x^2+2$, then $f(2)=$

$f(2) = 14$

Consider the function $f(x)=\dfrac{x^2-9}{x-3}$. $= \dfrac{9-9}{3-3} = \dfrac{0}{0}$ ← NOT UNDEFINED

we say "$\frac{0}{0}$" is INDETERMINANT – we need to figure it out.

$f(3)=$???????

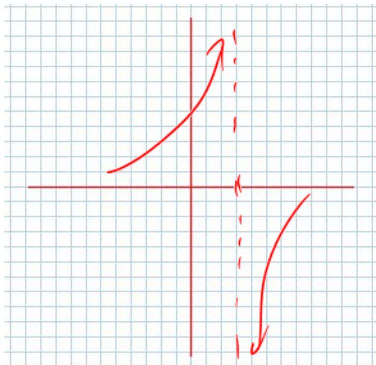
Now, we **can** calculate functional values such as:

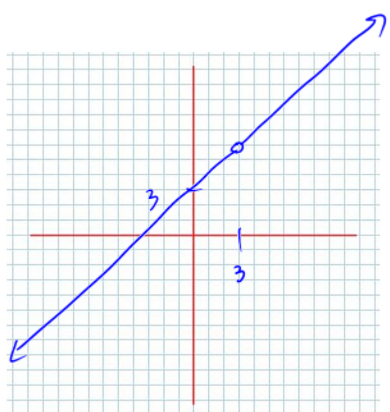
$f(2.999999999999999997)$ or $f(3.000000000000000000001)$, and these two functional values give hints to the **functional behaviour** of $f(x)$ **near** its problem domain value $x=3$

Two possible functional behaviours of $f(x)$ at $x=3$:

1) V.A

2) Hole





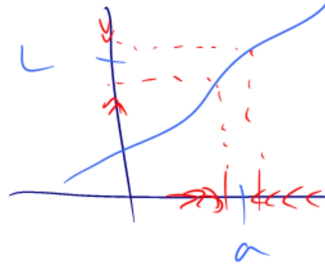
Note:

$\dfrac{x^2-9}{x-3}$

$= \dfrac{(x-3)(x+3)}{x-3}$

$= x+3$

IF $x \neq 3$.



Given $\varepsilon > 0$ $\exists$ $\delta > 0$ st. whenever $|x-a| < \delta$ $|f(x) - L| < \varepsilon$

Definition 1.4.1

Given $y = f(x)$ we write

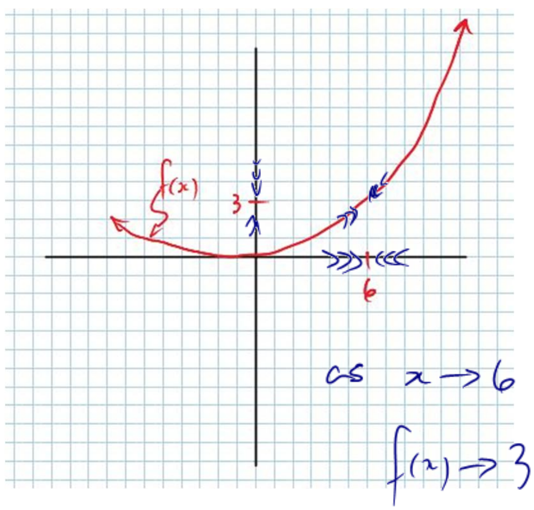
$$\lim_{x \to a} \left(f(x) \right) = L$$

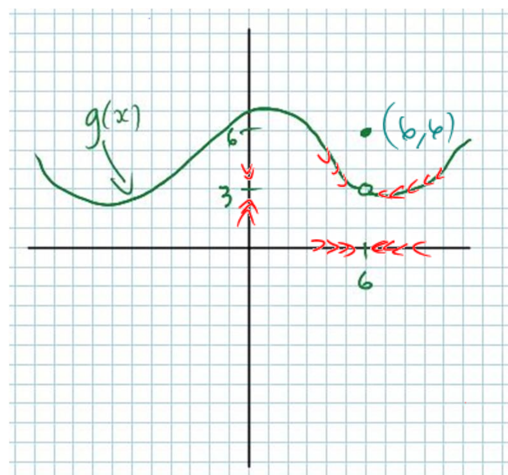
to mean as x gets RIDICULOUSLY close to the value 'a' $f(x)$ gets RIDICULOUSLY close to the value L.

L is called the limit.

CAUTION: WE CAN APPROACH NUMBERS FROM 2 SIDES (usually)

Pictures





$$\therefore \lim_{x \to 6} \left(f(x) \right) = 3$$

$$\lim_{x \to 6} \left(g(x) \right) = 3$$

even though $g(6) = 6$

Example 1.4.1

Consider the sketch of the Piece-wise define function

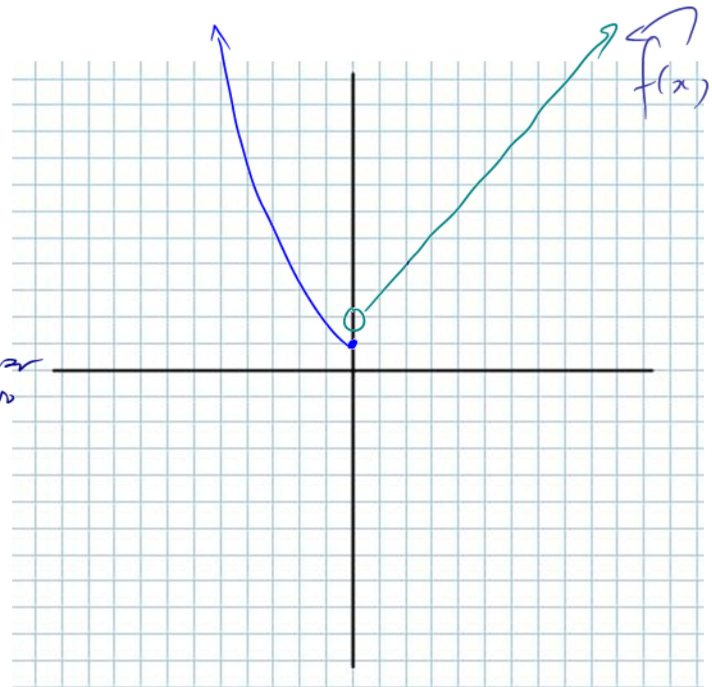
$$f(x)=\begin{cases} x^2+1, & x\le 0 \\ x+2, & x>0 \end{cases}$$

any # bigger than zero

Determine: a) $\lim\limits_{x\to 2}\left(f(x)\right)$

b) $\lim\limits_{x\to -1}\left(f(x)\right)$

c) $\lim\limits_{x\to 0}\left(f(x)\right)$



$f(0)=0^2+1$
$=1$

a) $\lim\limits_{x\to 2}\left(f(x)\right)$
$=\lim\limits_{x\to 2}\left(x+2\right)$
$=4$

b) $\lim\limits_{x\to -1}\left(f(x)\right)$
$=\lim\limits_{x\to -1}\left(x^2+1\right)$
$=2$

c) $\lim\limits_{x\to 0}\left(f(x)\right)$

which piece of the piece-wise defined fn do we plug in for f(x)?

Neither!!!

As it turns out

$\lim\limits_{x\to 0}\left(f(x)\right)$ d.n.e. does not exist.

We must consider **ONE SIDED LIMITS**

$\lim_{x \to a^-}$ $\lim_{x \to a^+}$ a

Notation: $\lim_{x \to a^-}$ means $x \to a$ from "below" or from the left

$\lim_{x \to a^+}$ means $x \to a$ from "above" or from the right

$$\lim_{x\to 0^-}\left(f(x)\right) = 1 \qquad \lim_{x\to 0^+}\left(f(x)\right) = 2$$

because the two one sided limits do not match

$$\therefore \lim_{x\to 0}\left(f(x)\right) = \text{dne}$$

no "− or +" on "0" means approaching from both sides simultaneously

Definition 1.4.2

Given a function $f(x)$, then

$\lim_{x\to a}\left(f(x)\right) = L$ exists iff

iff ⇒ if and only if (Notation is usually $\Longleftrightarrow$)

$$\lim_{x\to a^-}\left(f(x)\right) = L = \lim_{x\to a^+}\left(f(x)\right)$$

Thus, in **Example 1.4.1** c) $\lim_{x\to 0}\left(f(x)\right)$ dne.

Note: We really only need to calculate one sided limits if:

1) We are finding a limit at a "**break-point**" of a piece-wise defined function.
2) At "**restrictions**" in domain values.

e.g. for $f(x)=\sqrt{x}$,

$\lim_{x\to 0^-}\left(f(x)\right)$ has no meaning, and so we can only consider $\lim_{x\to 0^+}\left(f(x)\right)$

Example 1.4.2

Calculate: a) $\lim_{x\to 3}(3x) = 9$

b) $\lim_{x\to -2}\left(\frac{x^2}{4}\right) = \left(\frac{(-2)^2}{4}\right) = 1$

c) $\lim_{x\to \frac{5}{2}}\left(\frac{1}{2x-5}\right)$

Note: $x=5/2$ is a restriction

$\Rightarrow$ we 'should' take two one sided limits

$\lim_{x\to 5/2^-}\left(\frac{1}{2x-5}\right) = \frac{1}{\text{"}-0\text{"}} = -\infty$

$\lim_{x\to 5/2^+}\left(\frac{1}{2x-5}\right) = \frac{1}{\text{"}+0\text{"}} = +\infty$

"Since"

$\therefore \lim_{x\to 5/2^-}(f(x)) \neq \lim_{x\to 5/2^+}(f(x))$

$\Rightarrow \lim_{x\to 5/2}(f(x))$ dne.

To be continued…

Class/Homework for Section 1.4

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Hmk Check

3. Using a limit on the slope of a secant, determine the ~~slope~~ eqn of the tangent to each curve at the given domain value (Don't forget – a domain value isn't enough info…you need a ***point***!):

   a. $f(x) = -2x^2 + 5$, at $x = 1$

   b. $g(x) = -2x^2 + 5x$, at $x = 1$

   c. $h(x) = \sqrt{2x+1}$, at $x = 4$

   d. $p(x) = \frac{3}{x}$, at $x = 2$

$y = mx + b$

$y - y_1 = m(x - x_1)$

Point $(2, p(2)) = (2, \tfrac{3}{2})$

$$m_{tan} = \lim_{h \to 0}\left(\frac{p(2+h) - p(2)}{h}\right) = \lim_{h \to 0}\left(\frac{\frac{3}{2+h} - \frac{3}{2}}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\frac{3(2) - 3(2+h)}{2(2+h)}}{h}\right) = \lim_{h \to 0}\left(\frac{\frac{-3h}{2(2+h)}}{\frac{h}{1}}\right)$$

$$= \lim_{h \to 0}\left(\frac{-3\cancel{h}}{2(2+h)} \cdot \frac{1}{\cancel{h}}\right) = \lim_{h \to 0}\left(\frac{-3}{2(2+h)}\right) = -\frac{3}{4} \; (= m_{tan}$$

eqn of tangent : $y = -\frac{3}{4}x + b$ to find b use $(2, \tfrac{3}{2})$

$\rightarrow \frac{3}{2} = -\frac{3}{4}(2) + b \Rightarrow b = 3$

$\therefore$ eqn is $y = -\tfrac{3}{4}x + 3$.